Theory of Clean Rings and Matrices

Huanyin Chen
Marjan Sheibani Abdolyousefi

World Scientific

Theory of Clean Rings and Matrices

Other World Scientific Titles by the Author

Rings Related to Stable Range Conditions
ISBN: 978-981-4329-71-2

Theory of Clean Rings and Matrices

Huanyin Chen
Hangzhou Normal University, China

Marjan Sheibani Abdolyousefi
Semnan University, Iran

World Scientific

NEW JERSEY • LONDON • SINGAPORE • BEIJING • SHANGHAI • HONG KONG • TAIPEI • CHENNAI • TOKYO

Published by

World Scientific Publishing Co. Pte. Ltd.

5 Toh Tuck Link, Singapore 596224

USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601

UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

Library of Congress Control Number: 2022040233

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

THEORY OF CLEAN RINGS AND MATRICES

Copyright © 2023 by World Scientific Publishing Co. Pte. Ltd.

All rights reserved. This book, or parts thereof, may not be reproduced in any form or by any means, electronic or mechanical, including photocopying, recording or any information storage and retrieval system now known or to be invented, without written permission from the publisher.

For photocopying of material in this volume, please pay a copying fee through the Copyright Clearance Center, Inc., 222 Rosewood Drive, Danvers, MA 01923, USA. In this case permission to photocopy is not required from the publisher.

ISBN 978-981-126-012-4 (hardcover)
ISBN 978-981-126-013-1 (ebook for institutions)
ISBN 978-981-126-014-8 (ebook for individuals)

For any available supplementary material, please visit
https://www.worldscientific.com/worldscibooks/10.1142/12959#t=suppl

Typeset by Stallion Press
Email: enquiries@stallionpress.com

Printed in Singapore

Preface

An element in a ring is called clean if it can be written as the sum of an idempotent and a unit, and an element is further called strongly clean if such idempotent and unit commute. A ring is (strongly) clean if every its element is (strongly) clean. The notion of a clean ring was introduced by Nicholson in 1977 as an example of a ring with idempotents that can be lifted modulo any left (right) ideal. In the last decade, many authors have studied classes of rings defined by replacing "unit" and/or "idempotent" with different conditions on ring elements and/or possibly adding additional requirements to the above definition. For example, classes of strongly J-clean, uniquely clean, strongly nil-clean rings have all been obtained on this manner. Recently these classes have been further combined to produce new classes, such as almost clean, feckly clean, strongly $*$-clean rings, etc. The above mentioned classes of rings have been getting much attention. Their relations with other classes of rings, such as strongly π-regular rings, periodic rings and others, have been studied as well. Many kind of decompositions of matrices are obtained when applied to a single element.

In accordance with the logical development of the subject, the materials in this book fall roughly into five parts. The first part, consisting of Chapters 1 through 3, is concerned with strong cleanness. Here the basic properties of strongly clean rings are studied with a new perspective. In particular, we investigate this condition for full and triangular matrices over commutative and local rings. The second part, consisting of Chapters 4 through 6, is concerned with conditions stronger than strong cleanness. These allow us to factor a matrix into some simpler ones more accurately. In Chapters 7 through 11, we deal with conditions weaker than strong cleanness. From these, we consider analogues of strongly clean rings. Part four which consists of Chapter 12, Chapter 13 and Chapter 14 looks

at such rings with additional structures. This allows us to discuss strong cleanness and their extensions from the topological, continuous and involutional view of points. The final part, consisting of Chapter 15, deals with generalized Drazin inverse in a Banach algebra.

This book has evolved from our lectures and research over the past several years. It is based on the results of various published papers, particularly by the authors. We include many new results and almost all known ones, so that the book is self contained and reflects the recent progress in the subject. Most of main theorems involved are noted at the end of each chapter. We thank the many authors whose works have been used but not cited. Though these elementary facts are included, it was hard to prove them by more simpler manners as far as we can. The book is helpful for students and researchers in the fields of ring, matrix and operator theory. It is presumed that the reader is at the least acquainted with the most elementary facts of the theory of rings (for example as given in Anderson and Fuller's book [22]).

Huanyin Chen
Hangzhou Normal University
Hangzhou, China

Marjan Sheibani Abdolyousefi (corresponding author)
Farzanegan Campus, Semnan University
Semnan, Iran
Corresponding Email: m.sheibani@semnan.ac.ir

June 1, 2022

Notational Conventions

$\emptyset$	empty set
$\mathbb{C}$	field of complex numbers
$\mathbb{H}$	Hamilton ring
$\mathbb{N}$	set of natural numbers
$\mathbb{R}$	field of real numbers
$\mathbb{Z}$	ring of integers
$\mathbb{Z}_m$	ring of integers modulo m
$\mathbb{Z}[a]$	set $\{f(a) \mid f(t) \in \mathbb{Z}[t]\}$
$\widehat{A}$	set of all diagonal entries of the triangular matrix A
A^{-1}	inverse of the matrix A
A^d	g-Drazin inverse of the matrix A
A^T	transpose of the matrix A
(a)	the principal ideal generated by an element a
$[a, b]$	the commutator $ab - ba$
$\left(\frac{a}{p}\right)$	the Legendre symbol
$Aut_R(A)$	automorphism group of the R-module A
$A \cong B$	A is isomorphic to B
$A \subseteq_e B$	A is an essential submodule of B
$A \bigoplus B$	direct sum of modules A and B
$A \underset{R}{\bigotimes} B$	tensor product of R-modules A and B
$B(R)$	set of all central idempotents in the ring R
$C(R)$	center of the ring R

$char(R)$	characteristic of the ring R
$D(R)$	set of divisors of the ring R
$deg(f)$	the degree of the polynomial f
$det(A)$	determinant of the matrix A
$dim_D(V)$	dimension of the D-vector space V
$E(R)$	set of all idempotents in the ring R
$End_R(A)$	endomorphism ring of the R-module A
(f, g)	the ideal generated by polynomials f and g
$[f, g]$	the least common multiple of polynomials f and g
$gcd(f, g)$	the greatest common divisor of polynomials f and g
$GF(p^n)$	the Galois field of order p^n
$GL_n(R)$	group of all invertible $n \times n$ matrices over the ring R
$[G : H]$	the index of normal subgroup H in the group G
$Im(f)$	image of the function f
$I \lhd R$	I is an ideal of the ring R
I_n	$n \times n$ identity matrix
$J(R)$	Jacobson radical of the ring R
$Ker(f)$	kernel of the homomorphism f
$\ell(x)$	left annihilator of x
$Max(R)$	set of all maximal ideals of the ring R
$M_n(R)$	ring of all $n \times n$ matrices with entries from the ring R
nA	direct sum of n copies of the module A
$N_D(R)$	set of all non-zero-divisors of R
$N_D(X)$	set of all non-zero-divisors of $C(X)$
$r(x)$	right annihilator of x
$R[x]$	ring of polynomials over the ring R
$R[[x]]$	ring of formal power series over the ring R
$Spec(R)$	set of all prime ideals of the ring R
$\|S\|$	cardinal number of the set S
$T_n(R)$	ring of $n \times n$ lower triangular matrices with entries from R
$tr(A)$	trace of the matrix A
$U(R)$	group of all units in the ring R
$X \setminus Y$	The difference of sets X from Y
$Z(f)$	zero set $\{x \in X \mid f(x) = 0\}$ of a function $f : X \to \mathbb{R}$
$Z(R)$	set of all zero-divisors in R

Contents

Preface v

Notational Conventions vii

1. Strongly Clean Conditions 1

   1.1 Strongly Clean Rings 1
   1.2 2×2 Matrices over Local Rings 11
   1.3 2×2 Matrices over Integral Domains 22
   1.4 Rings of Power Series 29

2. Matrices over Commutative Rings 39

   2.1 Companion Matrices 39
   2.2 Strongly Clean $n \times n$ Matrices 44
   2.3 Explicit Criteria 57
   2.4 Projective-free Rings 66

3. Triangular and Generalized Matrix Rings 77

   3.1 Sufficient Conditions 77
   3.2 Necessary Conditions 87
   3.3 Triangular Matrices over Commutative Rings 97
   3.4 Generalized Matrix Rings 102

4. Strong J-Cleanness 119

   4.1 Strongly J-Clean Rings 120
   4.2 Solvability of Quadratic Equations 126
   4.3 Strongly $J^{\#}$-Clean Matrices 141
   4.4 Uniquely Clean Rings 148

5. Strong Nil-cleanness 175

5.1 Strongly Nil-clean Rings 176
5.2 Examples 189
5.3 Unique Decompositions 199
5.4 Strongly Nil-clean Matrices 209

6. Classes of Strongly Clean Rings 219

6.1 Strongly π-Regular Rings 219
6.2 Strongly J_n-Clean Rings 239
6.3 Strongly J_π-Clean Rings 253
6.4 Quasipolar Rings 262

7. Clean Properties 285

7.1 Clean Rings 285
7.2 Exchange Rings 301
7.3 Quasi-morphic Rings 317
7.4 Countably Linear Transformations 325

8. Rings Generated by Certain Elements 331

8.1 2-Good and 2-Clean Rings 331
8.2 Perspective Rings 349
8.3 Elementary Divisor Rings 360
8.4 Nil-clean and 2-Nil-clean Rings 375

9. Weakly Clean and Weakly Nil-clean Properties 389

9.1 Weakly Clean Rings 389
9.2 Weakly Nil-clean Rings 404
9.3 Strongly Weak Nil-clean Rings 420
9.4 Uniquely Weak Nil-clean Rings 427

10. Periodic and Weakly Periodic Rings 433

10.1 Periodic Rings 433
10.2 Generalized n-Like Rings 441
10.3 Uniquely π-Clean Rings 447
10.4 Weakly Periodic Rings 458

11. Conditions on Zero-divisors 465

11.1 Zero-divisors, Idempotents and Nilpotents 465
11.2 D-Rings and Their Generalizations 471
11.3 Decompositions of Zero-divisors 476
11.4 Almost Clean Rings 480

12. Topological Structures 491

12.1 Nearly Clean Rings 491
12.2 Commutative Rings 499
12.3 PM Rings 509
12.4 Feckly Clean Rings 524

13. Rings of Continuous Functions 537

13.1 Clean Functions 538
13.2 $C(X, \mathbb{C})$ and $C(X, A)$ 549
13.3 Classical Rings of Quotients 558
13.4 Feebly Clean Rings 565

14. Rings with Involutions 571

14.1 Strongly *-Clean Rings 571
14.2 The Uniqueness 579
14.3 Strongly Nil-*-clean Rings 588
14.4 *-Spaces 594

15. The Generalized Drazin Inverse 605

15.1 The g-Drazin Inverse 605
15.2 Additive Properties 614
15.3 Anti-triangular Matrices 625
15.4 Block Operator Matrices 637

Bibliography 645

Index 677

This page intentionally left blank

Chapter 1

Strongly Clean Conditions

Let R be an associative ring with an identity. We say that $a \in R$ is strongly clean provided that there exists an idempotent $e \in R$ such that $a - e \in U(R)$ and $ae = ea$. A ring R is strongly clean if every element in R is strongly clean. In Section 1.1, we investigate elementary strongly clean properties. A criterion for an endomorphism to be strongly clean is given. In Section 1.2, we develop necessary and sufficient conditions for a 2×2 matrix over local rings to be strongly clean. In Section 1.3, we consider strongly clean 2×2 matrices over integral domains. Finally, in Section 1.4, we discuss the problem of determining when a power series is strongly clean.

1.1 Strongly Clean Rings

The main purpose of this section is to explore elementary properties of strongly clean rings which will be used repeatedly in the sequel. A ring R is strongly π-regular provided that for any $a \in R$ there exists $n \in \mathbb{N}$ such that $a^n \in a^{n+1}R$. We start by recording the following crucial characterization.

Lemma 1.1.1. *Let R be a ring. Then the following are equivalent:*

(1) *R is strongly π-regular.*
(2) *For any $a \in R$, there exist $n \in \mathbb{N}, b \in R$ such that $a^n = a^{n+1}b, b = b^2a$ and $ab = ba$.*

Proof. (1) $\Rightarrow$ (2) Let $a \in R$. Then there exist $n \in \mathbb{N}$ and $r \in R$ such that $a^n = a^{n+1}r$, and so $a^n = a^{n+i}r^i$ for any $i \in \mathbb{N}$. Also we have some $m \geq n$ and $c \in R$ such that $r^m = r^{m+1}c$. Thus, $r^m = r^{m+j}c^j$ for any $j \in \mathbb{N}$, and so $a^m = a^{m-n}a^n = a^{m-n}a^{n+m}r^m = a^{2m}r^m$. Let $x = a^m, y = r^m$ and $z = c^m$. Then $x = x^2y$ and $y = y^2z$. By hypothesis, there exist $k \in \mathbb{N}, d \in R$ such that $(z-x)^k = (z-x)^{k+1}d$. Clearly, $xz = (x^2y)z = x(x^2y)yc = x^3(y^2z) = x^3y = x^2$ and $xyz = x^2(y^2z) = x^2y = x$. This implies that $(z-x)^2 = z^2 - zx - xz + x^2 = z(z-x)$; whence, $xy(z-x)^2 = xyz(z-x) = x(z-x) = 0$. Since $y^2(z-x)^2 = y^2z(z-x) = y(z-x)$, we have $y(z-x) = y\big(y(z-x)\big)(z-x) = y\big(y^2(z-x)^2\big)(z-x) = y^3(z-x)^3$. Similarly, $y^k(z-x)^k = y(z-x)$. Hence $y(z-x)^2d = y^k(z-x)^{k+1}d = y^k(z-x)^k = y(z-x)$, and so $0 = xy(z-x)^2d = xy(z-x)$. This implies that $x = xyz = xyx$. In addition, $0 = xy(z-x)d = xy^2(z-x)^2d = xy\big(y(z-x)\big) = xy^2(z-x)$. This means that $xy^2z = xy^2x$, and then $x = xyx = x(y^2z)x = xy^2x^2$. Therefore $x \in x^2R \cap Rx^2$, whence $a^m \in a^{m+1}R \cap Ra^{m+1}$.

Write $a^m = a^{m+1}s = ta^{m+1}$. Then $a^ms = ta^{m+1}s = ta^m$. By induction, we get $a^ms^l = t^la^m$ for any $l \in \mathbb{N}$. Let $b = a^ms^{m+1}$. Then $b = t^{m+1}a^m$. One easily checks that $ab = a^{m+1}s^{m+1} = (a^{m+1}s)s^m = a^ms^m = t^ma^m = t^m(ta^{m+1}) = ba; a^{m+1}b = a^{2m+1}s^{m+1} = a^m(a^{m+1}s)s^m = a^{2m}s^m = \cdots = a^{m+1}s = a^m; b^2a = b(ab) = b(a^{m+1}s^{m+1}) = a^{m+1}bs^{m+1} = a^ms^{m+1} = b$, as required.

(2) $\Rightarrow$ (1) is obvious. □

We say that an element $a \in R$ is strongly π-regular provided that there exist $n \in \mathbb{N}, b \in R$ such that $a^n = a^{n+1}b, b = b^2a$ and $ab = ba$. By the preceding discussion, $a \in R$ is strongly π-regular if and only if $a^m \in a^{m+1}R \cap Ra^{m+1}$ for some $m \in \mathbb{N}$. Thus, a ring R is strongly π-regular if and only if every element in R is strongly π-regular. As an immediate consequence of Lemma 1.1.1, we see that strongly π-regular rings are left-right symmetric. That is, a ring R is strongly π-regular if and only if for any $a \in R$, there exists $n \in \mathbb{N}$ such that $a^n \in Ra^{n+1}$. The following was proved by Nicholson [432, Theorem 1].

Theorem 1.1.2. *Every strongly π-regular ring is strongly clean.*

Proof. Let R be strongly π-regular. By virtue of Lemma 1.1.1, there exist $n \in \mathbb{N}, b \in R$ such that $a^n = a^{n+1}b, b = b^2a$ and $ab = ba$. Let $c = b^na^nb^n$. Then $a^n = a^{2n}c, ac = ca$ and $c = c^2a^n$. Set $v = c + 1 - a^nc$. Then $v^{-1} = a^n + 1 - a^nc$. Set $f = a^nc$. Then $f = f^2 \in R$ and $fv = vf$. Choose

$w = v^{-1}$. Then $a^n = fw = wf$, where f, w and a all commutate. Let $e = 1 - f$ and $u = a - e$. Choose $z = a^{n-1}w^{-1}f - (1 + a + \cdots + a^{n-1})e$. Then

$$\begin{aligned} uz &= (a-e)\big(a^{n-1}w^{-1}f - (1+a+\cdots+a^{n-1})e\big) \\ &= a^n w^{-1} f + (1-a)(1+a+\cdots+a^{n-1})e \\ &= f + (1-a^n)e \\ &= f + e \\ &= 1. \end{aligned}$$

As $uz = zu$, we get $zu = 1$. Therefore $a = e + u, e = e^2, u \in U(R)$ and $ae = ea$, as asserted. □

The converse of Theorem 1.1.2 is not true. Let $R = \prod_{n=1}^{\infty} \mathbb{Z}_{p^n}$ (p is a prime). Then $J(R) = \prod_{n=1}^{\infty} p\mathbb{Z}_{p^n}$, and so $p \in J(R)$. If R is strongly π-regular, then we have some $m \in \mathbb{N}$ such that $p^m = p^{m+1}x$ for an $x \in R$. Hence $p^m = 0$ in R, a contradiction. Therefore R is not strongly π-regular, while it is strongly clean.

Strongly π-regular rings form a large subclass of strongly clean rings. Let A is an algebra over a field F. An element a of an algebra A over a field F is said to be algebraic over F if a is the root of some non-constant polynomial in $F[x]$. A is said to be an algebraic algebra over F if every element of A is algebraic over F.

Example 1.1.3. Every algebraic algebra over a field is strongly π-regular.

Proof. Let R be an algebraic algebra over a field F, and let $a \in R$. Then a is the root of some non-constant polynomial in $F[x]$. That is, we can find some $a_m, \ldots, a_n \in F$ such that $a_n a^n + a_{n-1}a^{n-1} + \cdots + a_m a^m = 0$, where $a_m \neq 0$. Hence, $a^m = -a_m^{-1}(a_n a^n + \cdots + a_{m+1}a^{m+1}) = -a_m^{-1}(a_n a^{n-m-1} + \cdots + a_{m+1})a^{m+1}$. Set $b = -a_m^{-1}(a_n a^{n-m-1} + \cdots + a_{m+1})$. Then $a^m = ba^{m+1}$; hence, R is strongly π-regular. □

An algebra over a field F that is finitely dimensional as a vector space over F is called a finitely dimensional algebra over F. For instance, the 2×2 matrix ring over $F[x]/(x^2)$, where F is any field, is a finitely dimensional algebra.

Example 1.1.4. Every finitely dimensional algebra over a field is strongly π-regular.

Proof. Let R be a finitely dimensional algebra over a field F, and let $a \in R$. Write $dim_F(R) = n$. Then $1, a, a^2, \ldots, a^n$ is linearly dependent. Thus, we can find some $a_0, a_1, \ldots, a_n \in F$ such that $a_0 + a_1 a + \cdots + a_n a^n = 0$, where not all of the a_i are zero. Thus, a is the root of some non-constant polynomial in $F[x]$, and so R is an algebraic algebra over F. According to Example 1.1.3, R is strongly π-regular. □

Let p be a prime. Then the ring $T_n(\mathbb{Z}_p)$ is strongly π-regular. Clearly, $T_2(\mathbb{Z}_p)$ is a $\frac{n(n+1)}{2}$ dimensional algebra over $\mathbb{Z}_p$. Thus, we are done from Example 1.1.4. A ring R is called right (left) artinian if it does not contain an infinite descending chain of right (left) ideals, and artinian if it is both left and right artinian. Further, we claim that every right (left) artinian ring is strongly π-regular. Let R be right artinian. For any $a \in R$, we have a descending chain of right ideals $aR \supseteq a^2R \supseteq \cdots$, and so we can find some $n \in \mathbb{N}$ such that $a^nR = a^{n+1}R$. Thus, $a^n = a^n b$ for some $b \in R$, and then we are through. As every finite ring is artinian, we see that $M_n(\mathbb{Z}_m)$ is strongly π-regular for all $m \geq 2$, e.g., $M_n(\mathbb{Z}_4)$. Furthermore, every simisimple ring is artinian, hence strongly π-regular, e.g., $M_n(D)$ where D is a division ring.

A ring R is called an exchange ring provided that for any $a \in R$, there exists an idempotent $e \in R$ such that $e \in aR$ and $1-e \in (1-a)R$. Let I be a subset of a ring R. I is right T-nilpotent if for every sequence $a_1, a_2, \ldots$ in I there is $n \in \mathbb{N}$ such that $a_n \cdots a_1 = 0$. I is locally nilpotent if its every finitely generated subring is nilpotent. We now derive

Theorem 1.1.5. *Let R be an exchange ring with artinian primitive factors. If the Jacobson radical of any homomorphic image of R is right T-nilpotent or locally nilpotent, then R is a strongly π-regular ring.*

Proof. Suppose that R is not a strongly π-regular ring. Then there exists some $x \in R$ such that $x^n \notin x^{n+1}R$ for any $n \in \mathbb{N}$. Set

$$\Omega = \{A \trianglelefteq R \,|\, x^n \neq x^{n+1}y \,(mod\, A) \text{ for any } n \in \mathbb{N} \text{ and any } y \in R\}.$$

Clearly, Ω is a nonempty inductive set. By Zorn's Lemma, we have an ideal Q which is maximal in Ω. Let $S = R/Q$. We then see that S is indecomposable as a ring.

Case I. $J(S)$ is right T-nilpotent. If Q is a prime ideal of R, then S is a prime ring. Assume that $0 \neq a \in J(S)$. Then there exist $x_1, x_2, \ldots, x_n, \ldots \in S$ such that

$$ax_1a \neq 0, ax_2ax_1a \neq 0, \ldots, ax_na \cdots ax_1a \neq 0, \ldots.$$

This gives a contradiction, thus $J(S) = 0$. By using [563, Lemma 3.7], S is simple artinian; hence, it is strongly π-regular. This contradicts the choice of Q. So we may assume that Q is not prime. Thus, we have two ideals K and L of R such that $Q \subsetneqq K, L$ and $KL \subseteq Q$. By the maximality of Q, there are $k, l \in \mathbb{N}$ and $s, t \in R$ such that $x^k = x^{k+l+1}s \,(mod\, K)$ and $x^l = x^{k+l+1}t \,(mod\, L)$. It is easy to verify that $(x^k - x^{k+l+1}s)(x^l - x^{k+l+1}t) \in KL \subseteq Q$. Hence we have some $z \in R$ such that $x^{k+l} = x^{k+l+1}z \,(mod\, Q)$, a contradiction.

Case II. $J(S)$ is locally nilpotent. If $J(S) \neq 0$, by the maximality of Q, we have $m \in \mathbb{N}$ such that $(x+Q)^m = (x+Q)^{m+1}w \,(mod\, J(S))$ for some $w \in R$. Write a for $x+Q$. Then $j := a^m - a^{m+1}w$ satisfies $j \in J(S)$. Hence, there is a positive integer i such that $(SjS)^i = 0$. One easily computes that, for $k \geq 1$,

$$a^m - a^{m+k}w^k = j + ajw + a^2jw^2 + \cdots + a^{k-1}jw^{k-1} \in SjS.$$

Thus $(a^m - a^{mi+1}v)^i = 0$ for some $v \in R$, and so $x^{mi} = x^{mi+1}q$ for some $q \in R$, a contradiction. This implies that $J(S) = 0$. By [563, Lemma 3.7], S is simple artinian. This contradicts the choice of Q.

Therefore R is strongly π-regular. □

Corollary 1.1.6. *Let R be a commutative exchange ring. Then the following are equivalent*:

(1) *R is a strongly π-regular ring.*
(2) *The Jacobson radical of any homomorphic image of R is nil.*

In [431, Proposition], Nicholson proved that every exchange ring is left-right symmetric. That is, we have

Proposition 1.1.7. *A ring R is an exchange ring if and only if for any $a \in R$, there exists an idempotent $e \in R$ such that $e \in Ra$, $1 - e \in R(1-a)$.*

Proof. Let R be an exchange ring, and let $a \in R$. Then there exists an idempotent $f \in aR$ such that $1 - f \in (1-a)R$. Set $g = 1 - f$ and $b = 1 - a$. Then $g \in bR$. Write $e = ar'$ and $g = bs'$. Set $r = r'e$ and $s = s'g$. Then we have $rar = r, sbs = s, rbs = 0$ and $sar = 0$. Let $r'' = 1 - sb + rb$ and $s'' = 1 - ra + sa$. Thus,

$$ar'' = a(1-sb) + arb = a(1-sb) + (1-bs)b = a(1-sb) + b(1-sa) = 1 - sb.$$

Clearly, $r''s = 0$ and $s''r = 0$. Hence $r''ar'' = r''$. Similarly, $bs'' = 1 - ra$, and so $s''bs'' = s''$. It follows from $xy = yx$ that

$$r''a + s''b = (1 - sb + rb)a + (1 - ra + sa)b = a + b = 1.$$

Set $h = r''a$. Then $h = h^2 \in Ra$ and $1 - h = s''b \in R(1-a)$, as required. The converse is clear by applying the preceding discussion to the opposite ring R^{op}. □

Further, we note that a ring R is an exchange ring if and only if for any $a \in R$ there exists an idempotent $e \in aR \cap Ra$ such that $1 - e \in (1-a)R \cap R(1-a)$ (see [359, Theorem A]). For strongly clean rings, we have

Theorem 1.1.8. *Let R be a ring. Then the following are equivalent:*

(1) *R is strongly clean.*
(2) *For any $a \in R$, there exists an idempotent $e \in R$ such that $ae = ea \in U(eRe)$ and $(1-a)(1-e) = (1-e)(1-a) \in U\big((1-e)R(1-e)\big)$.*
(3) *For any $a \in R$, there exist an idempotent $e \in R$ and $x, y \in R$ such that $e = ax = xa$ and $1 - e = (1-a)y = y(1-a)$.*

Proof. (1) ⇒ (3) For any $a \in R$, write $a = f + v$, where $f = f^2, v \in U(R)$ and $fv = vf$. Set $e = 1 - f$. Then $e = e^2 \in R$. Clearly, $a - e = (a^2 - a)v^{-1}$. Hence, $e = a\big(1 - (a-1)v^{-1}\big)$ and $1 - e = (1-a)\big(1 - av^{-1}\big)$. Set $x = 1 - (a-1)v^{-1}$ and $y = 1 - av^{-1}$. Then $e = xa = ax$ and $1 - e = (1-a)y = y(1-a)$.

(3) ⇒ (2) For any $a \in R$, there exist an idempotent $e \in R$ and $x, y \in R$ such that $e = ax = xa$ and $1-e = (1-a)y = y(1-a)$. Thus, $ea = axa = ae$. Further, $e = (xa)(xa) = x(ax)a = (exe)(ea)$. Likewise, $e = (ea)(exe)$. This shows that $eae \in U(eRe)$. Similarly, $(1-a)(1-e) = (1-e)(1-a) \in U\big((1-e)R(1-e)\big)$.

(2) ⇒ (1) For any $a \in R$, there exists an idempotent $e \in R$ such that $ae = ea \in U(eRe)$ and $(1-a)(1-e) = (1-e)(1-a) \in U\big((1-e)R(1-e)\big)$. Thus $a = (1-e) + \big(ea - (1-e)(1-a)\big)$. Clearly, $ea - (1-e)(1-a) \in U(R)$, and so $a \in R$ is strongly clean, as required. □

In view of Theorem 1.1.8, we see that every strongly clean ring is an exchange ring. A ring R is abelian provided that all idempotents in R are central. Further, we now derive

Corollary 1.1.9. *Every abelian exchange ring is strongly clean.*

Proof. Let R be an abelian exchange ring, and let $a \in R$. Then there exists an idempotent $e \in R$ and $x, y \in R$ such that $e = ax$ and $1-e = (1-a)y$. We may assume that $x = ex = xe$ and $y = (1-e)y = y(1-e)$. It is easy to verify

that $(xa)^2 = x(ax)a = xea = xa$ and $xa = (ex)a = (ax)(xa) = a(xa)x = e$. Likewise, $y(1-a) = 1-e$. Therefore R is strongly clean by Theorem 1.1.8. □

Lemma 1.1.10. *Let R be a ring, and let $a = e + u$ with $e = e^2 \in R, u \in U(R)$ and $ae = ea$. Then $\ell(a) \subseteq \ell(1-e)$ and $r(a) \subseteq r(1-e)$.*

Proof. Let $r \in \ell(a)$. Then $ra = 0$. By hypothesis, $r(1-e) = r(1+u)$; hence, $r(1-e) = r(1+u)(1-e) = r(1-e)(1+u)$. This implies that $r(1-e)u = 0$, and so $r \in \ell(1-e)$. Therefore $\ell(a) \subseteq \ell(1-e)$. Likewise, we show that $r(a) \subseteq r(1-e)$. □

Lemma 1.1.11. *Let R be a ring, and let $e \in R$ be an idempotent. Then $a \in eRe$ is strongly clean in R if and only if a is strongly clean in eRe.*

Proof. Suppose that $a \in eRe$ is strongly clean in eRe. Then $a = f+u, f = f^2 \in eRe, u \in U(eRe)$ and $af = fa$. Then $a = (f+1-e) + (u-1+e)$. It is easy to check that $f+1-e \in R$ is an idempotent and $u-1+e \in U(R)$. In addition, $a(f+1-e) = (f+1-e)a$. Thus, $a \in eRe$ is strongly clean in R.

Conversely, suppose that $a = f+u, f = f^2 \in R, u \in U(R)$ and $af = fa$. As $a \in eRe$, we see that $1-e \in \ell(a) \cap r(a) \subseteq \ell(1-f) \cap r(1-f) = Rf \cap fR = fRf$. Hence $f(1-e) = 1-e = (1-e)f$, and so $fe = ef$. Therefore $a = efe + eue$, $(fef)^2 = fef$ and $a(efe) = af = fa = (efe)a$. In addition, $eu = a - ef = a - efe = eue$. Likewise, $ue = eue$. Therefore $eue \in U(eRe)$, as required. □

We are now ready to prove the following result of Chen [187, Theorem 2.4].

Theorem 1.1.12. *Let R be strongly clean. Then eRe is strongly clean for any idempotent $e \in R$.*

Proof. Let $e^2 = e \in R$ and $a \in eRe$. Then $a \in eRe$ is strongly clean in R. In view of Lemma 1.1.11, $a \in eRe$ is strongly clean in eRe. Hence, eRe is strongly clean. □

Corollary 1.1.13. *Let R be a ring. If $M_n(R)$ is strongly clean, then so is R.*

Proof. Choose $e = diag(1, 0, \ldots, 0) \in M_n(R)$. Then $R \cong eM_n(R)e$, and the result follows from Theorem 1.1.12. □

We come now to characterize strongly clean endomorphisms by means of the decompositions of related modules.

Theorem 1.1.14. *Let $E = End(_RM)$, and let $\alpha \in E$. Then the following are equivalent:*

(1) *$\alpha \in E$ is strongly clean.*
(2) *There exists $\pi^2 = \pi \in E$ such that $\alpha\pi = \pi\alpha$, $\alpha\pi \in U\big(\pi E\pi\big)$ and $(1-\alpha)(1-\pi) \in U\big((1-\pi)E(1-\pi)\big)$.*
(3) *$M = P \oplus Q$ where P and Q are α-invariant, and $\alpha|_P \in U\big(End(P)\big)$ and $(1-\alpha)|_Q \in U\big(End(Q)\big)$.*
(4) *$M = P_1 \oplus \cdots \oplus P_n$ for some $n \geq 1$ where P_i is α-invariant and $\alpha|_{P_i}$ is strongly clean in $End(P_i)$ for each i.*

Proof. (1) $\Rightarrow$ (2) Since α is strongly clean in E, there exists an idempotent $\pi \in E$ and a $u \in U(E)$ such that $\alpha = (1-\pi) + u$ and $\pi u = u\pi$. Thus, $\pi\alpha = \pi u \in U\big(\pi E\pi\big)$. Further, $1-\alpha = \pi + (-u)$, and so $(1-\pi)(1-\alpha) = (1-\pi)(-u) \in U\big((1-\pi)E(1-\pi)\big)$.

(2) $\Rightarrow$ (3) Set $P = M\pi$ and $Q = M(1-\pi)$. Then $M = P \oplus Q$. Since $\alpha\pi = \pi\alpha$, we see that P and Q are α-invariant. As $\alpha\pi \in U\big(\pi E\pi\big)$, we can construct $\beta : P \to P$ given by $(x)\beta = (x)(\alpha\pi)^{-1}$. Then $\alpha|_P = \beta^{-1} \in U\big(End(P)\big)$. Likewise, one easily checks that $(1-\alpha)|_Q \in U\big(End(Q)\big)$.

(3) $\Rightarrow$ (4) is obvious.

(4) $\Rightarrow$ (1) Let $\lambda_i \in End(P_i)$. We construct an R-morphism $\overline{\lambda_i} \in E$ given by $\left(\sum\limits_{j=1}^{n} p_j\right)\overline{\lambda_i} = (p_i)\lambda_i$ for any $p_j \in P_j$. By hypothesis, there exist $\pi_j^2 = \pi_j \in End(P_j), \sigma_j \in U\big(End(P_j)\big)$ such that $\alpha|_{P_j} = \pi_j + \sigma_j$ and $\pi_j\sigma_j = \sigma_j\pi_j$. Thus, $\alpha = \sum\limits_{j=1}^{n} \overline{\alpha|_{P_j}} = \sum\limits_{j=1}^{n} \overline{\pi_j} + \sum\limits_{j=1}^{n} \overline{\sigma_j}$. Obviously, $\overline{\pi_j\pi_k} = \overline{\pi_k\pi_j} = 0 (j \neq k)$, we see that $\sum\limits_{j=1}^{n} \overline{\pi_j} \in E$ is an idempotent. In addition, $\overline{\pi_j\sigma_k} = \overline{\sigma_k\pi_j} = 0 (j \neq k)$. Hence,

$$\left(\sum_{j=1}^{n} \overline{\pi_j}\right)\left(\sum_{j=1}^{n} \overline{\sigma_j}\right) = \sum_{j=1}^{n} \overline{\pi_j\sigma_j} = \sum_{j=1}^{n} \overline{\sigma_j\pi_j} = \left(\sum_{j=1}^{n} \overline{\sigma_j}\right)\left(\sum_{j=1}^{n} \overline{\pi_j}\right).$$

Clearly, $\overline{\sigma_j\sigma_k} = \overline{\sigma_k\sigma_j} = 0 (j \neq k)$. Since $\sigma_j \in U\big(End(P_j)\big)$, we see that $\sum\limits_{j=1}^{n} \overline{\sigma_j} \in U\big(E\big)$. This yields that $\alpha \in E$ is strongly clean, as asserted. □

Corollary 1.1.15. *Let $E = End(_RM)$, and let $\alpha \in E$. Then the following are equivalent:*

(1) *$\alpha \in E$ is strongly clean.*
(2) *There exists left E-module decomposition $E = A \oplus B$ where A and B are α-invariant, $\alpha|_A \in U\big(End(A)\big)$ and $(1_M - \alpha)|_B \in U\big(End(B)\big)$.*

Proof. (1) $\Rightarrow$ (2) In view of Theorem 1.1.14, there exists $\pi^2 = \pi \in E$ such that $\alpha\pi = \pi\alpha$, $\alpha\pi \in U\big(\pi E\pi\big)$ and $(1-\alpha)(1-\pi) \in U\big((1-\pi)E(1-\pi)\big)$. Set $A = E\pi$ and $B = E(1_M - \pi)$. Then $E = A \oplus B$ where A and B are α-invariant. Moreover, we check that $\alpha|_A \in U\big(End(A)\big)$ and $(1_M - \alpha)|_B \in U\big(End(B)\big)$.

(2) $\Rightarrow$ (1) By hypothesis, $E = A \oplus B$, where A and B are α-invariant, and $\alpha|_A \in U\big(End(A)\big)$ and $(1_M - \alpha)|_B \in U\big(End(B)\big)$. Let $p : E \to A$ be the projection, and let $\pi = (1_M)p$. Then $\pi = \pi^2 \in E$. In addition, $A = E\pi$ and $B = E(1_M - \pi)$. Thus $\alpha\pi = \pi\alpha$. Further, we see that $\alpha\pi \in U\big(\pi R\pi\big)$ and $(1-\alpha)(1-\pi) \in U\big((1-\pi)R(1-\pi)\big)$. According to Theorem 1.1.14, we complete the proof. □

Let $f(x) \in R[x]$. We say that R is strongly $g(x)$-clean provided that for any $a \in R$ there exists $e \in R$ such that $a - e \in U(R), ae = ea$ and $g(e) = 0$. Clearly, R is strongly clean if and only if R is strongly $(x^2 - x)$-clean, if and only if R is strongly $(x^2 + x)$-clean. We end this section by certain relations between strong cleanness and strong $g(x)$-cleanness (see [250]).

Lemma 1.1.16. *Let R be a ring, and let $a, b \in C(R)$. Then the following are equivalent:*

(1) *R is strongly $(x-a)(x-b)$-clean.*
(2) *R is strongly clean and $a - b \in U(R)$.*

Proof. (1) $\Rightarrow$ (2) For any $c \in R$, we can find $e \in R$ and $u \in U(R)$ such that $c(b-a) + a = e + u, eu = ue$ and $(e-a)(e-b) = 0$. Thus, $c = (e-a)(b-a)^{-1} + u(b-a)^{-1}$. It is easy to verify that $\big((e-a)(b-a)^{-1}\big)^2 = (e-a)(b-a)^{-1}$, and that

$$(e-a)(b-a)^{-1}u(b-a)^{-1} = u(b-a)^{-1}(e-a)(b-a)^{-1}.$$

Thus, R is strongly clean. For $b \in R$, we can find an $e \in R$ such that $b = e + u$ and $(e-a)(e-b) = 0$. This shows that $e - b = -u \in U(R)$, and so $e = a$. Therefore $a - b = -u \in U(R)$.

(2) $\Rightarrow$ (1) For any $c \in R$, there exist $e = e^2 \in R$ and $u \in U(R)$ such that $(c-a)(b-a)^{-1} = e + u$ and $eu = ue$. Hence, $c = \big(a + e(b-a)\big) + u(b-a)$.

One easily checks that

$$\big((a+e(b-a))-a\big)\big((a+e(b-a))-b\big)=e(b-a)(e-1)(b-a)=0$$

and

$$\big(a+e(b-a)\big)u(b-a)=u(b-a)\big(a+e(b-a)\big),$$

hence the result. □

Lemma 1.1.17. *Let R be a ring, and let $m,n\in\mathbb{N}$. Then the following are equivalent:*

(1) *R is strongly (x^2-nx)-clean.*
(2) *R is strongly (x^2-n^mx)-clean.*
(3) *R is strongly clean and $n\cdot 1_R\in U(R)$.*

Proof. (2) $\Rightarrow$ (3) By hypothesis, there exists $e\in R$ such that $n^m-e\in U(R)$ and $e^2-n^me=0$. This shows that $e=0$, and so $n\in U(R)$. As R is strongly $(x-0)(x-n)$-clean, it follows by Lemma 1.1.16 that R is strongly clean.

(3) $\Rightarrow$ (2) Since $n\in U(R)$, we see that $0-n^m\in U(R)$. In light of Lemma 1.1.16, R is strongly $(x-0)(x-n^m)$-clean. That is, R is strongly (x^2-n^mx)-clean.

(1) $\Leftrightarrow$ (3) is obvious by choosing $m=1$ in (2) $\Leftrightarrow$ (3). □

Theorem 1.1.18. *Let R be a ring, and let $m\in\mathbb{N}$. Then the following are equivalent:*

(1) *R is strongly (x^2-1)-clean.*
(2) *R is strongly (x^2-2x)-clean.*
(3) *R is strongly (x^2-2^mx)-clean.*
(4) *R is strongly (x^2-4x+3)-clean.*
(5) *R is strongly clean and $2\in U(R)$.*

Proof. (1) $\Rightarrow$ (2) Let $a\in R$. Then we have an $e\in R$ and a $u\in U(R)$ such that $1-a=e+u, eu=ue$ and $e^2=1$. Thus, $a=(1-e)-u$ and $(1-e)u=u(1-e)$. Further, $(1-e)^2-2(1-e)=0$. This shows that R is strongly (x^2-2x)-clean.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) is clear from Lemma 1.1.16 and Lemma 1.1.17.

(5) $\Rightarrow$ (1) Let $a\in R$. Then we have an idempotent $e\in R$ and a $u\in U(R)$ such that $\frac{a+1}{2}=e+u$ and $eu=ue$. Hence, $a=(2e-1)+2u$ and $(2e-1)a=a(2e-1)$. As $(2e-1)^2=1$, we conclude that R is strongly (x^2-1)-clean. □

1.2 2×2 Matrices over Local Rings

A ring R is local provided that R has only one maximal right ideal. As is well known, a ring R is local if and only if $R/J(R)$ is a division ring, if and only if for every $a \in R$, either a or $1-a$ is invertible. For instance, Hamilton ring $\mathbb{H}$ is a noncommutative local ring. The aim of this section is to establish necessary and sufficient conditions under which a 2×2 matrix A over a local ring R is strongly clean. Let $x \in R$, $i \neq j$, and let $B_{ij}(x)$ be the matrix with 1 in the diagonal entries and x in the (i,j)-entry, otherwise, 0. For convenience, we use $[a,b]$ to denote the diagonal matrix $diag(a,b)$. We begin with

Lemma 1.2.1. *Let R be a local ring, and let $A \in M_2(R)$. Then either $A \in GL_2(R)$, or $I_2 - A \in GL_2(R)$, or A is similar to a matrix $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in J(R), \mu \in 1 + J(R)$.*

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Case I. $c \in U(R)$. Then one easily checks that $[c,1]B_{12}(-ac^{-1})AB_{12}(ac^{-1})[c^{-1},1] = \begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$ for some $\lambda, \mu \in R$. If $\lambda \in U(R)$, then $A \in GL_2(R)$. If $\lambda \in J(R), 1-\mu \in U(R)$, $[c,1]B_{12}(-ac^{-1})(I_2 - A)B_{12}(ac^{-1})[c^{-1},1] = \begin{pmatrix} 1 & -\lambda \\ -1 & 1-\mu \end{pmatrix} = B_{21}(-1)\begin{pmatrix} 1 & -\lambda \\ 0 & 1-\lambda-\mu \end{pmatrix}$; hence, $I_2 - A \in GL_2(R)$. Otherwise, A is similar to a matrix $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in J(R), \mu \in 1 + J(R)$. Case II. $b \in U(R)$. Then $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$. Case III. $c, b \in J(R), a-d \in U(R)$. Then $B_{21}(-1)AB_{21}(1) = \begin{pmatrix} a+b & b \\ a-d+b-c & b-d \end{pmatrix}$. Case IV. $c, b \in J(R), a, d \in U(R)$. Then $B_{21}(-ca^{-1})A = \begin{pmatrix} a & b \\ 0 & d-ca^{-1}b \end{pmatrix}$; hence, $A \in GL_2(R)$. Case V. $c, b, a, d \in J(R)$. Then $I_2 - A \in GL_2(R)$. In any case, we obtain the result. □

Lemma 1.2.2. *Let I be an ideal of R such that $I \subseteq J(R)$, and let P and Q be finitely generated projective left R-module. If $P/IP \cong Q/IQ$, then $P \cong Q$.*

Proof. Let $\sigma : P/IP \cong Q/IQ$. Since P is projective, there is $\tau : P \to Q$ such that the diagram

$$\begin{array}{ccc} P & \stackrel{\pi_1}{\to} & P/IP \\ \tau\downarrow & & \downarrow\sigma \\ Q & \stackrel{\pi_2}{\to} & Q/IQ \end{array}$$

commutes, where π_1, π_2 are canonical maps, i.e., $\pi_i(x) = \overline{x}$. For any $q \in Q$, we have $p \in P$ such that $\sigma\pi_1(p) = \pi_2(q)$; hence, $q = \tau(p) + \big(q - \tau(p)\big)$. Clearly, $\pi_2\big(q - \tau(p)\big) = \sigma\pi_1(p) - \pi_2\tau(p) = 0$, and so $q \in \tau(P) + Ker\pi_2$. Thus, $Q = \tau(P) + Ker\pi_2$. As $Ker\pi_2 = IQ$, it follows by Nakayama's Lemma, we get $Q = \tau(P)$, i.e., τ is an epimorphism.

Since Q is projective, we can find $i : Q \to P$ such that $\tau i = 1_Q$, and then i is a monomorphism. Obviously, $\sigma\pi_1 i = \pi_2\tau i = \pi_2$; whence, $\pi_1 i = \sigma^{-1}\pi_2$. Similarly to the preceding discussion, we get $P = iQ$, i.e., i is an epimorphism. Thus i is an isomorphism. Accordingly, $\tau : P \cong Q$. □

Lemma 1.2.3. *Every finitely generated projective left module over a local ring is free.*

Proof. Let P be a finitely generated projective left module over a local ring R. Then $\big(R/J(R)\big)\bigotimes\limits_R P$ is a finitely generated projective $R/J(R)$-module. Clearly, every finitely generated projective module over a division ring is free. Write $\big(R/J(R)\big)\bigotimes\limits_R P \cong n\big(R/J(R)\big)$. Construct $f : R/J(R) \times P \to \big(R/J(R)\big)\bigotimes\limits_R P, f\big(\overline{r}, p\big) = \overline{r} \otimes p$ and $\varphi : R/J(R) \times P \to P/J(R)P, \varphi(\overline{r}, p) = \overline{rp}$. Then there exists a unique R-morphism $\phi : \big(R/J(R)\big)\bigotimes\limits_R P \to P/J(R)P$ such that $\phi f = \varphi$. Clearly, ϕ is an epimorphism. If $\phi\Big(\sum\limits_{i=1}^{m} \overline{r_i} \otimes p_i\Big) = 0$, then $\sum\limits_{i=1}^{m} \overline{r_i p_i} = 0$. Write $\sum\limits_{i=1}^{m} r_i p_i = \sum\limits_{i=1}^{s} a_i q_i$ where $a_i \in J(R), q_i \in P$. Then $\sum\limits_{i=1}^{s} \big(\overline{r_i} \otimes p_i\big) = \sum\limits_{i=1}^{s} \overline{1} \otimes (r_i p_i) = \overline{1} \otimes \Big(\sum\limits_{i=1}^{n} r_i p_i\Big) = \overline{1} \otimes \Big(\sum\limits_{i=1}^{s} a_i q_i\Big) = \sum\limits_{i=1}^{s} \big(\overline{a_i} \otimes q_i\big) = 0$; hence, ϕ is a monomorphism. Thus, $\big(R/J(R)\big)\bigotimes\limits_R P \cong P/J(R)P$. Likewise, $n\big(R/J(R)\big) \cong \big(R/J(R)\big)\bigotimes\limits_R (nR) \cong nR/J(R)(nR)$. This implies that $P/J(R)P \cong nR/J(R)(nR)$. In light of Lemma 1.2.2, $P \cong nR$, as required. □

We say that a polynomial $f(x) \in R[x]$ has a root $\alpha \in R$ provided that $f(\alpha) = 0$. Solvability of an equation or a system of equations is defined

by a similar manner. Choose $R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}_2, b \in \mathbb{Z}_2[x] \right\}$. Then R is a commutative ring. Choose $f(x) = x^2 - 1 \in R[x]$. Then $f(x)$ has infinitely many roots in R, e.g., $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ for all $b \in \mathbb{Z}_2[x]$. As in well known, a monic polynomial $f(t) \in R[t]$ has a root $\alpha \in R$ if and only if there exists $g(t) \in R[t]$ such that $f(t) = (t - \alpha)g(t)$, where $deg\big(g(t)\big) < deg\big(f(t)\big)$. As in the proof of [256, Theorem 2.2], we now extend [392, Theorem 2.7] and [549, Theorem 7] to a single strongly clean matrix over a local ring which is not necessarily commutative.

Theorem 1.2.4. *Let R be a local ring. Then $A \in M_2(R)$ is strongly clean if and only if*

(1) $A \in GL_2(R)$, *or*
(2) $I_2 - A \in GL_2(R)$, *or*
(3) A *is similar to* $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$ *where* $\lambda \in J(R), \mu \in 1 + J(R)$, *and the equation* $x^2 - x\mu - \lambda = 0$ *has a root in* $J(R)$ *and a root in* $1 + J(R)$.

Proof. Suppose that $A \in M_2(R)$ is strongly clean and $A, I_2 - A \notin GL_2(R)$. In view of Lemma 1.2.1, A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$ where $\lambda \in J(R), \mu \in 1 + J(R)$. Thus, $B := \begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$ is strongly clean. According to Theorem 1.1.14, there exists decomposition $2R = C \oplus D$ such that $B|_C$ and $(I_2 - B)|_D$ are R-isomorphisms. As $B, I_2 - B \notin GL_2(R)$, we know that $C, D \neq 0$. In view of Lemma 1.2.3, C and D are free modules of rank 1. Write $C = R(a, b)$ and $D = R(c, d)$. Set $\overline{R} = R/J(R)$. As $R(a, b)\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix} = R(a, b)$, we get $\overline{R}(\overline{a}, \overline{b}) \subseteq \overline{R}(\overline{1}, \overline{1})$. As $R(c, d)\begin{pmatrix} 1 & -\lambda \\ -1 & 1 - \mu \end{pmatrix} = R(c, d)$, we get $\overline{R}(\overline{c}, \overline{d}) \subseteq \overline{R}(\overline{1}, \overline{0})$. Write $(\overline{a}, \overline{b}) = \overline{s}(\overline{1}, \overline{1})$ and $(\overline{c}, \overline{d}) = \overline{t}(\overline{1}, \overline{0})$. Then

$$(\overline{1}, \overline{1}) = \overline{x}(\overline{a}, \overline{b}) + \overline{y}(\overline{c}, \overline{d}) = \overline{xs}(\overline{1}, \overline{1}) + \overline{yt}(\overline{1}, \overline{0}) \text{ in } 2\overline{R}.$$

This infers that $\overline{y}(\overline{c}, \overline{d}) = \overline{yt}(\overline{1}, \overline{0}) = (\overline{0}, \overline{0})$ in $2\overline{R}$. Hence, $\overline{R}(\overline{1}, \overline{1}) \subseteq \overline{R}(\overline{a}, \overline{b})$. This yields that $\overline{R}(\overline{a}, \overline{b}) = \overline{R}(\overline{1}, \overline{1})$. Likewise, $\overline{R}(\overline{c}, \overline{d}) = \overline{R}(\overline{1}, \overline{0})$. Write $(\overline{1}, \overline{1}) = \overline{r}(\overline{a}, \overline{b})$. Then $1 - ra \in J(R)$, and so $a \in R$ is left invertible. As R is local, it is Dedekind-finite (i.e., $xy = 1, x, y \in R \Longrightarrow yx = 1$); hence, $a \in U(R)$. It follows from $(\overline{a}, \overline{b}) = \overline{s}(\overline{1}, \overline{1})$ that $a - s, b - s \in J(R)$. Hence $1 - a^{-1}b \in J(R)$. Thus, $C = R(1, \alpha)$ where $\alpha = a^{-1}b \in 1 + J(R)$.

Analogously, $D = R(1, \beta)$ where $\beta = c^{-1}d \in J(R)$. As C is B-invariant, we can find some $r \in R$ such that

$$(1, \alpha)\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix} = r(1, \alpha),$$

and so $\alpha^2 - \alpha\mu - \lambda = 0$. Likewise, $\beta^2 - \beta\mu - \lambda = 0$. Therefore the equation $x^2 - x\mu - \lambda = 0$ has a root $\beta \in J(R)$ and a root $\alpha \in 1 + J(R)$.

If $A \in GL_2(R)$ or $I_2 - A \in GL_2(R)$, then A is strongly clean. Assume now that A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$ where $\lambda \in J(R), \mu \in 1 + J(R)$, and the equation $x^2 - x\mu - \lambda = 0$ has a root $\alpha \in J(R)$ and a root $\beta \in 1 + J(R)$. It suffices to show that $B := \begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$ is strongly clean. Set $C = R(1, \alpha)$ and $D = R(1, \beta)$. Clearly,

$$(1, \alpha)\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix} = \alpha(1, \alpha) \in C,$$

thus C is B-invariant. Likewise, D is B-invariant. Write $r(1, \alpha) = s(1, \beta) \in C \cap D$, then $r = s$ and $r\alpha = s\beta$, and so $r(\alpha - \beta) = 0$. This implies that $r = 0$, and so $C \cap D = 0$. Let $(a, b) \in 2R$. Choose $s = (b - a\alpha)(\beta - \alpha)^{-1}$ and $r = a - s$. Then it is easy to verify that $(a, b) = r(1, \alpha) + s(1, \beta) \in C \oplus D$. Thus, $2R = C \oplus D$. Obviously,

$$\begin{aligned} 1_C - B|_C : C &\to C; \\ r(1, \alpha) &\mapsto r(1, \alpha) - r\alpha(1, \alpha). \end{aligned}$$

If $\big(r(1, \alpha)\big)\big(1_C - B|_C\big) = 0$, then $r(1, \alpha) - r\alpha(1, \alpha) = 0$, and so $r(1 - \alpha)(1, \alpha) = 0$. As $\alpha \in U(R)$, we deduce that $r = 0$. Hence, $r(1, \alpha) = 0$. This implies that $1_C - B|_C\gamma$ is an R-monomorphism. For any $r(1, \alpha) \in C$, one easily checks that $\big(r(1 - \alpha b)^{-1}(1, \alpha)\big)\big(r(1, \alpha)\big)\big(1_C - B|_C\big) = r(1, \alpha)$; hence, $1_C - B|_C$ is an R-epimorphism. Therefore $1_C - B|_C$ is an R-isomorphism. This infers that

$$\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}\Big|_C \in U\big(End(A)\big).$$

Likewise,

$$\left(I_2 - \begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}\right)\Big|_D \in U\big(End(B)\big).$$

In light of Theorem 1.1.14, B is strongly clean, as required. □

Corollary 1.2.5. *Let R be a local ring. Then $A \in M_2(R)$ is strongly clean if and only if*

(1) $A \in GL_2(R)$, *or*
(2) $I_2 - A \in GL_2(R)$, *or*
(3) A *is similar to* $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$ *where* $\lambda \in J(R), \mu \in 1 + J(R)$, *and the equation* $x^2 - \mu x - \lambda = 0$ *has a root in* $J(R)$ *and a root in* $1 + J(R)$.

Proof. Suppose that $A \in M_2(R)$ is strongly clean. Then $(A^T)^{op} \in M_2(R^{op})$ is strongly clean. Clearly, R^{op} is local. In view of Theorem 1.2.4, $(A^T)^{op} \in GL_2(R^{op})$, or $I_2^{op} - (A^T)^{op} \in GL_2(R^{op})$, or $(A^T)^{op}$ is similar to $\begin{pmatrix} 0 & \lambda^{op} \\ 1^{op} & \mu^{op} \end{pmatrix}$ where $\lambda^{op} \in J(R^{op}), \mu^{op} \in 1^{op} + J(R^{op})$, and the equation $x^2 - x\mu^{op} - \lambda^{op} = 0$ has a root in $J(R^{op})$ and a root in $1 + J(R^{op})$. This implies that $A \in GL_2(R)$, or $I_2 - A \in GL_2(R)$, or A is similar to $\begin{pmatrix} 0 & 1 \\ \lambda & \mu \end{pmatrix}$ where $\lambda \in J(R), \mu \in 1+J(R)$, and the equation $x^2 - \mu x - \lambda = 0$ has a root in $J(R)$ and a root in $1 + J(R)$. If A is similar to $\begin{pmatrix} 0 & 1 \\ \lambda & \mu \end{pmatrix}$, then we can find some $P \in GL_2(R)$ such that $PAP^{-1} = \begin{pmatrix} 0 & 1 \\ \lambda & \mu \end{pmatrix}$. Choose $Q = \begin{pmatrix} 1 & -\mu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $QPAP^{-1}Q^{-1} = \begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, as required.

Conversely, $A \in M_2(R)$ is strongly clean if $A \in GL_2(R)$, or $I_2 - A \in GL_2(R)$. We now assume that A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$ where $\lambda \in J(R), \mu \in 1+J(R)$, and the equation $x^2 - \mu x - \lambda = 0$ has a root in $J(R)$ and a root in $1 + J(R)$. Then $(A^{op})^T$ is similar to $\begin{pmatrix} 0 & 1^{op} \\ \lambda^{op} & \mu^{op} \end{pmatrix}$ where $\lambda^{op} \in J(R^{op}), \mu^{op} \in 1^{op} + J(R^{op})$, and the equation $x^2 - x\mu^{op} - \lambda^{op} = 0$ has a root in $J(R^{op})$ and a root in $1^{op} + J(R^{op})$. As in the preceding discussion, we see that $(A^{op})^T$ is similar to $\begin{pmatrix} 0 & \lambda^{op} \\ 1^{op} & \mu^{op} \end{pmatrix}$. According to Theorem 1.2.4, $(A^{op})^T \in M_2(R^{op})$ is strongly clean, and therefore $A \in M_2(R)$ is strongly clean, as asserted. □

As a consequence, we can derive the following (see [395, Theorem 2.6]).

Corollary 1.2.6. *Let R be a commutative local ring, and let $A \in M_2(R)$. Then $A \in M_2(R)$ is strongly clean if and only if*

(1) $A \in GL_2(R)$, *or*
(2) $I_2 - A \in GL_2(R)$, *or*

(3) *the equation* $x^2 - tr(A)x + det(A) = 0$ *has a root in* $J(R)$ *and a root in* $1 + J(R)$.

Proof. Suppose that $A \in M_2(R)$ is strongly clean and $A, I_2 - A \notin GL_2(R)$. In view of Corollary 1.2.5, A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$ where $\lambda \in J(R), \mu \in 1 + J(R)$, and the equation $x^2 - \mu x - \lambda = 0$ has a root in $J(R)$ and a root in $1 + J(R)$. This implies that $\mu = tr(A)$ and $-\lambda = det(A)$. Therefore the equation $x^2 - tr(A)x + det(A) = 0$ has a root in $J(R)$ and a root in $1 + J(R)$.

Conversely, assume that the equation $x^2 - tr(A)x + det(A) = 0$ has a root in $J(R)$ and a root in $1 + J(R)$. In light of Lemma 1.2.1, A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$ where $\lambda \in J(R), \mu \in 1 + J(R)$. This implies that $\mu = tr(A)$ and $-\lambda = det(A)$. As a result, the equation $x^2 - x\mu - \lambda = 0$ has a root in $J(R)$ and a root in $1 + J(R)$. According to Corollary 1.2.5, we complete the proof. □

Let R be an integral domain and F be a field, and let $\sigma : R \to F$ be a ring map, and let $p(x) = \sum_{i=0}^{n} a_i x^i \in R[x] (a_n \neq 0)$. If $\sigma(a_n) \neq 0$ and if $p^*(x) = \sum_{i=0}^{n} \sigma(a_i) x^i$ is irreducible in $F[x]$, we note that $p(x)$ is not a product of two polynomials in $R[x]$ each of degree $< n$ (cf. [456, Theorem 34]).

Example 1.2.7. Let $\mathbb{Z}_{(2)} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, (m, n) = 1, 2 \nmid n\}$, and let $A = \begin{pmatrix} 5 & 6 \\ 3 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \in M_2(\mathbb{Z}_{(2)})$. Then $A \in M_2(\mathbb{Z}_{(2)})$ is strongly clean, but B is not strongly clean.

Proof. Clearly, $\mathbb{Z}_{(2)}$ is a commutative local ring with $J(\mathbb{Z}_{(2)}) = 2\mathbb{Z}_{(2)}$. As $tr(A) = 7$ and $det(A) = -8$, we see that the equation $x^2 - tr(A)x + det(A) = 0$ has a root $-1 \in 1 + J(\mathbb{Z}_{(2)})$ and a root $8 \in J(\mathbb{Z}_{(2)})$. In light of Corollary 1.2.6, $A \in M_2(\mathbb{Z}_{(2)})$ is strongly clean. Clearly, $B, I_2 - B \notin GL_2(\mathbb{Z}_{(2)})$. In addition, $tr(B) = 5$ and $det(B) = -2$. Taking $\sigma : \mathbb{Z}_{(2)} \to \mathbb{Q}$ to be the natural map, and $p(x) = x^2 - 5x - 2 \in \mathbb{Z}_{(2)}[x]$. Clearly, $p^*(x) = x^2 - 5x - 2 \in \mathbb{Q}[x]$ is irreducible, and so $x^2 - 5x - 2 = 0$ is not solvable in $\mathbb{Z}_{(2)}$. In light of Corollary 1.2.6, $B \in M_2(\mathbb{Z}_{(2)})$ is not strongly clean. □

Example 1.2.8. Let $R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a^2 & 0 \\ 0 & 0 & a \end{pmatrix} \;\middle|\; a,b,c \in \mathbb{Z}_4 \right\}$. Then $A = \begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix} \in M_2(R)$ is strongly clean, where $\lambda = \begin{pmatrix} \overline{2} & \overline{1} & \overline{2} \\ \overline{0} & \overline{0} & \overline{0} \\ \overline{0} & \overline{0} & \overline{2} \end{pmatrix}$ and $\mu = \begin{pmatrix} \overline{1} & \overline{1} & \overline{1} \\ \overline{0} & \overline{1} & \overline{0} \\ \overline{0} & \overline{0} & \overline{1} \end{pmatrix}$.

Proof. Clearly, $J(R) = \left\{ \begin{pmatrix} 2a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 2a \end{pmatrix} \;\middle|\; a,b,c \in \mathbb{Z}_4 \right\}$. Then $R/J(R) \cong \mathbb{Z}_2$. Hence, R is local. One easily checks that the equation $x^2 - \mu x - \lambda = 0$ has a root $\begin{pmatrix} \overline{2} & \overline{1} & \overline{0} \\ \overline{0} & \overline{0} & \overline{0} \\ \overline{0} & \overline{0} & \overline{2} \end{pmatrix} \in J(R)$ and a root $\begin{pmatrix} \overline{3} & \overline{0} & \overline{1} \\ \overline{0} & \overline{1} & \overline{0} \\ \overline{0} & \overline{0} & \overline{3} \end{pmatrix} \in 1_R + J(R)$. According to Corollary 1.2.5, $A \in M_2(R)$ is strongly clean. In this case, R is not commutative. □

Now we give a new simple way to look at the following known results which were observed by many authors, e.g., Chen et al. [182], Li [395], and Yang [549].

Proposition 1.2.9. *Let R be a commutative local ring. Then the following are equivalent:*

(1) *$M_2(R)$ is strongly clean.*
(2) *For any $w \in J(R)$, the equation $x^2 - x = w$ is solvable in R.*
(3) *For any $\lambda \in J(R), \mu \in 1+J(R)$, the equation $x^2 - \mu x - \lambda = 0$ is solvable in R.*

Proof. (1) ⇒ (2) Let $w \in J(R)$. Choose $A = \begin{pmatrix} 0 & w \\ 1 & 1 \end{pmatrix}$. Then A and $I_2 - A$ are nonunits. By virtue of Corollary 1.2.6, the equation $x^2 - x - w = 0$ is solvable, as required.

(2) ⇒ (3) For any $\lambda \in J(R), \mu \in 1+J(R)$, we can find some $a \in R$ such that $a^2 - a = \lambda\mu^{-2}$. Set $b = \mu a$. Then $\left(\mu^{-1}b\right)^2 - \mu^{-1}b = \lambda\mu^{-2}$. Hence, $b^2 - \mu b - \lambda = 0$, and so the equation $x^2 - \mu x - \lambda = 0$ is solvable in R.

(3) ⇒ (1) Let $A \in M_2(R)$. In view of Lemma 1.2.1, A is invertible or $I_2 - A$ is invertible, or A is similar to a matrix $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in J(R), \mu \in 1+J(R)$. By hypothesis, the equation $x^2 - \mu x - \lambda = 0$ is solvable in R. Let $a \in R$ be a root of the preceding equation. Then $a^2 - \mu a - \lambda = 0$; hence,

$a(a-\mu)=\lambda\in J(R)$. Thus, $a\in J(R)$ or $a\in\mu+J(R)\subseteq 1+J(R)$. It is easy to verify that $(\mu-a)^2-\mu(\mu-a)-\lambda=0$. This means that $\mu-a\in R$ is an other root. In addition, $\mu-a\in 1+J(R)$ or $J(R)$. Therefore the equation $x^2-\mu x-\lambda=0$ has a root in $1+J(R)$ and a root in $J(R)$. According to Theorem 1.2.4, we complete the proof. □

Let $p\in\mathbb{Z}$ be prime, and let $\mathbb{Z}_{(p)}$, $\{\frac{m}{n}\mid m,n\in\mathbb{Z},(m,n)=1,p\nmid n\}$, the ring of integers localized at the prime ideal (p). Then $\mathbb{Z}_{(p)}$ is a commutative local domain. In addition, $J(\mathbb{Z}_{(p)})=\{\frac{pm}{n}\mid m,n\in\mathbb{Z},(m,n)=1,p\nmid n\}$. Clearly, $-p\in J(\mathbb{Z}_{(p)})$. Taking $\sigma:\mathbb{Z}_{(p)}\to\mathbb{Q}$ to be the natural map, and $p(x)=x^2-x+p\in\mathbb{Z}_{(p)}[x]$. Since $p^*(x)=x^2-x+p\in\mathbb{Q}[x]$ is irreducible, $x^2-x=-p$ is not solvable in $\mathbb{Z}_{(p)}$. According to Proposition 1.2.9, $M_2(\mathbb{Z}_{(p)})$ is not strongly clean. But $\mathbb{Z}_{(p)}$ is strongly clean, and so strong cleanness is not Morita invariant.

Corollary 1.2.10. *Let R be a commutative local ring. If $\frac{1}{2}\in R$, then the following are equivalent:*

(1) *$M_2(R)$ is strongly clean.*
(2) *For any $w\in J(R)$, the equation $x^2-1=w$ is solvable in R.*

Proof. (1) $\Rightarrow$ (2) Let $w\in J(R)$. By virtue of Proposition 1.2.9, the equation $x^2-x-\frac{1}{4}\big(1-(1+w)^{-1}\big)\big(1-(1+w)^{-1}\big)^{-1}=0$ is solvable. Thus, $\big(1-(1+w)^{-1}\big)x^2-\big(1-(1+w)^{-1}\big)x-\frac{1}{4}\big(1-(1+w)^{-1}\big)=0$. That is, $\big(1-(1+w)^{-1}\big)x^2-\big(1-(1+w)^{-1}\big)x+1-\frac{1}{4}\big(1-(1+w)^{-1}\big)=1$. Hence $x^2-x+1=\big(1-\frac{1}{4}(1-(1+w)^{-1})\big)^{-1}$. Therefore $x^2-x=\big(1-\frac{1}{4}(1-(1+w)^{-1})\big)^{-1}-1=w$, as desired.

(2) $\Rightarrow$ (1) For any $w\in J(R)$, we can find some $a\in R$ such that $a^2-1=4w$. Set $b=\frac{a-1}{2}$. Then $a=2b+1$, and so $4(b^2+b)=4w$; hence, $x^2+x=w$ has a solution $b\in R$. This implies that $x^2-x=w$ is solvable. According to Proposition 1.2.9, $M_2(R)$ is strongly clean. □

Example 1.2.11. Let $R=\left\{\begin{pmatrix}a&b\\0&a\end{pmatrix}\;\middle|\;a\in\mathbb{Z}_2,b\in\mathbb{Z}_2[x]\right\}$. Then $M_2(R)$ is strongly clean.

Proof. Clearly, $J(R)=\left\{\begin{pmatrix}0&b\\0&0\end{pmatrix}\;\middle|\;b\in\mathbb{Z}_2[x]\right\}$. Hence, $R/J(R)\cong\mathbb{Z}_2$, and so R is a commutative local ring. Write $w=\begin{pmatrix}0&z\\0&0\end{pmatrix}\in J(R)$. Choose

$\alpha = \begin{pmatrix} 0 & -z \\ 0 & 0 \end{pmatrix}$. Then the equation $x^2 - x = w$ has a root α. According to Proposition 1.2.9, we complete the proof. □

Lemma 1.2.12. *Every 2×2 idempotent matrix over a local ring is similar to a diagonal matrix.*

Proof. Let R be local, and let $E \in M_2(R)$ be an idempotent. In view of Lemma 1.2.1, A or $I_2 - A$ is invertible, or there exist some $\lambda \in J(R), \mu \in 1 + J(R)$ such that A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$. If $A \in GL_2(R)$, then $A = I_2$. If $I_2 - A \in GL_2(R)$, then $A = 0$. If A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, then

$$\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix} = \begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}^2,$$

and so $\lambda = 0, \mu = 1$. It is easy to verify that

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

thus A is similar to $diag(1, 0)$. The lemma is therefore established. □

Theorem 1.2.13. *Let R be a local ring. Then $A \in M_2(R)$ is strongly clean if and only if $A \in GL_2(R)$ or $I_2 - A \in GL_2(R)$, or A is similar to a diagonal matrix.*

Proof. If either A or $I_2 - A$ is invertible, then A is strongly clean. As $R = U(R) \cup J(R)$, we may assume that A is similar to $diag(a_1, a_2)$ where $a_1 \in U(R), a_2 \in J(R)$. It is easy to verify that $diag(a_1, a_2) = diag(0, 1) + diag(a_1, a_2 - 1)$, and so $diag(a_1, a_2) \in M_2(R)$ is strongly clean. Therefore $A \in M_2(R)$ is strongly clean.

Conversely, assume that $A \in M_2(R)$ is strongly clean. Then there exist an idempotent $E \in M_2(R)$ and an invertible $U \in M_2(R)$ such that $A = E + U$ with $EU = UE$. Suppose that A and $I_2 - A$ are nonunits. In view of Lemma 1.2.12, there exists $V \in GL_2(R)$ such that $VEV^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Thus, $VAV^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + VUV^{-1}$. Set $W = (w_{ij}) := VUV^{-1}$. It follows from $EU = UE$ that $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W = W \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$; hence, $w_{12} = w_{21} = 0$ and

$w_{11}, w_{22} \in U(R)$. Let $\alpha = 1 + w_{11}$ and $\beta = w_{22}$. Then A is similar to $diag(\alpha, \beta)$, as required. □

Example 1.2.14. Let $A \in M_2(\mathbb{Z}_4)$. Then $A \in GL_2(\mathbb{Z}_4)$, or $I_2 - A \in GL_2(\mathbb{Z}_4)$, or A is similar to one of the matrices in the set

$$\left\{ \begin{pmatrix} \overline{0} & \overline{0} \\ \overline{0} & \overline{1} \end{pmatrix}, \begin{pmatrix} \overline{0} & \overline{0} \\ \overline{0} & \overline{3} \end{pmatrix}, \begin{pmatrix} \overline{2} & \overline{0} \\ \overline{0} & \overline{1} \end{pmatrix}, \begin{pmatrix} \overline{2} & \overline{0} \\ \overline{0} & \overline{3} \end{pmatrix} \right\}.$$

Proof. Clearly, $\mathbb{Z}_4$ is a local ring with the Jacobson radical $2\mathbb{Z}_4$. In addition, $M_2(\mathbb{Z}_4)$ is strongly π-regular, and so it is strongly clean. Therefore we complete the proof by Theorem 1.2.13. □

Let A and B be rings, and let V be an A-B-bimodule. Then $R = \left\{ \begin{pmatrix} a & v \\ 0 & b \end{pmatrix} \;\middle|\; a \in A, b \in B, v \in V \right\}$ forms a ring with the usual addition and multiplication. We denote such ring by $\begin{pmatrix} A & V \\ 0 & B \end{pmatrix}$.

Theorem 1.2.15. *Let A and B be local rings and ${}_AV_B$ be an A-B-bimodule, let $R = \begin{pmatrix} A & V \\ 0 & B \end{pmatrix}$. Then $D = \begin{pmatrix} a & v \\ 0 & b \end{pmatrix} \in R$ is strongly clean if and only if $D \in U(R)$ or $1_R - D \in U(R)$, or the equation $ax - xb = v$ is solvable.*

Proof. Suppose that D is strongly clean and $D, 1_R - D \notin U(R)$. Then there exists some $E = \begin{pmatrix} e & w \\ 0 & f \end{pmatrix} \in R$ such that

$$\begin{pmatrix} a & v \\ 0 & b \end{pmatrix} - E \in U(R) \text{ and } \begin{pmatrix} a & v \\ 0 & b \end{pmatrix} E = E \begin{pmatrix} a & v \\ 0 & b \end{pmatrix}.$$

Since A and B are local rings, we see that $e = 0, 1_A$ and $f = 0, 1_B$. Thus, $E = \begin{pmatrix} 1_A & x \\ 0 & 0 \end{pmatrix}$ or $E = \begin{pmatrix} 0 & x \\ 0 & 1_B \end{pmatrix}$ where $x \in V$. If $E = \begin{pmatrix} 1_A & x \\ 0 & 0 \end{pmatrix}$, it follows from $\begin{pmatrix} a & v \\ 0 & b \end{pmatrix} \begin{pmatrix} 1_A & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1_A & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & v \\ 0 & b \end{pmatrix}$ that $v = ax - xb$. If $E = \begin{pmatrix} 0 & x \\ 0 & 1_B \end{pmatrix}$, it follows from $\begin{pmatrix} a & v \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 & x \\ 0 & 1_B \end{pmatrix} = \begin{pmatrix} 0 & x \\ 0 & 1_B \end{pmatrix} \begin{pmatrix} a & v \\ 0 & b \end{pmatrix}$ that $v = a(-x) - (-x)b$, as required.

Suppose that $D \in U(R)$ or $1_R - D \in U(R)$. Then $D \in R$ is strongly clean. Assume now that $D, \begin{pmatrix} 1_A & 0 \\ 0 & 1_B \end{pmatrix} - D \notin U(R)$. By hypothesis, there

exists $x \in V$ such that $v = ax - xb$. Since A is local, we may assume that $a \in U(A)$ or $a \in 1_A + U(A)$. Likewise, $b \in U(B)$ or $b \in 1_B + U(B)$. Assume that $a - 1_A \in U(A)$ and $b \in U(B)$. Choose $E = \begin{pmatrix} 1_A & x \\ 0 & 0 \end{pmatrix}$. Then $E = E^2, DE = ED$ and $D - E \in U(R)$. Assume that $a \in U(A)$ and $b - 1_B \in U(B)$. Choose $E = \begin{pmatrix} 0 & x \\ 0 & 1_B \end{pmatrix}$. Then $E = E^2, DE = ED$ and $D - E \in U(R)$. In any case, $D \in R$ is strongly clean. □

Corollary 1.2.16. *Let A and B be local rings and ${}_AV_B$ be a bimodule, let $R = \begin{pmatrix} A & V \\ 0 & B \end{pmatrix}$. Then R is strongly clean if and only if for any $a \in 1_A + J(A), b \in J(B)$ and $v \in V$, the equation $ax - xb = v$ is solvable.*

Proof. Suppose that R is strongly clean. Let $a \in 1_A + J(A), b \in J(B)$ and $v \in V$. Choose $D = \begin{pmatrix} a & v \\ 0 & b \end{pmatrix} \in R$. Then D is strongly clean. Since $D, 1_R - D \notin U(R)$, it follows from Theorem 1.2.15 that $v = ax - xb$ is solvable.

Let $D = \begin{pmatrix} a & v \\ 0 & b \end{pmatrix} \in R$. If D or $I_R - D$ is invertible, then D is strongly clean; otherwise, either $a \in U(A), b \in J(A)$ or $a \in J(A), b \in U(A)$. If $a \in U(A), b \in J(A)$, then $a \in 1 + J(A)$; otherwise, $a - 1_A \in U(A), b - 1_B \in U(B)$. Thus, by hypothesis, the equation $ax - xb = v$ is solvable. If $a \in J(A), b \in U(B)$, then $b \in 1 + J(B)$; otherwise, $a - 1_A \in U(A), b - 1_B \in U(B)$. Thus, $1_A - a \in 1 + J(A), 1_B - b \in J(B)$. Thus, the equation $(1_A - a)x - x(1_A - b) = -v$ is solvable. That is, the equation $ax - xb = v$ is solvable. According to Theorem 1.2.15, D is strongly clean, as required. □

Let $a \in R$. $l_a : R \to R$ and $r_a : R \to R$ denote, respectively, the abelian group endomorphisms given by $l_a(r) = ar$ and $r_a(r) = ra$ for all $r \in R$. Thus, $l_a - r_b$ is an abelian group endomorphism such that $(l_a - r_b)(r) = ar - rb$ for any $r \in R$.

Corollary 1.2.17. *Let R be a local ring, let $T_2(R) = \begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$. Then $T_2(R)$ is strongly clean if and only if for any $a \in 1 + J(R), b \in J(R)$, $l_a - r_b$ is surjective.*

Proof. It is easily proved by Corollary 1.2.16. □

1.3 2×2 Matrices over Integral Domains

An integral domain is a commutative ring with identity in which the product of any two non-zero elements is not equal to zero. Clearly, an integral domain is a ring with identity that is a subring of a field. The maim purpose of this section is concerned with the strong cleanness of 2×2 matrices over integral domains. Let p be a prime. The $\mathbb{Z}_{(p)} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, (m, n) = 1, p \nmid n\}$ is a local domain with the Jacobson radical $p\mathbb{Z}_{(p)}$. In particular, we determine a matrix over such localizations of the ring of integers is strongly clean.

Lemma 1.3.1. *Let R be an integral domain. Then $A \in M_2(R)$ is an idempotent if and only if $A = 0$ or $A = I_2$ or $A = \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}$ where $bc = a - a^2$ in R.*

Proof. Let $A \in M_2(R)$ be an idempotent. Since R is an integral domain, $det(A) = 1$ or $det(A) = 0$. If $det(A) = 1$, then $A \in GL_2(R)$, and so $A = I_2$. If $det(A) = 0$, then $ad - bc = 0$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Thus we have $a(a + d) = a, b(a + d) = b, c(a + d) = c$ and $d(a + d) = d$. If $A \neq 0$, then $a + d = 1$. Thus, $d = 1 - a$, and so $bc = a - a^2$, as required. The converse is obvious. □

Let R be a commutative local ring. Analogously, $A \in M_2(R)$ is an idempotent if and only if $A = 0$ or $A = I_2$, or $A = \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}$ where $bc = a - a^2$ in R.

Let $A = (a_{ij}) \in M_2(R)$, $s_A = a_{11} - a_{22}$ and $t_A = tr^2(A) - 4det(A)$. We have

Lemma 1.3.2. *Let R be an integral domain, and let $A \in M_2(R)$. If A is strongly clean in $M_2(R)$, then A is invertible or $I_2 - A$ is invertible, or the equation $t_A x^2 = s_A^2$ is solvable.*

Proof. Suppose that A and $I_2 - A$ are nonunits. In view of Lemma 1.3.1, there exist $a, b, c \in R$ such that $A = E + (A - E)$, where

$$E = E^2 = \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}, bc = a - a^2, EA = AE \text{ and } A - E \in GL_2(R).$$

As $EA = AE$, we get $s_A b = a_{12}(2a-1)$ and $s_A c = a_{21}(2a-1)$. Hence, $s_A^2(a-a^2) = s_A^2 bc = a_{12}a_{21}(2a-1)^2$. That is,

$$(s_A^2 + 4a_{12}a_{21})(a-a^2) = a_{12}a_{21}.$$

Obviously, $t_A = s_A^2 + 4a_{12}a_{21}$. Therefore $s_A^2 = t_A - 4a_{12}a_{21} = t_A - 4t_A(a-a^2) = t_A(1-2a)^2$, as required. □

Let A be a strongly clean 2×2 matrix over a commutative local ring. Analogously, we claim that A is invertible or $I_2 - A$ is invertible, or the equation $t_A x^2 = s_A^2$ is solvable. The following result was proved in [184, Corollary 2.2]. Here, we give an alternative proof.

Theorem 1.3.3. *Let $A \in M_2\big(\mathbb{Z}_{(2)}\big)$. Then A is strongly clean if and only if either A is invertible or $I_2 - A$ is invertible, or the equation $t_A x^2 = s_A^2$ is solvable.*

Proof. Set $R = \mathbb{Z}_{(2)}$. Suppose that $\frac{x_1}{s_1} + \frac{x_2}{s_2} = \frac{1}{1}$ where $2 \nmid s_1, s_2$. If $\frac{x_1}{s_1}, \frac{x_2}{s_2} \notin U(R)$, then $2 \mid x_1, x_2$. Obviously, we have some $t \in \mathbb{Z}$ such that $t(x_1 s_2 + x_2 s_1) = t s_1 s_2$ and $2 \nmid t$. But $2 \mid (x_1 s_2 + x_2 s_1)$, a contradiction. This shows that R is a local integral domain with $J(R) = 2R$.

One direction is obvious from Lemma 1.3.2. Conversely, assume that $A \in M_2(R)$ with $A, I_2 - A \notin GL_2(R)$, and that the equation $t_A x^2 = s_A^2$ has a root $\alpha \in R$. Then $det(A), det(I_2 - A) = 1 - tr(A) + det(A) \in J(R)$. Hence, $tr(A) \in 1 + J(R)$ and $t_A = tr^2(A) - 4det(A) \in 1 + J(R)$, and so $s_A^2 = t_A - 4a_{12}a_{21} \in 1 + J(R)$. Thus, $s_A \in U(R)$. As $R/J(R) \cong \mathbb{Z}_2$, $s_A \in 1 + J(R)$. This implies that $\alpha \in 1 + J(R)$; hence, $tr(A) - s_A\alpha^{-1} \in J(R) = 2R$. Write $tr(A) - s_A\alpha^{-1} = 2\beta$. One easily checks that

$$\begin{aligned} 4\big(\beta^2 - tr(A)\cdot\beta + det(A)\big) &= \big(2\beta - tr(A)\big)^2 - tr^2(A) + 4det(A) \\ &= (s_A\alpha^{-1})^2 - tr^2(A) + 4det(A) \\ &= t_A - tr^2(A) + 4det(A) \\ &= 0. \end{aligned}$$

Since R is an integral domain and $2 \neq 0$, the equation $x^2 - tr(A)\cdot x + det(A) = 0$ has a root β. One easily checks that this equation has an other root $tr(A) - \beta$. As $\beta\big(tr(A) - \beta\big) = det(A) \in J(R)$, we see that the equation has a root in $J(R)$ and a root in $1 + J(R)$. According to Corollary 1.2.6, A is strongly clean. □

Especially, we see that $\begin{pmatrix} 1 & 1 \\ -\frac{2}{9} & 0 \end{pmatrix} \in M_2(\mathbb{Z}_{(2)})$ is strongly clean, while $\begin{pmatrix} 1 & 1 \\ \frac{2}{9} & 0 \end{pmatrix} \in M_2(\mathbb{Z}_{(2)})$ is not strongly clean. Also we note that $\begin{pmatrix} 8 & 6 \\ 3 & 7 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} \in M_2(\mathbb{Z}_{(2)})$ are not strongly clean (see [533]).

Lemma 1.3.4. *Let R be an integral domain, let $A, I_2 - A \notin GL_2(R)$, and let $s_A \in U(R)$. If $A \in M_2(R)$ is strongly clean, then t_A is an invertible square.*

Proof. In view of Lemma 1.3.2, there exists $x \in R$ such that $s_A^2 = t_A x^2$. Since $s_A \in U(R)$, $x \in U(R)$. Therefore $t_A = \big(s_A x^{-1}\big)^2$, as asserted. □

We come now to prove the main result in this section.

Theorem 1.3.5. *Let $A \in M_2(\mathbb{Z}_{(p)})$($p$ is a prime). Then A is strongly clean if and only if A is invertible or $I_2 - A$ is invertible, or t_A is an invertible square.*

Proof. Set $R = \mathbb{Z}_{(p)}$. Then R is a local integral domain.

Case I. $p = 2$. Assume that A is strongly clean and $A, I_2 - A \notin GL_2(R)$. As in the proof of Theorem 1.3.3, $s_A \in 1 + J(R) \subseteq U(R)$. By virtue of Lemma 1.3.4, t_A is an invertible square, as required.

Conversely, assume that $A, I_2 - A \notin GL_2(R)$ and $t_A = u^2$ for a $u \in U(R)$. Then $tr(A) \in 1 + J(R)$. As $R/J(R) \cong \mathbb{Z}_2$, $u \in 1 + J(R)$. Thus, $tr(A) - u \in J(R) = 2R$. Write $tr(A) - u = 2\beta$. As in the proof of Theorem 1.3.3, the equation $x^2 - tr(A) \cdot x + det(A) = 0$ has a root β and a root $tr(A) - \beta$. Furthermore, one root is in $J(R)$, and the other is in $1 + J(R)$. Therefore A is strongly clean from Corollary 1.2.6.

Case II. $p \neq 2$. Then $\frac{1}{2} \in R$. Assume that $A, I_2 - A \notin GL_2(R)$ and $t_A = u^2$ for some $u \in U(R)$. One directly checks that $x^2 - tr(A)x + det(A) = 0$ has two roots $\alpha = \frac{1}{2}\big(tr(A) + u\big)$ and $\beta = \frac{1}{2}\big(tr(A) - u\big)$. As in the proof of Theorem 1.3.3, $tr(A) \in 1 + J(R)$. As $\alpha\beta \in J(R)$ and $\alpha + \beta = tr(A)$, we see that one root is in $J(R)$, and the other is in $1 + J(R)$. Accordingly, $A \in M_2(R)$ is strongly clean by Corollary 1.2.6.

Conversely, assume that $A \in M_2(R)$ is strongly clean and $A, I_2 - A \notin GL_2(R)$. In view of Corollary 1.2.6, the equation $x^2 - tr(A)x + det(A) = 0$ has a root $\alpha \in J(R)$. It is easy to check that $t_A = tr^2(A) - 4det(A) = tr^2(A) - 4\big(tr(A) \cdot \alpha - \alpha^2\big) = \big(tr(A) - 2\alpha\big)^2$. Clearly, $tr(A) \in 1 + J(R)$, and therefore t_A is an invertible square. □

Let $A = \begin{pmatrix} p+1 & p \\ q & p \end{pmatrix} \in M_2(\mathbb{Z}_{(2)}), p, q \in \mathbb{Z}, p \neq \pm 1$. If $1 + 4pq$ is not a square of a prime, then $A \in M_2(\mathbb{Z}_{(2)})$ is not strongly clean. Clearly, $\mathbb{Z}_{(2)}$ is an integral domain. As $t_A = 1 + 4pq$ is not a square of an invertible element, it follows from Theorem 1.3.5 that $A \in M_2(\mathbb{Z}_{(2)})$ is not strongly clean. For instance, $\begin{pmatrix} 8 & 7 \\ 3 & 7 \end{pmatrix} \in M_2(\mathbb{Z}_{(2)})$ is not strongly clean. This is the case for $p = 7, q = 3$. Furthermore, we can derive the following.

Corollary 1.3.6. *Let $A \in M_2(\mathbb{Z}_{(2)})$. Then A is strongly clean if and only if A is invertible or $I_2 - A$ is invertible, or $det(A) = \frac{2m}{n} t_A$, where $(m, n) = 1, 2m = k(k+1) - l(l+1), n = (2l+1)^2$ for some $k, l \in \mathbb{Z}$.*

Proof. Assume that $A \in M_2(\mathbb{Z}_{(2)})$ is strongly clean and $A, I_2 - A \notin GL_2(\mathbb{Z}_{(2)})$. Then $det(A) \in J(\mathbb{Z}_{(2)}), tr(A) \in 1 + J(\mathbb{Z}_{(2)})$. By virtue of Theorem 1.3.5, $t_A = u^2$ for a $u \in U(\mathbb{Z}_{(2)})$. Write $t_A^{-1} tr^2(A) = \left(\frac{2k+1}{2l+1}\right)^2$ with $(2k+1, 2l+1) = 1$. On the other hand, $t_A^{-1} det(A) \in J(\mathbb{Z}_{(2)})$. Write $t_A^{-1} det(A) = \frac{2m}{n}, (m, n) = 1, m, n \in \mathbb{Z}$. Then $det(A) = \frac{2m}{n} t_A$. It is easy to check that $\frac{tr^2(A)}{t_A} = 1 + 4t_A^{-1} det(A) = 1 + \frac{8m}{n} = \frac{n+8m}{n} = \left(\frac{2k+1}{2l+1}\right)^2$. Observing that $(n + 8m, n) = 1$, we get $n = (2l+1)^2$ and $8m = (2k+1)^2 - (2l+1)^2$. Therefore $2m = k(k+1) - l(l+1)$, as required.

Conversely, assume that $A, I_2 - A \notin GL_2(\mathbb{Z}_{(2)})$, and that $det(A) = \frac{2m}{n} t_A$, where $(m, n) = 1, 2m = k(k+1) - l(l+1), n = (2l+1)^2$ for some $k, l \in \mathbb{Z}$. Then $det(A) \in J(\mathbb{Z}_{(2)}), tr(A) \in 1 + J(\mathbb{Z}_{(2)})$. Hence, $t_A^{-1} det(A) = \frac{2m}{n}$. Thus $1 + 4t_A^{-1} det(A) = 1 + \frac{8m}{n} = \frac{(2l+1)^2 + 4k(k+1) - 4l(l+1)}{(2l+1)^2} = \left(\frac{2k+1}{2l+1}\right)^2$. Clearly, $\frac{tr^2(A)}{t_A} = 1 + 4t_A^{-1} det(A)$. Therefore $t_A = \left(\frac{2l+1}{2k+1}\right)^2 tr^2(A) \in U(\mathbb{Z}_{(2)})$. In view of Theorem 1.3.3, $A \in M_2(\mathbb{Z}_{(2)})$ is strongly clean, as desired. □

Lemma 1.3.7. *Let R be an integral domain, and let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(R)$. Then A is strongly clean in $M_2(R)$ if and only if A is invertible or $I_2 - A$ is invertible, or there exists $u \in U(R)$ such that the system of equations*

$$\begin{aligned} (a-d)x + cy + bz &= a - det(A) + u; \\ yz &= x - x^2; \\ (a-d)y &= b(2x-1); \\ (a-d)z &= c(2x-1) \end{aligned}$$

is solvable.

Proof. Suppose that A is strongly clean and $A, I_2 - A \notin GL_2(R)$. Then there exists $E = \begin{pmatrix} x & y \\ z & 1-x \end{pmatrix} \in M_2(R)$ such that $A - E \in GL_2(R)$ and $AE = EA$, where $yz = x - x^2$. It follows from $AE = EA$ that $(a-d)y = b(2x-1)$ and $(a-d)z = c(2x-1)$. One easily checks that $u := det(A-E) = (a-x)(d-1+x) - (b-y)(c-z) = det(A) - a + (a-d)x + bz + cy \in U(R)$. Therefore $(a-d)x + cy + bz = a - det(A) + u$, as desired.

Conversely, if either $A \in GL_2(R)$ or $I_2 - A \in GL_2(R)$, then A is strongly clean; otherwise, there exists $u \in U(R)$ such that the preceding system of equations is solvable. Set $E = \begin{pmatrix} x & y \\ z & 1-x \end{pmatrix}$. Then $E = E^2$ and $AE = EA$. Further, $det(A-E) = det(A) - a + (a-d)x + bz + cy = u \in U(R)$. Therefore $A - E \in GL_2(R)$, hence the result. □

Theorem 1.3.8. *Let R be an integral domain, and let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $s_A \in U(R)$, then $A \in M_2(R)$ is strongly clean if and only if A is invertible or $I_2 - A$ is invertible, or the equation $t_A(x^2 - x) + bc = 0$ has a root α such that $t_A\alpha + s_A\big(det(A) - a\big) - 2bc \in U(R)$.*

Proof. Suppose that A is strongly clean in $M_2(R)$. If $A, I_2 - A \notin GL_2(R)$, it follows from Lemma 1.3.7 that there exists $u \in U(R)$ such that the preceding system of equations is solvable. Hence, $s_A^2 yz = bc(2x-1)^2$, and so $s_A^2(x - x^2) = 4bc(x^2 - x) + bc$. Thus, $(s_A^2 + 4bc)(x - x^2) = bc$, whence $t_A(x - x^2) = bc$. Further,

$$s_A^2 x + c(s_A y) + b(s_A z) = s_A(a - det(A) + u).$$

As a result,

$$s_A^2 x + cb(2x-1) + bc(2x-1) = s_A(a - det(A) + u),$$

then $(s_A^2 + 4bc)x = 2bc + s_A(a - det(A) + u)$. Therefore $t_A x + s_A\big(det(A) - a\big) - 2bc = s_A u \in U(R)$, as required.

Conversely, we may assume that the equation $t_A(x^2 - x) + bc = 0$ has a root α such that $t_A\alpha + s_A\big(det(A) - a\big) - 2bc \in U(R)$. Set $y = s_A^{-1} b(2\alpha - 1)$ and $z = s_A^{-1} c(2\alpha - 1)$. Then

$$\begin{aligned} s_A^2 yz &= bc(2\alpha - 1)^2 \\ &= 4bc\big(\alpha^2 - \alpha\big) + bc \\ &= s_A^2\big(\alpha - \alpha^2\big) - t_A\big(\alpha - \alpha^2\big) + bc \\ &= s_A^2\big(\alpha - \alpha^2\big). \end{aligned}$$

We infer that $yz = \alpha - \alpha^2$. Let $E = \begin{pmatrix} \alpha & y \\ z & 1-\alpha \end{pmatrix}$. Then $E = E^2 \in M_2(R)$. One easily checks that

$$\begin{aligned} det(A-E) &= (a-d)x + cy + bz - a + det(A) \\ &= (a-d)x + cs_A^{-1}b\big(2\alpha - 1\big) + bs_A^{-1}c\big(2\alpha - 1\big) - a + det(A) \\ &= s_A^{-1}\big(s_A^2 x + 2bc(2x-1) - s_A(a - det(A))\big) \\ &= s_A^{-1}\big((s_A^2 + 4bc)\alpha - 2bc - s_A(a - det(A))\big) \\ &= s_A^{-1}\big(t_A\alpha - 2bc - s_A(a - det(A))\big) \\ &\in U(R). \end{aligned}$$

Thus, $A - E \in GL_2(R)$. Moreover,

$$AE = \begin{pmatrix} a\alpha + bz & ay + b(1-\alpha) \\ c\alpha + dz & cy + d(1-\alpha) \end{pmatrix} = \begin{pmatrix} a\alpha + cy & b\alpha + dy \\ az + c(1-\alpha) & bz + d(1-\alpha) \end{pmatrix} = EA.$$

Therefore $A \in M_2(R)$ is strongly clean. □

Corollary 1.3.9. *Let R be an integral domain, and let $p, q \in R$. If $p \in J(R)$, then the following are equivalent:*

(1) $\begin{pmatrix} p & p \\ q & p+1 \end{pmatrix}$ *is strongly clean.*
(2) *The equation $x^2 - x + \frac{pq}{1+4pq} = 0$ has a root in $U(R)$.*

Proof. Let $A = \begin{pmatrix} p & p \\ q & p+1 \end{pmatrix}$. Then $A, I_2 - A \notin GL_2(R)$. Clearly, $s_A = -1$ and $t_A = 1 + 4pq \in U(R)$. If A is strongly clean, by virtue of Theorem 1.3.8, there exists $u \in U(R)$ such that the equation $x^2 - x + \frac{pq}{1+4pq} = 0$ has a root $\alpha = t_A^{-1}\big(s_A(p - det(A)) - 2pq\big) + u \in U(R)$.

Conversely, assume that the equation $x^2 - x + \frac{pq}{1+4pq} = 0$ has a root α in $U(R)$. Then $t_A\alpha - s_A\big(p - det(A)\big) - 2pq \in U(R)$. According to Theorem 1.3.8, A is strongly clean. □

Corollary 1.3.10. *Let R be an integral domain, and let $\lambda \in J(R), \mu \in 1 + J(R)$. Then the following are equivalent:*

(1) $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$ *is strongly clean.*
(2) *The equation $x^2 - x + \frac{\lambda}{\mu^2 + 4\lambda} = 0$ has a root in $U(R)$.*

Proof. Let $A = \begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$. Then $s_A = -\mu \in U(R)$ and $t_A = \mu^2 + 4\lambda \in U(R)$. If $A \in M_2(R)$ is strongly clean, by virtue of Theorem 1.3.8, the equation $x^2 - x + \frac{\lambda}{\mu^2+4\lambda} = 0$ has a root $\alpha = t_A^{-1}\big(s_A\lambda + 2\lambda + u\big)$ for a $u \in U(R)$. Therefore $\alpha \in U(R)$, as required.

Conversely, assume that the equation $x^2 - x + \frac{\lambda}{\mu^2+4\lambda} = 0$ has a root α in $U(R)$. Then $t_A\alpha + s_A det(A) - 2\lambda = \big(\mu^2 + 4\lambda\big)\alpha + \mu\lambda - 2\lambda \in U(R)$. In light of Theorem 1.3.8, $A \in M_2(R)$ is strongly clean. □

Let $p, q \neq 2$ be distinct primes. We use $\mathbb{Z}_{(p)} \cap \mathbb{Z}_{(q)}$ to denote the subring ring $\{\frac{m}{n} \mid m, n \in \mathbb{Z}; p, q \nmid n\}$ of $\mathbb{Q}$. We claim that the ring $\mathbb{Z}_{(p)} \cap \mathbb{Z}_{(q)}$ is not strongly clean. As $(p, q) = 1$, we see that $\frac{p(q+1)}{p+q} \in R$. Observing that R is an integral domain, the set of all idempotents in R is $\{\frac{0}{1}, \frac{1}{1}\}$. As $q \nmid q + 1$, we see that $\frac{p(q+1)}{p+q} \notin U(R)$. As $p \nmid (p-1)q$, we see that $\frac{p(q+1)}{p+q} - \frac{1}{1} \notin U(R)$. This shows that $\frac{p(q+1)}{p+q} \in R$ is not strongly clean, and we are done.

Theorem 1.3.11. *Let $p, q \neq 2$ be distinct primes. Then $\begin{pmatrix} 0 & pq \\ 1 & 1 - pq \end{pmatrix} \in M_2\big(\mathbb{Z}_{(p)} \cap \mathbb{Z}_{(q)}\big)$ is strongly clean.*

Proof. Set $R = \mathbb{Z}_{(p)} \cap \mathbb{Z}_{(q)}$. Suppose that M is an ideal of R such that $pR \subsetneq M \subseteq R$. Choose $\frac{m}{n} \in M$ while $\frac{m}{n} \notin pR$. Then $p \nmid m$; hence, $(p, m) = 1$. Thus, we can find some $k, l \in \mathbb{Z}$ such that $kp + lm = 1$. Clearly, $\frac{m}{1} = \frac{m}{n}\frac{n}{1} \in M$, hence, $\frac{1}{1} = p \cdot \frac{k}{1} + l\frac{m}{1} \in M$. This implies that $M = R$. Therefore pR is a maximal ideal of R. Likewise, qR is a maximal ideal of R. As $p, q \neq 2$, we see that $p, q \nmid 2$, and so $\frac{1}{2} \in R$. Obviously, $J(R) \subseteq pR \cap qR$. For any $\frac{m}{n} \in pR \cap qR$ and $\frac{a}{b} \in R$, then $\frac{1}{1} - \frac{m}{n}\frac{a}{b} = \frac{nb-ma}{nb}$. Write $\frac{m}{n} = \frac{ps}{t}$. Then $psn = mt$; hence, $p \mid mt$. As $p \nmid t$, we see that $p \mid m$. Likewise, $q \mid m$. Clearly, $p \nmid nb$, and so $p \nmid (nb - ma)$. Similarly, we see that $q \nmid (nb - ma)$. This yields that $\frac{nb}{nb-ma} \in R$.

Thus, $\big(\frac{1}{1} - \frac{m}{n}\frac{a}{b}\big) \cdot \frac{nb}{nb-ma} = \frac{1}{1}$. This means that $\frac{1}{1} - \frac{m}{n}\frac{a}{b} \in U(R)$, and then $\frac{m}{n} \in J(R)$. Therefore $J(R) = pR \cap qR$. Assume that M is a maximal ideal of R and $M \neq pR, qR$. Then $pR + M = R$ and $qR + M = R$. It follows that $R = (pR + M)(qR + M) \subseteq pR \cap qR + M = J(R) + M \subseteq M$, and so $R = M$. This gives a contradiction. Therefore R be an integral domain with exactly two maximal ideals pR and qR.

Choose $\lambda = pq \in J(R)$ and $\mu = 1 - pq \in 1 + J(R)$. It is easy to check that $\frac{1}{1+pq} \in U(R)$ is a root of the equation $x^2 - x + \frac{\lambda}{\mu^2+4\lambda} = 0$. According to Corollary 1.3.10, we complete the proof. □

For instance, $\begin{pmatrix} 0 & 15 \\ 1 & -14 \end{pmatrix} \in M_2\big(\mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)}\big)$ is strongly clean, and that $\begin{pmatrix} 0 & 21 \\ 1 & -20 \end{pmatrix} \in M_2\big(\mathbb{Z}_{(3)} \cap \mathbb{Z}_{(7)}\big)$ is strongly clean.

1.4 Rings of Power Series

Let $R[[x]]$ denote the ring of formal series over a ring R, that is all formal power series in x with coefficients from R. Let $A(x) = \big(a_{ij}(x)\big) \in M_n\big(R[[x]]\big)$. We use $A(0)$ to denote the matrix $A(0) = \big(a_{ij}(0)\big) \in M_n(R)$. Following Shifflet, we say that an element $a \in R$ is optimally clean provided that there exists an idempotent $e \in R$ such that $a - e \in U(R)$ and $ae = ea$, and that for any $b \in R$, there exists $c \in R$ such that $[a, c] = [e, b]$. A ring R is optimally clean provided that every element in R is optimally clean. The aim of this section is to explore the strong cleanness for the rings of power series by virtue of optimal cleanness. We start by

Lemma 1.4.1. *Let R be a ring, and let $a \in R$. Then the following are equivalent:*

(1) *$a \in R$ is optimally clean.*
(2) *There exists an idempotent $e \in R$ such that $a - e \in U(R)$ and $ae = ea$, and that for any $b \in R$, there exists $c \in eR(1-e) + (1-e)Re$ such that $[a, c] = [e, b]$.*

Proof. (1) $\Rightarrow$ (2) Since $a \in R$ is optimally clean, there exists an idempotent $e \in R$ such that $a - e \in U(R)$ and $ae = ea$, and that for any $b \in R$, there exists $c \in R$ such that $[a, c] = [e, b]$. It is easy to check that

$$\begin{aligned} [a, ec(1-e) + (1-e)ce] &= [a, ec(1-e)] + [a, (1-e)ce] \\ &= e[a, c](1-e) + (1-e)[a, c]e \\ &= e[e, b](1-e) + (1-e)[e, b]e \\ &= [e, b], \end{aligned}$$

as required.

(2) $\Rightarrow$ (1) is trivial. □

Lemma 1.4.2. *Let R be a ring, and let $n \geq 2$. If $e_0 = e_0^2 \in R$ and $e_k = \sum_{i=0}^{k} e_i e_{k-i} (0 < k < n)$, then $e_0 \left(\sum_{i=1}^{n-1} e_i e_{n-i} \right) = \left(\sum_{i=1}^{n-1} e_i e_{n-i} \right) e_0$.*

Proof. One easily checks that

$$\begin{aligned}
&e_0\left(\sum_{i=1}^{n-1} e_ie_{n-i}\right) - \left(\sum_{i=1}^{n-1} e_ie_{n-i}\right)e_0\\
&= e_0\left(\sum_{i=1}^{n-1} e_ie_{n-i}\right) - \sum_{i=1}^{n-1} e_i(e_{n-i}e_0)\\
&= e_0\big(e_1e_{n-1} + \cdots + e_{n-1}e_1\big) - \sum_{i=1}^{n-1} e_i\big((1-e_0)e_{n-i} - \cdots - e_{n-i-1}e_1\big)\\
&= (e_0e_1 - e_1 + e_1e_0)e_{n-1} + \cdots + (e_0e_{n-1} + e_1e_{n-2} + \cdots - e_{n-1} + e_{n-1}e_0)e_1\\
&= 0,
\end{aligned}$$

and thus the proof is true. □

Lemma 1.4.3. *Let R be a ring, and let $n \geq 2$. If $e_0 = e_0^2 \in R, e_k = \sum_{i=0}^{k} e_ie_{k-i}$ and $[r_0, e_k] + [r_1, e_{k-1}] + \cdots + [r_k, e_0] = 0$ for all $0 < k < n$. Then*

$$\left[r_0, \sum_{i=1}^{n-1} e_ie_{n-i}\right] = (1-e_0)\left(\sum_{i=1}^{n-1}[e_i, r_{n-i}]\right) - \left(\sum_{i=1}^{n-1}[e_i, r_{n-i}]\right)e_0.$$

Proof. Set $t_n = \sum_{i=1}^{n-1}[e_i, r_{n-i}]$. Then $t_n = e_1r_{n-1} + \cdots + e_{n-1}r_1 - r_{n-1}e_1 - \cdots - r_1e_{n-1}$. Let $\alpha_k = e_1r_{k-1} + \cdots + e_{k-1}r_1$ and $\beta_k = r_{k-1}e_1 + \cdots + r_1e_{k-1}$. We check that

$$\begin{aligned}
&\alpha_ne_0 + e_0\alpha_n\\
&= \big(e_1r_{n-1} + \cdots + e_{n-1}r_1\big)e_0 + e_0\big(e_1r_{n-1} + \cdots + e_{n-1}r_1\big)\\
&= \big(e_1r_{n-1} + \cdots + e_{n-1}r_1\big)e_0 + (e_0e_1)r_{n-1} + \cdots + (e_0e_{n-1})r_1\\
&= \big(e_1r_{n-1} + \cdots + e_{n-1}r_1\big)e_0 + (e_1 - e_1e_0)r_{n-1} + (e_2 - e_2e_0\\
&\quad -e_1e_1)r_{n-2} + \cdots + (e_{n-1} - e_{n-1}e_0 - \cdots - e_1e_{n-2})r_1\\
&= e_1\big(r_{n-1}e_0 + r_{n-1} - e_0r_{n-1} - e_1r_{n-2} - \cdots - e_{n-2}r_1\big) + e_2\big(r_{n-2}e_0\\
&\quad + r_{n-2} - e_0r_{n-2} - \cdots - e_{n-3}r_1\big) + \cdots + e_{n-1}\big(r_1e_0 + r_1 - e_0r_1\big)\\
&= \alpha_n + e_1\big(r_{n-1}e_0 - e_0r_{n-1} - e_1r_{n-2} - \cdots - e_{n-2}r_1\big)\\
&\quad + e_2\big(r_{n-2}e_0 - e_0r_{n-2} - \cdots - e_{n-3}r_1\big) + \cdots + e_{n-1}\big(r_1e_0 - e_0r_1\big)\\
&= \alpha_n - e_1\big(r_{n-2}e_1 + r_{n-3}e_2 + \cdots + r_1e_{n-2} + r_0e_{n-1} - e_{n-1}r_0\big) - e_2\big(r_{n-3}e_1\\
&\quad + r_{n-4}e_2 + \cdots + r_1e_{n-3} + r_0e_{n-2} - e_{n-2}r_0\big) - \cdots - e_{n-1}\big(r_0e_1 - e_1r_0\big)\\
&= \alpha_n - \big(e_1r_{n-2} + e_2r_{n-3} + \cdots + e_{n-1}r_0\big)e_1 - \big(e_1r_{n-3} + e_2r_{n-4} + \cdots\\
&\quad + e_{n-2}r_0\big)e_2 - \cdots - \big(e_1r_0\big)e_{n-1} + \big(e_1e_{n-1} + \cdots + e_{n-1}e_1\big)r_0\\
&= \alpha_n - \gamma_n + e_{n-1}r_0,
\end{aligned}$$

where $\gamma_n = \big(e_1r_{n-2} + e_2r_{n-3} + \cdots + e_{n-1}r_0\big)e_1 + \big(e_1r_{n-3} + e_2r_{n-4} + \cdots + e_{n-2}r_0\big)e_2 + \cdots + \big(e_1r_0\big)e_{n-1}$. Likewise, $\beta_ne_0 + e_0\beta_n = \beta_n - \lambda_n + r_0e_{n-1}$,

where $\lambda_n = e_1\big(r_{n-2}e_1 + r_{n-3}e_2 + \cdots + r_0e_{n-1}\big) + e_2\big(r_{n-3}e_1 + r_{n-4}e_2 + \cdots + r_0e_{n-2}\big) + \cdots + e_{n-1}\big(r_0e_1\big)$.

Further, we verify that $\gamma_n - \lambda_n = \big(e_1r_{n-2}e_1 + e_1r_{n-3}e_2 + \cdots + e_1r_0e_{n-1}\big) + \big(e_2r_{n-3}e_1 + e_2r_{n-4}e_2 + \cdots + e_2r_0e_{n-2}\big) + \cdots + \big(e_{n-1}r_0e_1\big) - \lambda_n = 0$. Therefore we get

$$\begin{aligned} & e_0\big(\alpha_n - \beta_n\big) + \big(\alpha_n - \beta_n\big)e_0 \\ &= \big(\alpha_ne_0 + e_0\alpha_n\big) - \big(\beta_ne_0 + e_0\beta_n\big) \\ &= \big(\alpha_n - \gamma_n + e_{n-1}r_0\big) - \big(\beta_n - \lambda_n + r_0e_{n-1}\big) \\ &= \alpha_n - \beta_n - \big(\gamma_n - \lambda_n\big) + e_{n-1}r_0 - r_0e_{n-1}. \end{aligned}$$

Hence, $e_0t_n + t_ne_0 = t_n + e_{n-1}r_0 - r_0e_{n-1}$, so $e_0\left(\sum\limits_{i=1}^{n-1}[e_i, r_{n-i}]\right) + \left(\sum\limits_{i=1}^{n-1}[e_i, r_{n-i}]\right)e_0 = \sum\limits_{i=1}^{n-1}[e_i, r_{n-i}] - \left[r_0, \sum\limits_{i=1}^{n-1}e_ie_{n-i}\right]$, and therefore the result follows. □

We now come to the main result of this section.

Theorem 1.4.4. *Let R be a ring, and let $f(x) \in R[[x]]$. If $f(0) \in R$ is optimally clean, then $f(x) \in R[[x]]$ is strongly clean.*

Proof. Write $f(x) = \sum\limits_{i=0}^{\infty} r_ix^i$. Then we can find an idempotent e_0 such that $r_0 = e_0 + (r_0 - e_0)$ is an optimally clean decomposition of r_0. By virtue of Lemma 1.4.1, there exists some $e_1 \in (1-e_0)Re_0 + e_0R(1-e_0)$ such that $[r_0, e_1] + [e_0, r_1] = 0$. In addition, $e_1 = e_0e_1 + e_1e_0$. Further, we will show that there exist $e_2, \ldots, e_k, \ldots \in R$ such that

$$e_k = e_0e_k + e_1e_{k-1} + \cdots + e_ke_0 \text{ and } [r_0, e_k] + [r_1, e_{k-1}] + \cdots + [r_k, e_0] = 0.$$

Assume that we have some $e_1, \ldots, e_{n-1} \in R (n \geq 2)$ such that the preceding conditions hold for all $1 \leq k \leq n-1$. Set $f_n = (1-2e_0)(e_1e_{n-1} + e_2e_{n-2} + \cdots + e_{n-1}e_1)$ and $s_n = r_n + \big[e_0, [e_1, r_{n-1}] + [e_2, r_{n-2}] + \cdots + [e_{n-1}, r_1]\big]$. In view of Lemma 1.4.1, we can find some $g_n \in (1-e_0)Re_0 + e_0R(1-e_0)$ such that $[r_0, g_n] = [e_0, s_n]$. Let $e_n = f_n + g_n$. By virtue of Lemma 1.4.2, we get

$$\begin{aligned} \sum_{i=1}^{n-1} e_ie_{n-i} &= (1-e_0)\left(\sum_{i=1}^{n-1} e_ie_{n-i}\right) + \left(\sum_{i=1}^{n-1} e_ie_{n-i}\right)e_0 \\ &= (1-e_0)f_n - f_ne_0 \\ &= (1-e_0)f_n + \big((1-e_0)g_n - g_ne_0\big) - f_ne_0 \\ &= (1-e_0)e_n - e_ne_0. \end{aligned}$$

Thus, $e_n = \sum_{i=1}^{n} e_i e_{n-i}$. Further, we show that

$$\begin{aligned}[r_0, f_n] &= \left[r_0, (1-e_0)\left(\sum_{i=1}^{n-1} e_i e_{n-i}\right)\right] - \left[r_0, \left(\sum_{i=1}^{n-1} e_i e_{n-i}\right) e_0\right] \\ &= (1-e_0)\left[r_0, \left(\sum_{i=1}^{n-1} e_i e_{n-i}\right)\right](1-e_0) - e_0\left[r_0, \left(\sum_{i=1}^{n-1} e_i e_{n-i}\right)\right] e_0.\end{aligned}$$

In view of Lemma 1.4.3, we get

$$\left[r_0, \sum_{i=1}^{n-1} e_i e_{n-i}\right] = (1-e_0)\left(\sum_{i=1}^{n-1}[e_i, r_{n-i}]\right) - \left(\sum_{i=1}^{n-1}[e_i, r_{n-i}]\right) e_0,$$

and so

$$[r_0, f_n] = (1-e_0)\left(\sum_{i=1}^{n-1}[e_i, r_{n-i}]\right)(1-e_0) + e_0\left(\sum_{i=1}^{n-1}[e_i, r_{n-i}]\right) e_0.$$

Moreover,

$$\begin{aligned}[r_0, g_n] &= [e_0, s_n] \\ &= [e_0, r_n] + \left[e_0, \left[e_0, \sum_{i=1}^{n-1}[e_i, r_{n-i}]\right]\right] \\ &= [e_0, r_n] + e_0\left(\sum_{i=1}^{n-1}[e_i, r_{n-i}]\right)(1-e_0) + (1-e_0)\left(\sum_{i=1}^{n-1}[e_i, r_{n-i}]\right) e_0.\end{aligned}$$

Hence, $[r_0, e_n] = [r_0, f_n] + [r_0, g_n] = [e_0, r_n] + e_0\left(\sum_{i=1}^{n-1}[e_i, r_{n-i}]\right)(1 - e_0) + (1-e_0)\left(\sum_{i=1}^{n-1}[e_i, r_{n-i}]\right) e_0 + (1-e_0)\left(\sum_{i=1}^{n-1}[e_i, r_{n-i}]\right)(1-e_0) + e_0\left(\sum_{i=1}^{n-1}[e_i, r_{n-i}]\right)(1-e_0) = \sum_{i=0}^{n-1}[e_i, r_{n-i}]$, and so $\sum_{i=0}^{n}[r_i, e_{n-i}] = 0$. By induction, the affirmation above is true. Therefore $\sum_{i=0}^{\infty} e_i x^i = \left(\sum_{i=0}^{\infty} e_i x^i\right)^2 \in R[[x]]$ and $f(x)\left(\sum_{i=0}^{\infty} e_i x^i\right) = \left(\sum_{i=0}^{\infty} e_i x^i\right) f(x)$. Obviously, $f(x) - \sum_{i=0}^{\infty} e_i x^i \in U\big(R[[x]]\big)$. Consequently, $f(x) \in R[[x]]$ is strongly clean. □

Corollary 1.4.5. *Let R be an abelian exchange ring. Then $R[[x]]$ is strongly clean.*

Proof. By virtue of Corollary 1.1.9, R is strongly clean. Let $a \in R$. Then there exists an idempotent $e \in R$ such that $a - e \in U(R)$ and $ae = ea$. For any $b \in R$, we see that $[a, 0] = [e, b]$, and thus $a \in R$ is optimally clean.

Therefore R is optimally clean. According to Theorem 1.4.4, we complete the proof. □

Corollary 1.4.6. *Let R be strongly π-regular. Then $R[[x]]$ is strongly clean.*

Proof. Let $a \in R$. As in the proof of Theorem 1.1.2, there exist $n \in \mathbb{N}, x \in R$ such that $a^n = a^{2n}x, ax = xa$ and $x = x^2a^n$. Clearly, $v = x + 1 - a^nx \in U(R)$. Set $f = a^nx$. Then $f = f^2 \in R$ and $fv = vf$. Choose $w = v^{-1}$. Then $a^n = fw = wf$, where f, w and a all commutate. Let $e = 1 - f$ and $u = a - e$. Then $a = e + u, e = e^2, u \in U(R)$ and $ae = ea$. Obviously, $faf = fuf \in U(fRf)$. In addition, $(eae)^n = (1 - a^nx)a^n = 0$. For any $b \in R$, we set $y = [e, b]$. Then $y = eb - be = eb(1 - e) - (1 - e)be$. Choose $c = \sum_{i=1}^{n} \big((faf)^{-i}y(eae)^{i-1} - (eae)^{i-1}y(faf)^{-i}\big) \in R$, where $(eae)^0$ means that e. Then we check that $ac - ca = \big(fye - aey(faf)^{-1}\big) - \big((faf)^{-1}yea - eyf\big) + \big((faf)^{-1}y(eae)^1 - (eae)^2y(faf)^{-2}\big) - \big((faf)^{-2}y(eae)^2 - (eae)^1y(faf)^{-1}\big) + \cdots + \big((faf)^{-(n-2)}y(eae)^{n-2} - (eae)^{n-1}y(faf)^{-(n-1)}\big) - \big((faf)^{-(n-1)}y(eae)^{n-1} - (eae)^{n-2}y(faf)^{-(n-2)}\big) + \big((faf)^{-(n-1)}y(eae)^{n-1} - (eae)^ny(faf)^{-n}\big) - \big((faf)^{-n}y(eae)^n - (eae)^{n-1}y(faf)^{-(n-1)}\big) = fye + eyf = y$. That is, $[a, c] = [e, b]$. Thus, R is optimally clean. According to Theorem 1.4.4, we obtain the result. □

Lemma 1.4.7. *Let R be a ring, and let $a \in R$ be optimally clean. If $u \in U(R)$, then $uau^{-1} \in R$ is optimally clean.*

Proof. By hypothesis, there exists an idempotent $e \in R$ such that $a - e \in U(R)$ and $ae = ea$, and that for any $b \in R$, there exists $c \in R$ such that $[a, c] = [e, u^{-1}bu]$. Thus, $uau^{-1} - ueu^{-1} \in U(R)$, $(uau^{-1})(ueu^{-1}) = (ueu^{-1})(uau^{-1})$ and $ueu^{-1} \in R$ is an idempotent. In addition, we check that

$$[uau^{-1}, ucu^{-1}] = u[a, c]u^{-1} = u[e, u^{-1}bu]u^{-1} = [ueu^{-1}, b].$$

Therefore $uau^{-1} \in R$ is optimally clean. □

A ring R is called weakly bleached if for any $a \in 1 + J(R), b \in J(R)$, $l_a - r_b$ is surjective. For instance, every commutative local ring is weakly bleached. Clearly, a ring R is weakly bleached if and only if for any $a \in 1 + J(R), b \in J(R)$, $l_a - r_b$ and $l_b - r_a$ are surjective. In view of Corollary 1.2.16, a local ring R is weakly bleached if and only if $T_2(R)$ is strongly clean.

Theorem 1.4.8. *Let R be a local ring, and let $A \in M_2(R)$. If R is weakly bleached, then the following are equivalent:*

(1) *$A \in M_2(R)$ is optimally clean.*
(2) *$A \in M_2(R)$ is strongly clean.*

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1) Let $A \in M_2(R)$ be strongly clean. In view of Theorem 1.2.13, $A \in GL_2(R)$, or $I_2 - A \in GL_2(R)$, or A is similar to a diagonal matrix. If $A \in GL_2(R)$, then $A - 0 \in GL_2(R)$. For any $B \in M_2(R)$, we see that $[A, 0] = [0, B]$, and so $A \in M_2(R)$ is optimally clean. If $I_2 - A \in GL_2(R)$, then $A - I_2 \in GL_2(R)$. For any $B \in M_2(R)$, we see that $[A, 0] = [I_2, B]$, and so $A \in M_2(R)$ is optimally clean. Thus $UAU^{-1} = diag(\alpha, \beta)$ for some $U \in GL_2(R)$. We may assume that $\alpha \in 1 + J(R)$ and $\beta \in J(R)$. For any $B = (b_{ij}) \in M_2(R)$, there are $c_1, c_2 \in R$ such that $\alpha c_1 - c_1\beta = -b_{12}$ and $\beta c_2 - c_2\alpha = b_{21}$. Choose $C = \begin{pmatrix} 0 & c_1 \\ c_2 & 0 \end{pmatrix}$. It is easy to check that

$$\big[diag(\alpha, \beta), C\big] = \begin{pmatrix} 0 & -b_{12} \\ b_{21} & 0 \end{pmatrix} = \big[diag(0, 1), B\big].$$

In addition, $diag(\alpha, \beta) - diag(0, 1) \in GL_2(R)$. Therefore UAU^{-1} is optimally clean. According to Lemma 1.4.7, $A \in M_2(R)$ is optimally clean, as asserted. □

Corollary 1.4.9. *Let R be a weakly bleached local ring. Then the following are equivalent:*

(1) *$A(x) \in M_2\big(R[[x]]\big)$ is strongly clean.*
(2) *$A(0) \in M_2(R)$ is strongly clean.*

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1) Clearly, $M_2\big(R[[x]]\big) \cong M_2(R)[[x]]$. We may assume that $A(x) \in M_2(R)[[x]]$. Then $A(0) \in M_2(R)$. By hypothesis, $A(0) \in M_2(R)$ is strongly clean. According to Theorem 1.4.8, $A(0) \in M_2(R)$ is optimally clean. In light of Theorem 1.4.4, $A(x) \in M_2\big(R[[x]]\big)$ is strongly clean. □

Let $\mathbb{Z}_4 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$ be the ring of all integers modulo 4. Then $\mathbb{Z}_4$ is a commutative local ring with $J\big(\mathbb{Z}_4\big) = \{\overline{0}, \overline{2}\}$. Since $M_2\big(\mathbb{Z}_4\big)$ is strongly π-regular, it is strongly clean. According to Corollary 1.4.9, $M_2\big(\mathbb{Z}_4[[x]]\big)$ is strongly clean.

Proposition 1.4.10. *Let R be a ring, and let $\overline{f(x)} \in R[x]/(x^n)$. If $f(0) \in R$ is optimally clean, then $\overline{f(x)} \in R[x]/(x^n)$ is strongly clean.*

Proof. Write $f(x) = \sum_{i=0}^{m} a_i x^i \in R[x]$. Since $f(0) \in R$ is optimally clean, it follows from Theorem 1.4.4 that $f(x) \in R[[x]]$ is strongly clean. Thus, $\overline{f(x)} \in R[[x]]/(x^n)$ is strongly clean, and we are through. □

Corollary 1.4.11. *Let R be a local ring, and let $\overline{A(x)} \in M_2\big(R[[x]]/(x^m)\big)$ $(m \geq 1)$. If R is weakly bleached, then the following are equivalent:*

(1) *$\overline{A(x)} \in M_2\big(R[[x]]/(x^m)\big)$ is strongly clean.*
(2) *$A(0) \in M_2(R)$ is strongly clean.*

Proof. It is obvious from Corollary 1.4.9 and Proposition 1.4.10. □

Example 1.4.12. Let $R = \mathbb{Z}_{(2)}[x]/(x^2)$, and let $A(x) = \begin{pmatrix} \overline{2} + \overline{2}x & \overline{3}x \\ x & \overline{3} + \overline{2}x \end{pmatrix} \in M_2(R)$. Obviously, $\mathbb{Z}_{(2)}$ is a local ring. It is easy to verify that $R/J(R) \cong \mathbb{Z}_2$, and then $R = \mathbb{Z}_{(2)}[[x]]/(x^2)$ is a weakly bleached local ring. Since we have the strongly clean decomposition $A(0) = \begin{pmatrix} \overline{1} & \overline{0} \\ \overline{0} & \overline{0} \end{pmatrix} + \begin{pmatrix} \overline{1} & \overline{0} \\ \overline{0} & \overline{3} \end{pmatrix}$ in $M_2(\mathbb{Z}_{(2)})$, it follows by Corollary 1.4.11 that $A(x) \in M_2(R)$ is strongly clean.

Theorem 1.4.13. *Let A and B be local rings and ${}_AV_B$ is a A-B-bimodule, and let $R = \begin{pmatrix} A & V \\ 0 & B \end{pmatrix}$. Then the following are equivalent:*

(1) *R is optimally clean.*
(2) *R is strongly clean.*

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1) Let $D = \begin{pmatrix} a & v \\ 0 & b \end{pmatrix} \in R$. Then $D \in R$ is strongly clean. If D or $I_R - D$ is invertible, then $D \in R$ is optimally clean; otherwise, either $a \in U(A), b \in J(B)$ or $a \in J(A), b \in U(B)$. If $a \in U(A), b \in J(B)$, then $a \in 1_A + J(A)$; otherwise, $a - 1_A \in U(A), b - 1_B \in U(B)$. According to Theorem 1.2.15, we have some $x \in R$ such that $ax - xb = v$. Choose $E = \begin{pmatrix} 0 & -x \\ 0 & 1_B \end{pmatrix} \in R$. Then $D - E \in U(R), E = E^2$ and $DE = ED$. For any $H = \begin{pmatrix} c & w \\ 0 & d \end{pmatrix} \in R$, it follows from Theorem 1.2.15 that there exists some

$z \in V$ such that $az - zb = cx - xd - w$. Choose $K = \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} \in R$. Then it is easy to verify that

$$[D, K] = \begin{pmatrix} 0 & az - zb \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & cx - xd - w \\ 0 & 0 \end{pmatrix} = [E, H].$$

Therefore $D \in R$ is optimally clean. If $a \in J(A), b \in U(B)$, analogously, $D \in R$ is optimally clean. Therefore the result follows. □

Corollary 1.4.14. *Let R be a local ring. Then the following are equivalent:*

(1) *$T_2(R)$ is optimally clean.*
(2) *R is weakly bleached.*

Proof. (1) $\Rightarrow$ (2) Since $T_2(R)$ is optimally clean, it is strongly clean. In view of Corollary 1.2.16, R is weakly bleached.

(2) $\Rightarrow$ (1) By virtue of Corollary 1.2.16, $T_2(R)$ is strongly clean, and so the result follows from Theorem 1.4.13. □

Corollary 1.4.15. *Let R be a weakly bleached local ring. Then $T_2\big(R[[x]]\big)$ is strongly clean.*

Proof. Obviously, $T_2\big(R[[x]]\big) \cong T_2(R)[[x]]$. According to Corollary 1.4.14, $T_2(R)$ is optimally clean. Therefore $T_2(R)[[x]]$ is strongly clean from Theorem 1.4.4, and we are done. □

Corollary 1.4.16. *Let R be a local ring. If $J(R)$ is nil, then $T_2\big(R[[x]]\big)$ is strongly clean.*

Proof. By virtue of Corollary 1.4.15, it will suffice to show that R is weakly bleached. Let $a \in 1 + J(R), b \in J(R)$. Write $b^n = 0$. Choose $\varphi = l_{a^{-1}} + l_{a^{-2}r_b} + \cdots + l_{a^{-n}r_{b^{n-1}}} : R \to R$. For any $r \in R$, we see that

$$\begin{aligned} &\big(l_a - r_b\big)\varphi(r) \\ &= \big(l_a - r_b\big)\big(a^{-1}r + a^{-2}rb + \cdots + a^{-n}rb^{n-1}\big) \\ &= \big(r + a^{-1}rb + \cdots + a^{-n+1}rb^{n-1}\big) - \big(a^{-1}rb + a^{-2}rb^2 + \cdots + a^{-n+1}rb^{n-1}\big) \\ &= r. \end{aligned}$$

Thus, $\big(l_a - r_b\big)\varphi = 1_R$. This implies that $l_a - r_b : R \to R$ is surjective, as desired. □

Construct R as in Example 1.2.8. Then we claim that $T_2\big(R[[x]]\big)$ is strongly clean. Clearly, R is a local ring with nil Jacobson radical, and then we are through by Corollary 1.4.16.

Corollary 1.4.17. *Let R be a commutative local ring. Then $T_2\big(R[[x]]\big)$ is strongly clean.*

Proof. Clearly, R is weakly bleached. Therefore we obtain the result from Corollary 1.4.15. □

For instance, for any prime p ($\neq 2$), $T_2\big(\mathbb{Z}_{(p)}[[x]]\big)$ is strongly clean. It follows from Corollary 1.4.14 that strong cleanness and optimally cleanness coincide with each other for 2×2 triangular matrix ring over local rings. But this is not the case for a single triangular matrix over a local ring. We include a counterexample which is due to Shifflet [471].

Example 1.4.18. Let k be a field, and let $R = k[t_1, t_2, \ldots]_{(t_1)}$. Define $\sigma : R \to R, t_i \to t_{i+1}, \sigma \mid_k = 1_k$ and $\tau : R \to R, t_i \to t_{i+1} + 1, \tau \mid_k = 1_k$. Construct the left-right skew power series ring $A = \sum_{i,j\geq 0} y^i a_{ij} x^j$ where $xy = yx, ry = y\tau(r), xs = \sigma(s)x$. Let $S = T_2(A)$, and let $M_0 + M_1 x \in S[[x]]$ where $M_0 = \begin{pmatrix} t_1 & 0 \\ 0 & 1+t_1 \end{pmatrix}, M_1 = \begin{pmatrix} 0 & xy \\ 0 & 0 \end{pmatrix}$. As in the proof of [471, Example 4.2], A is local ring and $M_0 \in T_2(A)$ is strongly clean. If $M_0 + M_1 x \in T_2(A)[[x]]$ is strongly clean, then $M_0 + M_1 x \in T_2(A)[[x]]$ has the strongly clean decomposition

$$\left[\begin{pmatrix} 1 & c \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & z_1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & z_2 \\ 0 & 0 \end{pmatrix} x^2 + \cdots\right] +$$
$$\left[\begin{pmatrix} t_1 - 1 & -c \\ 0 & 1+t_1 \end{pmatrix} + \begin{pmatrix} 0 & xy - z_1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & -z_2 \\ 0 & 0 \end{pmatrix} x^2 + \cdots\right],$$

where $c = t_1 c - c t_1$. As the elements commute, we see that $xy = t_1 z_1 - z_1(1 + t_1)$. This also gives a contradiction. Therefore $M_0 + M_1 x \in T_2(A)[[x]]$ is not strongly clean. In view of Theorem 1.4.4, $M_0 \in T_2(A)$ is not optimally clean.

Notes. Theorem 1.1.8 was proved by Chen [187, Corollary 2.3]. Theorem 1.1.14 is due to Nicholson [432, Theorem 3]. Theorem 1.1.18 was proved by Fan [250, Theorem 3.2.14]. Theorem 1.2.13 was carried out by Li [395, Theorem 2.6]. Theorem 1.2.15 is due to Nicholson [432. Example 2]. Theorem 1.3.5 was first observed by Li [395, Corollary 3.3]. Theorem 1.4.4 was proved by Shifflet [471, Theorem 3.2.2]. Theorem 1.4.13 was obtained in [471, Proposition 4.1.8].

This page intentionally left blank

Chapter 2

Matrices over Commutative Rings

In this chapter we establish a number of equivalent characterizations of strong cleanness of matrices over commutative rings. We start by investigating when a companion matrix over a ring is strongly clean. We then completely characterize a strongly clean $n \times n$ matrix over a commutative ring R, in terms of factorization in $R[t]$. Next, we develop explicit criteria for matrices of lower ranks. Finally, we are concerned with a class of commutative rings over which every finitely generated projective module is free, and describe strongly clean matrices over such rings.

2.1 Companion Matrices

Every square matrix over a commutative ring is associated with a characteristic polynomial. If $\varphi \in M_n(R)$, we use $\chi(\varphi)$ to stand for the characteristic polynomial $det(tI_n - \varphi)$. Let $f, g \in R[t]$. The notation $(f, g) = 1$ means that there exist some $h, k \in R[t]$ such that $fh + gk = 1$, i.e., the ideal generated by f, g is $R[t]$. Let $h(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$. We say that the matrix

$$\varphi = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix} \in M_n(R)$$

is the companion matrix of h, and denote it by C_h. The characteristic polynomial of the companion matrix C_h over a commutative ring is equal to h. A polynomial $h(t) \in R[t]$ is a monic polynomial of degree n if $h(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$ where $a_{n-1}, \ldots, a_1, a_0 \in R$. In this section we investigate necessary and sufficient conditions under which companion matrices over commutative rings are strongly clean.

We first characterize a strongly clean companion matrix over an arbitrary ring. Let R be a ring, and let $h \in R[t]$ be a monic polynomial of degree n. We say that h has a right iSRC factorization if there exists a pairs of right ideals I and J of $R[t]$ such that

(1) $I + J = R[t]$;
(2) $I \cap J = hR[t]$;
(3) *There exist* $p \in I$ *and* $q \in J$ *such that* $p(0) = 1$ *and* $q(1) = 1$;
(4) $I = \{f \in R[t] \mid tf \in I\}$ *and* $J = \{f \in R[t] \mid (1-t)f \in J\}$.

Let $h \in R[t]$. Construct $m_t : R[t]/hR[t] \to R[t]/hR[t], \overline{f} \mapsto \overline{tf}$. Then $m_t \in End_R\big(R[t]/hR[t]\big)$.

Lemma 2.1.1. *Let R be an arbitrary ring, and let $h \in R[t]$ be monic. Then $m_t \in End_R\big(R[t]/hR[t]\big)$ is strongly clean if and only if h has a right iSRC factorization.*

Proof. Let $M = R[t]/hR[t]$. Then an m_t-invariant right R-submodule of M is the same as a right $R[t]$-submodule of M. Note that each decomposition of M as $M = A \oplus B$ into right $R[t]$-modules corresponds exactly two right ideals I and J of $R[t]$ such that $I + J = R[t]$ and $I \cap J = hR[t]$, where $A = J/hR[t]$ and $B = I/hR[t]$. Hence, $A \cong M/B \cong R[t]/I$ as right $R[t]$-modules. Likewise, $B \cong R[t]/J$. Clearly, $m_t|_A$ is surjective if and only if $\overline{1} \in R[t]/I$ is in the image m_t, i.e., $1 - tf \in I$ for some $f \in R[t]$. Set $p = 1 - tf$. This is equivalent to $p \in I$ and $p(0) = 1$. Furthermore, $m_t|_A$ is injective if and only if $I = \{f \in R[t] \mid tf \in I\}$. Likewise, $(1 - m_t)|_B$ is surjective if and only if $\overline{1} \in R[t]/J$ is in the image m_{1-t}, i.e., $1 - (1-t)g \in J$ for some $g \in R[t]$. Set $q = 1 - (1-t)g$. This is equivalent to $q \in J$ and $q(1) = 1$, and that $(1 - m_t)|_B$ is injective if and only if $J = \{f \in R[t] \mid (1-t)f \in J\}$. Therefore we complete the proof, by Corollary 1.1.15. □

Theorem 2.1.2. *Let R be an arbitrary ring, and let $h \in R[t]$ be monic. Then the following are equivalent:*

(1) *The companion matrix C_h of h is strongly clean.*
(2) *h has a right iSRC factorization.*

Proof. Let $h \in R[t]$ be a monic polynomial of degree n. Then $R[t]/hR[t]$ is a free right R-module of rank n, with a basis by the image of $\{1, t, \ldots, t^{n-1}\}$. With respect to this basis, the matrix of m_t is C_h. This completes the proof, by Lemma 2.1.1. □

Lemma 2.1.3. *Let R be a commutative ring, and let $h \in R[t]$ be monic. If there exist ideals I and J such that $I + J = R[t]$ and $I \cap J = (h)$. Then $I = (h_0), J = (h_1), h = h_0h_1$ and $(h_0, h_1) = 1$ for some $h_0, h_1 \in R[t]$.*

Proof. Let $A = J/(h)$ and $B = I/(h)$. Then we have a direct sum decomposition $M = A \oplus B$. Since M is a free module with a basis $\{1, t, \ldots, t^{n-1}\}$ where $deg(h) = n$, we see that h is the characteristic polynomial of m_t. Let $h_0 = \chi_{m_t|_A}$ and $h_1 = \chi_{m_t|_B}$. Then $h = h_0h_1$. By the Cayley-Hamilton Theorem, every square matrix over a commutative ring satisfies its own characteristic equation. Thus, $h_0(m_t|_A) = 0$, and so $h_0 \in I$. Likewise, $h_1 \in J$. Let $f \in I$. Then $fh_1 \in I \cap J = (h)$, and so $fh_1 = hk$ for some $k \in R[t]$. It follows that $fh_0h_1 = h_0kh = fh$. Thus, $f = h_0k \in (h_0)$, as h is not a zero-divisor (being monic). Hence, $I \subseteq (h_0)$. This implies that $I = (h_0)$. Likewise, $J = (h_1)$. As $I + J = R[t]$, we get $(h_0, h_1) = 1$, as desired. □

Lemma 2.1.4. *Let R be a commutative ring, and let $h \in R[t]$ be monic. Then the following are equivalent:*

(1) *The image of t in $R[t]/(h)$ is strongly clean.*
(2) *h has a right iSRC factorization.*

Proof. (1) $\Rightarrow$ (2) For any $\overline{f} \in R[t]/(h)$, we define $\phi_{\overline{f}} : R[t]/(h) \to R[t]/(h), \overline{g} \mapsto \overline{fg}$. Let $\varphi : R[t]/(h) \to End_R\big(R[t]/(h)\big), \overline{f} \mapsto \phi_{\overline{f}}$. One easily checks that φ is a ring homomorphism. Since $\overline{t} \in R[t]/(h)$ is strongly clean, we see that $m_t = \varphi(\overline{t}) \in End_R\big(R[t]/(h)\big)$ is strongly clean. Thus, (2) is proved by Lemma 2.1.1.

(2) $\Rightarrow$ (1) Since h has a right iSRC factorization, there exist two ideals I and J of $R[t]$ such that $I + J = R[t], I \cap J = (h)$, and there exist $p \in I$ and $q \in J$ such that $p(0) = 1$ and $q(1) = 1$. Then $R[t]/(h) \cong R[t]/(p) \times R[t]/(q)$ as rings. It follows from $p(0) = 1$ that $p = 1 + tk \in I$ for some $k \in R[t]$; hence, $\overline{t(-k)} = \overline{1}$ in $R[t]/I$. Thus, the image of t in R/I is a unit. Likewise, the image of $1 - t$ in $R[t]/J$ is a unit. Accordingly, the images of t in $R[t]/I$ and $R[t]/J$ are both clean. Therefore the image of t in $R/(h)$ is strongly clean, as asserted. □

For $r \in R$, define

$$\mathbb{S}_r = \{f \in R[t] \mid f \text{ is monic and } f(r) \in U(R)\}.$$

Theorem 2.1.5. *Let R be a commutative ring, and let $h \in R[t]$ be monic. Then the following are equivalent:*

(1) *The companion matrix C_h of h is strongly clean.*
(2) *There exists a factorization $h = h_0h_1$ such that $h_0 \in \mathbb{S}_0, h_1 \in \mathbb{S}_1$ and $(h_0, h_1) = 1$.*

Proof. (1) $\Rightarrow$ (2) In view of Theorem 2.1.2, h has an iSRC factorization given by ideals I and J. In view of Lemma 2.1.3, there exist $h_0, h_1 \in R[t]$ such that $I = (h_0), J = (h_1)$ and $(h_0, h_1) = 1$. Since we have $p \in I$ such that $p(0) = 1$, there exists some $k \in R[t]$ such that $p = h_0k$, and so $p(0) = h_0(0)k(0)$. This implies that $h_0(0) \in U(R)$, i.e., $h_0 \in \mathbb{S}_0$. Likewise, $h_1 \in \mathbb{S}_1$. This proves (2).

(2) $\Rightarrow$ (1) Since $(h_0, h_1) = 1$, we see that $\frac{R[t]}{(h_0)\cap(h_1)} \cong R[t]/(h_0) \times R[t]/(h_1)$. As $h_0 \in \mathbb{S}_0$, the image of t in $R[t]/(h_0)$ is a unit. Similarly, the image of $1 - t$ in $R[t]/(h_1)$ is a unit. Thus, the image of t in $\frac{R[t]}{(h_0)\cap(h_1)}$ is strongly clean. Since $h = h_0h_1$, $\frac{R[t]}{(h_0)\cap(h_1)} \cong \frac{R[t]/(h)}{\big((h_0)\cap(h_1)\big)/(h)}$. Additionally, $\big(\frac{(h_0)\cap(h_1)}{(h)}\big)^2 = 0$. Thus, the image of t in $R[t]/(h)$ is clean, hence the result follows by Lemma 2.1.4. □

A matrix $\varphi \in M_n(R)$ is cyclic matrix! if there exists a column α such that $(\alpha, \varphi\alpha, \ldots, \varphi^{n-1}\alpha) \in GL_n(R)$. Let $\varphi = (\varphi_{ij}) \in M_n(R)$ be a matrix which coincides with a companion matrix below the main diagonal, i.e., $\varphi_{ij} = 1, i = j + 1$ and $\varphi_{ij} = 0$ when $i \geq j + 2$. Then φ is cyclic [525, Proposition 6].

Corollary 2.1.6. *Let R be a commutative ring, and let $\varphi \in M_n(R)$ be a cyclic matrix. Then the following are equivalent:*

(1) *$\varphi \in M_n(R)$ is strongly clean.*
(2) *There exists a factorization $\chi(\varphi) = h_0h_1$ such that $h_0 \in \mathbb{S}_0, h_1 \in \mathbb{S}_1$ and $(h_0, h_1) = 1$.*

Proof. Let $F = nR$ be free as a right R-module, and let $\{e_1, \ldots, e_n\}$ be the standard basis of F. Define $\sigma : F \to F$ given by $\sigma(e_1, \ldots, e_n) = (e_1, \ldots, e_n)\varphi$. Since φ is cyclic, there exists a column α such that

$(\alpha, \varphi\alpha, \ldots, \varphi^{n-1}\alpha) \in GL_n(R)$. Thus, $\{\alpha, \varphi\alpha, \ldots, \varphi^{n-1}\alpha\}$ is a basis of F. So there are some $c_1, \ldots, c_n \in R$ such that $\varphi^n\alpha = \alpha c_1 + \varphi\alpha c_2 + \cdots + \varphi^{n-1}\alpha c_n$. Clearly, $(\alpha, \varphi\alpha, \ldots, \varphi^{n-1}\alpha) = (e_1, \ldots, e_n)(\alpha, \varphi\alpha, \ldots, \varphi^{n-1}\alpha)$. Thus,

$$\begin{aligned}\sigma(\alpha, A\alpha, \ldots, \varphi^{n-1}\alpha) &= \big(\sigma(e_1, \ldots, e_n)\big)(\alpha, \varphi\alpha, \ldots, \varphi^{n-1}\alpha)\\ &= (e_1, \ldots, e_n)\varphi(\alpha, \varphi\alpha, \ldots, \varphi^{n-1}\alpha)\\ &= (\varphi\alpha, \varphi^2\alpha, \ldots, \varphi^{n-1}\alpha, \varphi^n\alpha)\\ &= (\alpha, \varphi\alpha, \ldots, \varphi^{n-1}\alpha)\begin{pmatrix} 0 & 0 & \cdots & 0 & c_1\\ 1 & 0 & \cdots & 0 & c_2\\ 0 & 1 & \cdots & 0 & c_3\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & c_n \end{pmatrix}.\end{aligned}$$

Let $\gamma = (\alpha, \varphi\alpha, \ldots, \varphi^{n-1}\alpha) \in GL_n(R)$. Then

$$\gamma^{-1}\varphi\gamma = \begin{pmatrix} 0 & 0 & \cdots & 0 & c_1\\ 1 & 0 & \cdots & 0 & c_2\\ 0 & 1 & \cdots & 0 & c_3\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & c_n \end{pmatrix}.$$

Since strong cleanness and characteristic polynomial for matrices over a commutative ring are invariant under similarity, we complete the proof, by Theorem 2.1.5. □

Corollary 2.1.7. *Let R be a commutative ring, and let $A = (a_{ij}) \in M_2(R)$ with $a_{21} \in U(R)$. Then the following are equivalent:*

(1) *$A \in M_2(R)$ is strongly clean.*
(2) *$A \in GL_2(R)$, or $I_2 - A \in GL_2(R)$, or the equation $t^2 - tr(A)t + det(A) = 0$ has a root $\alpha \in U(R)$ and a root $\beta \in 1 + U(R)$ with $\alpha - \beta \in U(R)$.*

Proof. (1) $\Rightarrow$ (2) Clearly,

$$diag(1, a_{21}^{-1})Adiag(1, a_{21}) = \begin{pmatrix} a_{11} & a_{12}a_{21}\\ 1 & a_{21}^{-1}a_{22}a_{21} \end{pmatrix}.$$

Thus, A is cyclic. In view of Corollary 2.1.6, $\chi(A) = t^2 - tr(A)t + det(A) = h_0h_1$, $h_0 \in \mathbb{S}_0, h_1 \in \mathbb{S}_1$ and $(h_0, h_1) = 1$. If $deg(h_0) = 2$ and $deg(h_1) = 0$, then $A \in GL_2(R)$. If $deg(h_0) = 0$ and $deg(h_1) = 2$, then $I_2 - A \in GL_2(R)$. If $deg(h_0) = deg(h_1) = 1$, then $\chi(A) = (t - \alpha)(t - \beta)$ with $\alpha \in U(R)$,

$\beta \in 1+U(R)$. As $(t-\alpha, t-\beta)=1$, we can find $p(t), q(t) \in R[t]$ such that $p(t)(t-\alpha)+q(t)(t-\beta)=1$. Choose $t=\alpha$. Then $q(\alpha)(\alpha-\beta)=1$; hence, $\alpha-\beta \in U(R)$, as required.

(2) $\Rightarrow$ (1) By the preceding discussion, we may assume A is cyclic. If $A \in GL_2(R)$ or $I_2 - A \in GL_2(R)$, then A is strongly clean. Assume now that the equation $t^2 - tr(A)t + det(A) = 0$ has a root $\alpha \in U(R)$ and a root $\beta \in 1+U(R)$ with $\alpha - \beta \in U(R)$. Then $\alpha^2 - tr(A)\alpha + det(A) = 0$; whence, $\chi(A) = (x-\alpha)\big(x+\alpha-tr(A)\big) = 0$. This implies that $(\beta-\alpha)\big(\beta+\alpha-tr(A)\big)=0$. As $\alpha-\beta \in U(R)$, we deduce that $tr(A)-\alpha=\beta$. Thus, $\chi(A)=(x-\alpha)(x-\beta)$. Set $h_0 = x-\alpha$ and $h_1 = x-\beta$. Then $h_0 \in \mathbb{S}_0$ and $h_1 \in \mathbb{S}_1$. As $(\beta-\alpha)^{-1}h_0 - (\beta-\alpha)^{-1}h_1 = 1$, we see that $(h_0, h_1)=1$, which completes the proof by Corollary 2.1.6. □

Example 2.1.8. Let $R = \mathbb{Z}_{(3)} = \{\frac{m}{n} \mid m,n \in \mathbb{Z}, 3 \nmid n\}$. Then R is a commutative local ring with the Jacobson radical $J(R) = 3R$. Let $h = \big((t-1)(t^2+1)+9\big)\big(t(t^2+1)+9\big) \in R[t]$. As $h(0), h(1) \in J(R)$, h has no any trivial factorizations. In addition, $h \in \mathbb{Q}[t]$ has no any rational roots, and so h has no roots in R. This means that $h \in R[t]$ has no factors of degree 1. Assume that $h = h_0h_1, h_0 \in \mathbb{S}_0, h_1 \in \mathbb{S}_1$ and $(h_0, h_1) \equiv 1$. Then $deg(h_0), deg(h_1) \geq 2$. Clearly, $R/J(R)$ is a related field of R. But $\overline{h_0} = (t-1)(t^2+1)$ and $\overline{h_1} = t(t^2+1)$. As $gcd\big(\overline{h_0}, \overline{h_1}\big) = t^2+1$, a contradiction. By virtue of Theorem 2.1.5, $C_h \in M_6(R)$ is not strongly clean. But $h_0 = g_0g_1$, where $g_0 = (t-1)(t^2+1)+9 \in \mathbb{S}_0$ and $g_1 = 1$. In view of Theorem 2.1.5, C_{h_0} is strongly clean. Likewise, C_{h_1} is strongly clean. Therefore $\begin{pmatrix} C_{h_0} & \\ & C_{h_1} \end{pmatrix} \in M_6(R)$ is strongly clean.

2.2 Strongly Clean $n \times n$ Matrices

In this section we develop necessary and sufficient conditions under which an $n \times n$ matrix over commutative rings is strongly clean. Details may be found in papers and theses covering these topics, such as Dosey [241], Diesl and Dorsey [236], Lin [250] and Yang [546].

Lemma 2.2.1. *Let R be a ring, and let M be a left R-module. Suppose that $x, y, a, b \in End(M)$ such that $xa + yb = 1_M, xy = yx = 0, ay = ya$ and $xb = bx$. Then $M = Ker(x) \oplus Ker(y)$ as left R-modules.*

Proof. Given any $m \in Ker(x) \cap Ker(y)$, then $(m)x = (m)y = 0$, and so $m = (m)\big(xa + yb\big) = 0$. Hence, $Ker(x) \oplus Ker(y) \subseteq M$. For any $m \in M$, we see that $m = (m)yb + (m)xa$. As $xy = yx = 0, ay = ya$ and $xb = bx$, it is easy to check that $(m)yb \in Ker(x)$ and $(m)xa \in Ker(y)$. Therefore $M \subseteq Ker(x) \oplus Ker(y)$, as required. □

Lemma 2.2.2. *Let M be a module over a commutative ring R, and let $\varphi \in End(M)$. If there exists a polynomial $h \in \mathbb{S}_r$ such that $h(\varphi) = 0$, then $\varphi - l_r \in Aut(M)$.*

Proof. Write $h(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$. As $h \in \mathbb{S}_r$, we see that $h(r) \in U(R)$. Since $\varphi^n + a_{n-1}\varphi^{n-1} + \cdots + a_1\varphi + a_0 \cdot 1_M = h(\varphi) = 0$, we have

$$(\varphi - l_r + l_r)^n + a_{n-1}(\varphi - l_r + l_r)^{n-1} + \cdots + a_1(\varphi - l_r + l_r) + a_0 \cdot 1_M = 0,$$

and so $(\varphi - l_r)g(\varphi, r) + l_{h(r)} = 0$. This shows that $(\varphi - l_r)g(\varphi, r)l_{(h(r))^{-1}} = 1$. Likewise, $l_{(h(r))^{-1}}g(\varphi, r)(\varphi - l_r) = 1$. Therefore $\varphi - l_r \in Aut(M)$. □

Theorem 2.2.3. *Let R be a commutative ring, let $\varphi \in M_n(R)$, and let $h \in R[t]$ be a monic polynomial of degree n. If $h(\varphi) = 0$ and there exists a factorization $h = h_0h_1$ and idempotents $e_0, e_1 \in R$ such that $h_0 \in \mathbb{S}_{e_0}, h_1 \in \mathbb{S}_{e_1}$ and $(h_0, h_1) = 1$, then φ is strongly clean.*

Proof. Suppose that $h(\varphi) = 0$ and there exists a factorization $h = h_0h_1$ such that $h_0 \in \mathbb{S}_{e_0}, h_1 \in \mathbb{S}_{e_1}$ and $(h_0, h_1) = 1$. Then there exist $u_0, u_1 \in R[t]$ such that $u_0h_0 + u_1h_1 = 1$, and so $u_0(\varphi)h_0(\varphi) + u_1(\varphi)h_1(\varphi) = 1_{nR}$. By hypothesis, $h(\varphi) = h_0(\varphi)h_1(\varphi) = h_1(\varphi)h_0(\varphi) = 0$. Clearly, $u_0(\varphi)h_1(\varphi) = h_1(\varphi)u_0(\varphi)$ and $h_0(\varphi)u_1(\varphi) = u_1(\varphi)h_0(\varphi)$. In light of Lemma 2.2.1, $nR = Ker\big(h_0(\varphi)\big) \oplus Ker\big(h_1(\varphi)\big)$. As $h_0t = th_0$ and $h_1t = th_1$, we see that $Ker\big(h_0(\varphi)\big)$ and $Ker\big(h_1(\varphi)\big)$ are both φ-invariant. Further, $h_0(e_0) \in U(R)$ and $h_0\big(\varphi \mid_{Ker(h_0(\varphi))}\big) = 0$. It follows from Lemma 2.2.2 that $\varphi - l_{e_0} \mid_{Ker(h_0(\varphi))} \in Aut\big(Ker(h_0(\varphi))\big)$. Likewise, $\varphi - l_{e_1} \mid_{Ker(h_1(\varphi))} \in Aut\big(Ker(h_1(\varphi))\big)$. Choose $\pi = \big(l_{e_0} \mid_{Ker(h_0(\varphi))}, l_{e_1} \mid_{Ker(h_1(\varphi))}\big) \in End(nR)$. Then $\pi = \pi^2$. Furthermore, $\varphi\pi = \pi\varphi$, and that

$$\varphi - \pi = \big((\varphi - l_{e_0}) \mid_{Ker(h_0(\varphi))}, (\varphi - l_{e_1}) \mid_{Ker(h_1(\varphi))}\big) \in Aut(M).$$

Therefore the proof is complete. □

Corollary 2.2.4. *Let R be a commutative ring, let $\varphi \in M_n(R)$, and let $h \in R[t]$ be a monic polynomial of degree n. If $h(\varphi) = 0$ and there exists a*

factorization $h = h_0h_1$ such that $h_0 \in \mathbb{S}_0, h_1 \in \mathbb{S}_1$ and $(h_0, h_1) = 1$, then φ is strongly clean.

Proof. It is clear from Theorem 2.2.3. □

Corollary 2.2.5. *Let R be a commutative ring, let $\varphi \in M_n(R)$, and let $h \in R[t]$ be a monic polynomial of degree n. If $h(\varphi) = 0$ and $h(e) \in U(R)$ for an idempotent $e \in R$, then φ is strongly clean.*

Proof. As $h = h_0h_1$ with $h_0 = h$ and $h_1 = 1$, we complete the proof by Theorem 2.2.3. □

Corollary 2.2.6. *Let R be a commutative ring, let $A \in M_2(R)$, and let $f(t) = (t-\alpha)(t-\beta) \in R[t]$, where α, β in $J(R)$ or $1 + J(R)$. If $f(A) = 0$, then $A \in M_2(R)$ is strongly clean.*

Proof. Case I. $\alpha, \beta \in 1 + J(R)$. Clearly, $f(t) = t^2 - (\alpha + \beta)t + \alpha\beta$. Set $h_0 = t^2 - (\alpha + \beta)t + \alpha\beta$ and $h_1 = 1$. Then h has a trivial factorization $h = h_0h_1$ with $h_0 \in \mathbb{S}_0, h_1 \in \mathbb{S}_1$ and $(h_0, h_1) = 1$. In view of Theorem 2.2.3, $A \in M_2(R)$ is strongly clean.

Case II. $\alpha, \beta \in J(R)$. Then $g(t) := f(1-t) = \big((1-\alpha)-t\big)\big((1-\beta)-t\big)$. As $g(I_2 - A) = f(A) = 0$, it follows from Case I that $I_2 - A$ is strongly clean. Therefore $A \in M_2(R)$ is strongly clean.

Case III. $\alpha \in 1 + J(R), \beta \in J(R)$. Set $h_0 = t - \alpha$ and $h_1 = t - \beta$. Then $f(t) = h_0h_1$ with $h_0 \in \mathbb{S}_0$ and $h_1 \in \mathbb{S}_1$. Clearly, $(\beta-\alpha)^{-1}h_0-(\beta-\alpha)^{-1}h_1 = 1$, and so $(h_0, h_1) = 1$. In view of Theorem 2.2.3, $A \in M_2(R)$ is strongly clean.

Case IV. $\alpha \in J(R), \beta \in 1+J(R)$. Set $u = 1-t$. Then $f(t) = f(1-u) := g(u)$. Then $g(u) = \big(u-(1-\alpha)\big)\big(u-(1-\beta)\big)$. Further, $g\big(I_2-A\big) = f(A) = 0$. By the discussion in the Case III, $I_2 - A \in M_2(R)$ is strongly clean, and then so is $A \in M_2(R)$.

Consequently, $A \in M_2(R)$ is strongly clean. □

Corollary 2.2.7. *Let R be a commutative ring, and let $f(t) = (t-\alpha)(t-\beta)(t-\gamma) \in R[t]$, where α, β, γ in $J(R)$ or $1 + J(R)$. If $f(A) = 0$, then $A \in M_3(R)$ is strongly clean.*

Proof. Case I. $\alpha, \beta, \gamma \in 1 + J(R)$. Clearly, $f(t) = t^3 - (\alpha + \beta + \gamma)t^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)t - \alpha\beta\gamma$. Let $h_0 = f(t)$ and $h_1 = 1$. Then h has a trivial factorization h_0h_1 with $h_0 \in \mathbb{S}_0, h_1 \in \mathbb{S}_1$ and $(h_0, h_1) = 1$. By virtue of Theorem 2.2.3, $A \in M_3(R)$ is strongly clean.

Case II. $\alpha, \beta, \gamma \in J(R)$. Then $g(t) := f(1-t) = \big((1-\alpha)-t\big)\big((1-\beta)-t\big)\big((1-\gamma)-t\big)$. Then $g(I_3 - A) = f(A) = 0$. As in the proof of Case I, $I_3 - A$ is strongly clean, and then so is $A \in M_3(R)$.

Case III. $\alpha \in 1 + J(R), \beta, \gamma \in J(R)$. Choose $h_0 = t - \alpha$ and $h_1 = (t-\alpha)(t-\beta)$. Then $h_0 \in \mathbb{S}_0$ and $h_1 \in \mathbb{S}_1$. It is easy to verify that

$$\begin{aligned} &\big(t - (\beta+\gamma-\alpha)\big)(t-\alpha) - (t-\beta)(t-\gamma) \\ &= (\beta+\gamma-\alpha)\alpha - \beta\gamma \\ &\in U(R), \end{aligned}$$

and so $(h_0, h_1) = 1$. In light of Theorem 2.2.3, $A \in M_3(R)$ is strongly clean.

Case IV. $\alpha, \beta \in 1 + J(R), \gamma \in J(R)$. Then $g(t) = f(1-t) = \big((1-\gamma)-t\big)\big((1-\alpha)-t\big)\big((1-\beta)-t\big)$. Obviously, $g(I_3 - A) = f(A) = 0$. As in the proof of Case III, $I_3 - A \in M_3(R)$ is strongly clean. This implies that $A \in M_3(R)$ is strongly clean.

Therefore $A \in M_3(R)$ is strongly clean. □

Let R be a commutative ring, and let $A \in M_n(R)$. The characteristic polynomial of A is the polynomial $\chi(A) = det(tI_n - A)$. If $A \in M_3(R)$ and $\chi(A) = (t-\alpha)(t-\beta)(t-\gamma) \in R[t]$ where α, β, γ in $J(R)$ or $1 + J(R)$, then $A \in M_3(R)$ is strongly clean. In view of the Cayley-Hamilton Theorem, $\big(\chi(A)\big)(A) = 0$, and we are done by Corollary 2.2.7. We note that if $h(t) \in R[t]$ is the characteristic polynomial of $A \in M_n(R)$, then $f(A) = 0$. But the converse is not true, e.g., $A = \begin{pmatrix} \overline{2} & \overline{1} & \overline{1} \\ \overline{1} & \overline{2} & \overline{1} \\ \overline{1} & \overline{1} & \overline{2} \end{pmatrix} \in M_3(\mathbb{Z}_3), f(t) = t^3 + t^2 + t \in \mathbb{Z}_3[t]$.

The condition $(h_0, h_1) = 1$ in Theorem 2.2.3 can not be replaced by $gcd(h_0, h_1) = 1$ even for an integral domain as the following shows.

Example 2.2.8. Let $h = t^2 - t - 2 \in \mathbb{Z}[t]$. Then $h = h_0 h_1$ where $h_0 = t+1 \in \mathbb{S}_0$ and $h_1 = t - 2 \in \mathbb{S}_1$. Clearly, $s_{C_h} = -1$ and $t_{C_h} = 9$. If $C_h \in M_2(\mathbb{Z})$ is strongly clean, as in the proof of Lemma 1.3.2, the equation $t_{C_h} x^2 = s_{C_h}^2$ is solvable. This gives a contradiction. Thus, $C_h = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} \in M_2\big(\mathbb{Z}\big)$ is not strongly clean. In this case, $gcd(h_0, h_1) = 1$ in $\mathbb{Z}[t]$.

We have at our disposal all the information necessary to the demonstration of the main result in this section.

Theorem 2.2.9. *Let R be a commutative ring, and let $h \in R[t]$ be a monic polynomial of degree n. Then the following are equivalent:*

(1) *Every $\varphi \in M_n(R)$ with $\chi(\varphi) = h$ is strongly clean.*
(2) *There exists a factorization $h = h_0h_1$ such that $h_0 \in \mathbb{S}_0, h_1 \in \mathbb{S}_1$ and $(h_0, h_1) = 1$.*

Proof. (1) $\Rightarrow$ (2) Write $h = t^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0 \in R[t]$. Then

$$C_h = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix} \in M_n(R).$$

Clearly, $\chi(C_h) = h$. Thus, proving (2), in terms of Theorem 2.1.5.

(2) $\Rightarrow$ (1) For every $\varphi \in M_n(R)$ with $\chi(\varphi) = h$, it follows by the Cayley-Hamilton Theorem that $h(A) = 0$. Therefore φ is strongly clean from Corollary 2.2.4. □

Example 2.2.10. Let $A = \begin{pmatrix} 2 & 3 \\ 6 & 4 \end{pmatrix} \in M_2(\mathbb{Z})$. Then $\chi(A) = t^2 - 6t - 10$. It is easy to verify that there are no any $h_0 \in \mathbb{S}_0$ and $h_1 \in \mathbb{S}_1$ such that $\chi(A) = h_0h_1$. Accordingly, $A \in M_2(\mathbb{Z})$ is not strongly clean by Theorem 2.2.9.

Lemma 2.2.11. *Let R be a commutative ring, let M be a finitely generated R-module, and let $\sigma : M \to M$ be an R-morphism. Then there exist $a_0, a_1, \ldots, a_{n-1} \in R$ such that $\sigma^n + a_{n-1}\sigma^{n-1} + \cdots + a_1\sigma + a_0 = 0$.*

Proof. Suppose $M = x_1R + \cdots + x_sR$. Then we have some $a_{ij} \in R (1 \leq i, j \leq s)$ such that $\sigma(x_i) = \sum_{j=1}^{s} a_{ij}x_j$. Let $S = \mathbb{Z}[a_{11}, \ldots, a_{1s}, \ldots, a_{s1}, \ldots, a_{ss}] = \{f(a_{11}, \ldots, a_{1s}, \ldots, a_{s1}, \ldots, a_{ss}) \mid f(x_{11}, \ldots, x_{1s}, \ldots, x_{s1}, \ldots, x_{ss}) \in \mathbb{Z}[x_{11}, \ldots, x_{1s}, \ldots, x_{s1}, \ldots, x_{ss}]\}$. Let

$$\varphi : \mathbb{Z}[x_{11}, \ldots, x_{1s}, \ldots, x_{s1}, \ldots, x_{ss}] \to S,$$
$$f(x_{11}, \ldots, x_{1s}, \ldots, x_{s1}, \ldots, x_{ss}) \mapsto f(a_{11}, \ldots, a_{1s}, \ldots, a_{s1}, \ldots, a_{ss}).$$

Clearly, φ is surjective; hence,

$$S \cong \mathbb{Z}[x_{11}, \ldots, x_{1s}, \ldots, x_{s1}, \ldots, x_{ss}]/ker(\varphi).$$

In light of Hilbert's Basis Theorem, $\mathbb{Z}[x_{11}, \ldots, x_{1s}, \ldots, x_{s1}, \ldots, x_{ss}]$ is a noetherian ring. Thus, S is a noetherian subring of R. Choose $N = x_1S + \cdots + x_sS$. Then N is an S-module with $\sigma(N) \subseteq N$. Choose

$\tau = \sigma\mid_N : N \to N$. Then N is a finitely generated S-module, and that τ an S-endomorphism. Let $\psi : S^s \to N, (c_1, \ldots, c_s) \mapsto \sum_{i=1}^{s} c_i x_i$. Then ψ is an S-epimorphism. Furthermore, we have an S-monomorphism $\varphi^* : K \to Hom_S(S^s, N) \cong N^s$, where $K := Hom_S(N, N)$. Hence, K is a finitely generated S-module. Let $S_k = \sum_{i=0}^{k} S\tau^i, k = 0, 1, 2, \ldots$. Then we have $S_0 \subseteq S_1 \subseteq S_2 \subseteq \cdots \subseteq K$. Set $I = \bigcup_{i=0}^{\infty} S_i$. Then $I \subseteq K$; hence, I is finitely generated. Say $\{s_1, \ldots, s_t\}$ is the generating set of I. Then we have $S_{j_{s_i}}$ such that $s_i \in S_{j_{s_i}}$, and so $I = \bigcup_{i=1}^{t} S_{j_{s_i}}$. Choose $m = max\{j_{s_1}, \ldots, j_{s_t}\} + 1$. It is easy to verify that $S_{m-1} = S_m = \cdots$. Thus, $\tau^m \in S_m = S_{m-1}$. Hence, we have $b_0, b_1, \ldots, b_{m-1} \in S$ such that $\tau^m + b_{m-1}\tau^{m-1} + \cdots + b_1\tau + b_0 = 0$. Obviously, $b_{m-1}, \ldots, b_0 \in R$. Observing that $\big(\sigma^m + b_{m-1}\sigma^{m-1} + \cdots + b_1\sigma + b_0\big)^2(M) \subseteq \big(\sigma^m + b_{m-1}\sigma^{m-1} + \cdots + b_1\sigma + b_0\big)(NR) = \big(\tau^m + b_{m-1}\tau^{m-1} + \cdots + b_1\tau + b_0\big)(NR) = 0$, we get $\big(\sigma^m + b_{m-1}\sigma^{m-1} + \cdots + b_1\sigma + b_0\big)^2 = 0$. Accordingly, there are some $a_0, a_1, \ldots, a_{n-1} \in R$ such that $\sigma^n + a_{n-1}\sigma^{n-1} + \cdots + a_1\sigma + a_0 = 0$. □

The following lemma is known as the Generalized Nakayama's Lemma.

Lemma 2.2.12. *Let M be a finitely generated module over a commutative ring R. If A is an ideal of R with $M = AM$, then $R = A + ann_R(M)$, where $ann_R(M) = \{r \in R \mid Mr = 0\}$.*

Proof. Let $\sigma : M \to M, m \mapsto m$. In view of Lemma 2.2.11, there exist $a_0, a_1, \ldots, a_{n-1} \in R$ such that $\sigma^n + a_{n-1}\sigma^{n-1} + \cdots + a_1\sigma + a_0 = 0$. Thus, $\big(1 + a_{n-1} + \cdots + a_1 + a_0\big)(M) = 0$. As $M = AM$, by the proof of Lemma 2.2.11, we may assume that $a_0, a_1, \ldots, a_{n-1} \in A$. Hence, $1 = \big(1 + a_{n-1} + \cdots + a_1 + a_0\big) - \big(a_{n-1} + \cdots + a_1 + a_0\big) \in ann_R(M) + A$, and therefore $R = A + ann_R(M)$. □

A commutative local ring R is a Hensel ring provided that if $f \in R[t]$ is monic and satisfies $\overline{f} = GH$ with $G, H \in \big(R/J(R)\big)[t]$ monic and $(G, H) = \overline{1}$, then $f = gh$ for some monic $f, g \in R[t]$ with $\overline{g} = G$ and $\overline{h} = H$. Every field is a Hensel local ring. Complete Hausdorff local rings, such as the ring of p-adic integers and rings of formal power series over a field, are Hensel local rings

Theorem 2.2.13. *Let R be a Hensel local ring. Then $M_n(R)$ is strongly clean for all $n \in \mathbb{N}$.*

Proof. As R is a commutative local ring, $R/J(R)$ is a field, and so $M_n\big(R/J(R)\big)$ is artinian. Hence, $M_n\big(R/J(R)\big)$ is strongly π-regular. In view of Theorem 1.1.2, $M_n\big(R/J(R)\big)$ is strongly clean. Let $A \in M_n(R)$. According to Theorem 2.2.9, $\overline{\chi(A)} = \chi\big(\overline{A}\big) = GH$ with $G, H \in \big(R/J(R)\big)[t]$ monic, $G \in \mathbb{S}_0, H \in \mathbb{S}_1$ and $(G, H) = \overline{1}$. By hypothesis, $\chi(A) = gh$ for some monic $g, h \in R[t]$ with $\overline{g} = G$ and $\overline{h} = H$. Hence, $(g, h) + J(R)R[t] = R[t]$, and so $J(R)\big(R[t]/(g, h)\big) = \big(R[t]/(g, h)\big)$. Clearly, $R[t]/(g, h)$ is a finitely generated R-module. By virtue of Lemma 2.2.12, we get $(g, h) = R[t]$. Since every unit lifts modulo $J(R)$, one easily checks that $g \in \mathbb{S}_0, h \in \mathbb{S}_1$. By using Theorem 2.2.9 again, A is strongly clean, as desired. □

Let p be a prime. The ring $\widehat{\mathbb{Z}_p}$ of p-adic integers is described as the ring of numbers of the form $\sum\limits_{i=0}^{\infty} a_i p^i$, where $0 \leq a_i \leq p-1$. The addition is

$$\sum_{i=0}^{\infty} a_i p^i + \sum_{i=0}^{\infty} b_i p^i = \sum_{i=0}^{\infty} c_i p^i,$$

where $a_0 + b_0 = c_0 + k_0 p, a_1 + b_1 + k_0 = c_1 + k_1 p, \ldots$. The multiplication is

$$\left(\sum_{i=0}^{\infty} a_i p^i\right)\left(\sum_{i=0}^{\infty} b_i p^i\right) = \sum_{i=0}^{\infty} c_i p^i,$$

where $a_0 b_0 = c_0 + k_0 p, a_0 b_1 + a_1 b_0 + k_0 = c_1 + k_1 p, \ldots$. Then R is a commutative local ring with Jacobson radical $J(R) = pR$.

Corollary 2.2.14. *Let p be a prime. Then $M_n\big(\widehat{\mathbb{Z}_p}\big)$ is strongly clean for all $n \in \mathbb{N}$.*

Proof. In view of [245, Theorem 7.18], $\widehat{\mathbb{Z}_p}$ is a Hensel local ring. Therefore we are through by Theorem 2.2.13. □

Let R be a commutative ring. Given polynomials $f(t) = t^m + a_1 t^{m-1} + \cdots + a_m, g(t) = b_0 t^n + b_1 t^{n-1} + \cdots + b_n (b_0 \neq 0) \in R[t]$, the resultant of f

and g is defined by the determinant of the $(m+n) \times (m+n)$ matrix

$$res(f,g) = \begin{vmatrix} 1 & a_1 & \cdots & a_m & & & & \\ & 1 & a_1 & \cdots & a_m & & & \\ & & 1 & a_1 & \cdots & a_m & & \\ & & & \ddots & \ddots & \ddots & \ddots & \\ & & & & 1 & a_1 & \cdots & a_m \\ b_0 & b_1 & \cdots & b_n & & & & \\ & b_0 & b_1 & \cdots & b_n & & & \\ & & b_0 & b_1 & \cdots & b_n & & \\ & & & \ddots & \ddots & \ddots & \ddots & \\ & & & & b_0 & b_1 & \cdots & b_n \end{vmatrix}$$

where blank spaces consist of zeros. The following is called the Weyl Principal. Let f, g be polynomials in $\mathbb{Z}[x_1, x_2, \ldots, x_n]$ with $g \neq 0$ (i.e., g is not the zero polynomial). If for all $(\alpha_1, \ldots, \alpha_n) \in \mathbb{Q}^n$, $f(\alpha_1, \ldots, \alpha_n) = 0$ whenever $g(\alpha_1, \ldots, \alpha_n) \neq 0$, then $f = 0$ in any commutative ring R.

Lemma 2.2.15. *Let R be a commutative ring, and let $f \in R[t]$ be monic and $g, h \in R[t]$. Then the following are equivalent:*

(1) $res(f, g) = res(f, g + fh)$.
(2) $res(f, gh) = res(f, g)res(f, h)$.

Proof. (1) Write $h = c_0 t^s + \cdots + c_s \in R[t]$. It will suffice to show that $res(f, g) = res(f, g + c_{s-i} t^i f)$. Since any determinant in which every entry in a row is the sum of two elements is the sum of two corresponding determinants, the result follows.

(2) Write $f = t^m + a_1 t^{m-1} + \cdots + a_m, g = b_0 t^n + b_1 t^{n-1} + \cdots + b_n, h = c_0 t^s + c_1 t^{s-1} + \cdots + c_s$. Then

$$\begin{aligned} \alpha(a_1, \ldots, a_m; b_0, \ldots, b_n; c_0, \ldots, c_s) &:= res(f, gh) - res(f, g)res(f, h) \\ &\in \mathbb{Z}[a_1, \ldots, a_m; b_0, \ldots, b_n; c_0, \ldots, c_s]. \end{aligned}$$

Consider $\alpha(x_1, \ldots, x_m; y_0, \ldots, y_n; z_0, \ldots, z_s) \in \mathbb{Z}[x_1, \ldots, x_m; y_0, \ldots, y_n; z_0, \ldots, z_s]$. Clearly, the result holds if $R = \mathbb{Q}$. For any $u_1, \ldots, u_m; v_0, \ldots, v_n; w_0, \ldots, w_s \in \mathbb{Q}$, we see that

$$\alpha(u_1, \ldots, u_m; v_0, \ldots, v_n; w_0, \ldots, w_s) = 0.$$

Because of the Weyl Principal, $\alpha(x_1, \ldots, x_m; y_0, \ldots, y_n; z_0, \ldots, z_s) = 0$ in $\mathbb{Z}[x_1, \ldots, x_m; y_0, \ldots, y_n; z_0, \ldots, z_s]$. Therefore $\alpha(a_1, \ldots, a_m; b_0, \ldots, b_n; c_0, \ldots, c_s) = 0$, and so $res(f, gh) = res(f, g)res(f, h)$. □

Lemma 2.2.16. *Let R be a commutative ring, and let $f, g \in R[t]$ be monic. Then the following are equivalent:*

(1) $(f, g) = 1$.
(2) $res(f, g) \in U(R)$.

Proof. (1) $\Rightarrow$ (2) As $(f, g) = 1$, we can find some $u, v \in R[t]$ such that $uf + vg = 1$. By virtue of Lemma 2.2.15, one easily checks that $res(f, vg) = res(f, v)res(f, g) = res(f, vg + uf) = res(f, 1) = 1$. Accordingly, $res(f, g) \in U(R)$.

(2) $\Rightarrow$ (1) Let $m = deg(f)$ and $n = deg(g)$. Observing that

$$res(f,g)\begin{vmatrix} & & t^{m+n} \\ I_{m+n-1} & & \vdots \\ & & t \\ 0 \cdots & 0 & 1 \end{vmatrix} = \begin{vmatrix} t^n f \\ \vdots \\ f \\ * \quad t^m g \\ \vdots \\ g \end{vmatrix},$$

we can find some $u, v \in R[t]$ such that $res(f, g) = uf + vg$. Hence $\big(res(f,g)\big)^{-1}uf + \big(res(f,g)\big)^{-1}vg = 1$, as asserted. □

Theorem 2.2.17. *Let R be a commutative ring, and let $n \in \mathbb{N}$. Then the following are equivalent:*

(1) $M_n(R)$ *is strongly clean.*
(2) $M_n\big(R[[x]]\big)$ *is strongly clean.*

Proof. (1) $\Rightarrow$ (2) Let $H(x,t) = \chi\big(A(x)\big) \in R[[x]][t]$. Then $H(0,t) = \chi\big(A(0)\big) \in R[t]$. In view of Theorem 2.2.9, $H(0,t) = h_0h_1$, where $h_0 = t^m + \alpha_1 t^{m-1} + \cdots + \alpha_m \in \mathbb{S}_0$, $h_1 = t^s + \beta_1 t^{s-1} + \cdots + \beta_s \in \mathbb{S}_1$ and $(h_0, h_1) = 1$. Next, we will find a factorization $H(x,t) = H_0H_1$ where $H_0(x,t) = t^m + \sum_{i=0}^{m-1} A_i(x)t^i \in \mathbb{S}_0$ and $H_1(x,t) = t^s + \sum_{i=0}^{s-1} B_i(x)t^i \in \mathbb{S}_1$. Choose $H_0(0,t) \equiv h_0$ and $H_1(0,t) \equiv h_1$. Write $H(x,t) = \sum_{i=0}^{n}\Big(\sum_{j=0}^{\infty} c_{ij}x^j\Big)t^i$. Then

$$\begin{aligned} H(x,t) &= \sum_{j=0}^{\infty}\left(\sum_{i=0}^{n} c_{ij}t^i\right)x^j \\ &= H(0,t) + \sum_{j=1}^{\infty}\left(\sum_{i=0}^{n} c_{ij}t^i\right)x^j. \end{aligned}$$

Write $A_i(x) = \sum\limits_{j=0}^{\infty} a_{ij}x^j$ and $B_i(x) = \sum\limits_{j=0}^{\infty} b_{ij}x^j$. Then

$$\begin{aligned} H_0 &= t^m + \sum_{i=0}^{m-1} \left(\sum_{j=0}^{\infty} a_{ij}x^j \right) t^i \\ &= t^m + \sum_{j=0}^{\infty} \left(\sum_{i=0}^{m-1} a_{ij}t^i \right) x^j \\ &= h_0 + \sum_{j=1}^{\infty} \left(\sum_{i=0}^{m-1} a_{ij}t^i \right) x^j. \end{aligned}$$

Likewise,

$$H_1 = h_1 + \sum_{j=1}^{\infty} \left(\sum_{i=0}^{s-1} b_{ij}t^i \right) x^j.$$

Write $H_0H_1 = h_0h_1 + \sum\limits_{j=1}^{\infty} z_jx^j$. Thus, we should have

$$\begin{aligned} z_1 &= h_0 \left(\sum_{i=0}^{s-1} b_{i1}t^i \right) + \left(\sum_{i=0}^{m-1} a_{i1}t^i \right) h_1 \\ &= \sum_{i=0}^{n-1} c_{i1}t^i. \end{aligned}$$

This implies that

$$\big(b_{(s-1)1}, \ldots, b_{01}, a_{(m-1)1}, \ldots, a_{01}\big)A = \big(c_{(n-1)1}, \ldots, c_{01}\big),$$

where $A = \begin{pmatrix} 1 & \alpha_1 & \cdots & \alpha_m & & & \\ & 1 & \alpha_1 & \cdots & \alpha_m & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & \alpha_1 & \cdots & \alpha_m \\ 1 & \beta_1 & \cdots & \beta_s & & & \\ & 1 & \beta_1 & \cdots & \beta_s & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & \beta_1 & \cdots & \beta_s \end{pmatrix}$. As $(h_0, h_1) = 1$, it follows from Lemma 2.2.16 that $res(h_0, h_1) \in U(R)$. That is, $det(A) \in U(R)$, and so we can find $a_{i1}, b_{j1} \in R$.

$$\begin{aligned} z_2 &= h_0 \left(\sum_{i=0}^{s-1} b_{i2}t^i \right) + \left(\sum_{i=0}^{m-1} a_{i1}t^i \right) \left(\sum_{i=0}^{s-1} b_{i1}t^i \right) + \left(\sum_{i=0}^{m-1} a_{i2}t^i \right) h_1 \\ &= \sum_{i=0}^{n-1} c_{i2}t^i. \end{aligned}$$

Hence

$$h_0\left(\sum_{i=0}^{s-1} b_{i2}t^i\right)+\left(\sum_{i=0}^{m-1} a_{i2}t^i\right)h_1=\sum_{i=0}^{n-1} c_{i2}t^i-\left(\sum_{i=0}^{m-1} a_{i1}t^i\right)\left(\sum_{i=0}^{s-1} b_{i1}t^i\right)$$
$$=\sum_{i=0}^{n-1} d_{i2}t^i.$$

Thus,

$$\big(b_{(s-1)2},\ldots,b_{02},a_{(m-1)2},\ldots,a_{02}\big)A=\big(d_{(n-1)2},\ldots,d_{02}\big),$$

whence we can find $a_{i2},b_{j2}\in R$. By iteration of this process, we can find $a_{ij},b_{ij}\in R, j=3,4,\ldots$. Therefore we have H_0 and H_1 such that $H(x,t)=H_0H_1$. Further, $H_0(x,0)=H_0(0,0)+xf(x)=h_0(0)+xf(x)\in U\big(R[[x]]\big)$ and $H_1(x,1)=H_1(0,1)+xg(x)=h_1(1)+xg(x)\in U\big(R[[x]]\big)$. Thus, $H_0(x,t)\in\mathbb{S}_0$ and $H_1(x,t)\in\mathbb{S}_1$. As $(h_0,h_1)=1$, we get $\big(H_0,H_1\big)\equiv 1\big(mod\ (xR[[x]])[t]\big)$, and so $\big(H_0,H_1\big)+J\big(R[[x]]\big)R[[[x]][t]=R[[x]][t]$. Set $M=R[[x]][t]/\big(H_0,H_1\big)$. Then M is a finitely generated $R[[x]]$-module, and that $J\big(R[[x]]\big)M=M$. By Lemma 2.2.12, $M=0$, and so $\big(H_0,H_1\big)=1$. Accordingly, $A(x)\in M_n\big(R[[x]]\big)$ is strongly clean by Theorem 2.2.9.

(2) $\Rightarrow$ (1) This is obvious. □

Corollary 2.2.18. *Let R be a commutative ring, and let $m,n\in\mathbb{N}$. Then the following are equivalent:*

(1) *$M_n(R)$ is strongly clean.*
(2) *$M_n\big(R[x]/(x^m)\big)$ is strongly clean.*

Proof. (1) $\Rightarrow$ (2) Write $A(x)\in M_n\big(R[x]/(x^m)\big)$. Then $A(x)\in M_n\big(R[[x]]\big)$. In view of Theorem 2.2.17, there exist $E^2=E=\left(\sum_{k=0}^{\infty} e_k^{ij}x^k\right), U=\left(\sum_{k=0}^{\infty} u_k^{ij}x^k\right)\in GL_n\big(R[[x]]\big)$ such that $A(x)=E+U$ and $EU=UE$. As $R[[x]]/(x^m)\cong R[x]/(x^m)$, we see that $A(x)=\overline{E}+\overline{U}$ and $\overline{EU}=\overline{UE}$, where $\overline{E}^2=\overline{E}=\left(\sum_{k=0}^{m-1} e_k^{ij}x^k\right)\in M_n\big(R[x]/(x^m)\big)$ and $\overline{U}=\left(\sum_{k=0}^{m-1} u_k^{ij}x^k\right)\in GL_n\big(R[[x]]/(x^m)\big)$, as desired.

(2) $\Rightarrow$ (1) This is clear. □

Let R be a commutative ring, by induction, we easily prove that $M_n\big(R[x_1,\ldots,x_m]/(x_1^{s_1},\ldots,x_m^{s_m})\big)(s_1,\ldots,s_m\in\mathbb{N})$ is strongly clean if and only if so is $M_n(R)$.

Let R be a ring and G be a group. The group ring RG is the set $\Big\{ \sum\limits_{g\in G} r_g g \mid r_g \in R, \text{ where at most finitely many } r_g \neq 0 \Big\}$, the additive operation and multiplicative operation are defined as follow:

$$\sum_{g\in G} r_g g + \sum_{g\in G} s_g g = \sum_{g\in G} (r_g + s_g)g;$$
$$\left(\sum_{g\in G} r_g g\right)\left(\sum_{g\in G} s_g g\right) = \sum_{g\in G} t_g g;$$
$$t_g = \sum_{x,y\in G,\, xy=g} r_x s_y.$$

The zero is $0 \cdot e$ and the identity is $1_R \cdot e$, where e is the identity of G. Next, we include an elementary result which makes our discussion from $R[x]/\big(x^2-1\big)$ to a kind of group rings.

Lemma 2.2.19. *Let R be a commutative ring, and let $G = \{1, g\}$ be a group. Then $R[x]/(x^2-1) \cong RG$.*

Proof. Let $G = \{1, g\}$ be a group. Construct a map $\varphi : RG \to R[x]/(x^2-1)$ given by $\varphi(a+bg) = \overline{a+bx}$ for any $a + bg \in RG$. It is easy to verify that φ is a ring surjective morphism. If $\varphi(a+bg) = 0$, then $a + bx \in (x^2-1)$. Write $a + bx = (a_0 + a_1x + \cdots + a_nx^n)(x^2-1)$ for some $a_0, a_1, \ldots, a_n \in R$. Then we get

$$-a_0 = a, -a_1 = b;$$
$$a_0 - a_2 = a_1 - a_3 = \cdots = a_{n-2} - a_n = 0;$$
$$a_{n-1} = a_n = 0.$$

Thus, $a + bg = 0$, i.e., φ is injective. This completes the proof. □

Lemma 2.2.20. *Let R be a commutative ring with $char(R) = 2$, and let $G = \{1, g\}$ be a group. Then $a + bg \in U(RG)$ if and only if $a + b \in U(R)$.*

Proof. Clearly, $(a+bg)(a-bg) = a^2-b^2 = (a+b)(a-b)$. Then $(a+bg)^2 = (a+b)^2$, as $char(R) = 2$. If $a + bg \in U(RG)$, then $(a + bg)(x + yg) = 1$ for some $x, y \in R$. Hence, $(a + bg)^2(x + yg)^2 = 1$, and so $(a + b)^2(x + y)^2 = 1$. This shows that $x + y \in U(R)$. The converse is similar. □

Let $A(x) = \big(\overline{a_{ij}(x)}\big) \in M_n\big(R[x]/(x^2-1)\big)$ where $deg\big(a_{ij}(x)\big) \leq 1$, and let $r \in R$. We use $A(r)$ to stand for the matrix $\big(a_{ij}(r)\big) \in M_n(R)$.

Theorem 2.2.21. *Let R be a commutative ring with $char(R) = 2$, and let $n \in \mathbb{N}$. Then the following are equivalent:*

(1) $M_n(R)$ *is strongly clean.*
(2) $M_n\big(R[[x]]/(x^2-1)\big)$ *is strongly clean.*

Proof. (1) $\Rightarrow$ (2) In view of Lemma 2.2.19, $R[[x]]/(x^2-1) \cong RG$, where $G = \{1, g\}$. Let $A(g) \in M_n(RG)$, and let $H(g,t) = \chi\big(A(g)\big) \in (RG)[t]$. Then $H(1,t) = \chi\big(A(1)\big) \in R[t]$. In light of Theorem 2.2.9, $H(1,t) = h_0h_1$, where $h_0 = t^m + \alpha_{m-1}t^{m-1} + \cdots + \alpha_0 \in \mathbb{S}_0$, $h_1 = t^s + \beta_{s-1}t^{s-1} + \cdots + \beta_0 \in \mathbb{S}_1$ and $(h_0, h_1) = 1$. We shall find a factorization $H(g,t) = H_0H_1$ where $H_0(g,t) = t^m + \sum\limits_{i=0}^{m-1}\big(y_i + (\alpha_i - y_i)g\big)t^i \in \mathbb{S}_0$ and $H_1(g,t) = t^s + \sum\limits_{i=0}^{s-1}\big(z_i + (\beta_i - z_i)g\big)t^i \in \mathbb{S}_1$. Clearly, $H_0(1,t) \equiv h_0$ and $H_1(1,t) \equiv h_1$. We will suffice to find $y_i's$ and $z_i's$. Write $H(g,t) = \sum\limits_{i=0}^{n}\big(r_i + s_ig\big)t^i$. The equality $H(g,t) = H_0H_1$ is equivalent to

$$
\begin{aligned}
&t^n + \sum_{i=0}^{n-1} r_it^i \\
&= \left(t^m + \sum_{i=0}^{m-1} y_it^i\right)\left(t^s + \sum_{i=0}^{s-1} z_it^i\right) + \left(\sum_{i=0}^{m-1}(\alpha_i - y_i)t^i\right)\left(\sum_{i=0}^{s-1}(\beta_i - z_i)t^i\right) \quad (*) \\
&\sum_{i=0}^{n-1} s_it^i \\
&= \left(t^m + \sum_{i=0}^{m-1} y_it^i\right)\left(\sum_{i=0}^{s-1}(\beta_i - z_i)t^i\right) + \left(t^s + \sum_{i=0}^{s-1} z_it^i\right)\left(\sum_{i=0}^{m-1}(\alpha_i - y_i)t^i\right) (**).
\end{aligned}
$$

$(**)$ holds from $H(1,t) = h_0h_1 = H_0(1,t)H_1(1,t)$. $(*)$ is equivalent to

$$
\begin{gathered}
y_0z_0 + (\alpha_0 - y_0)(\beta_0 - z_0) = r_0, \\
y_0z_1 + y_1z_0 + (\alpha_0 - y_0)(\beta_1 - z_1) + (\alpha_1 - y_1)(\beta_0 - z_0) = r_1, \\
\vdots \\
y_{m-2} + y_{m-1}z_{s-1} + z_{s-2} + (\alpha_{m-1} - y_{m-1})(\beta_{s-1} - z_{s-1}) = r_{n-2}, \\
y_{m-1} + z_{s-1} = r_{n-1}.
\end{gathered}
$$

As $char(R) = 2$, we have

$$
\begin{gathered}
\beta_0y_0 + \alpha_0z_0 = r_0 + \alpha_0\beta_0, \\
\beta_0y_1 + \beta_1y_0 + \alpha_0z_1 + \alpha_1z_0 = r_1 + \alpha_0\beta_1 + \alpha_1\beta_0, \\
\vdots \\
\beta_{s-1}y_{m-1} + y_{m-2} + \alpha_{m-1}z_{s-1} + z_{s-2} = r_{n-2} + \alpha_{m-1}\beta_{s-1}, \\
y_{m-1} + z_{s-1} = r_{n-1}.
\end{gathered}
$$

This implies that

$$\big(y_{m-1}, \ldots, y_0, z_{s-1}, \ldots, z_0\big)A = \big(*, \ldots, *\big),$$

where

$$A = \begin{pmatrix} 1 & \beta_{m-1} & \cdots & \beta_0 & & & & \\ & 1 & \beta_{m-1} & \cdots & \beta_0 & & & \\ & & \ddots & \ddots & \ddots & \ddots & & \\ & & & 1 & \beta_{m-1} & \cdots & \beta_0 & \\ 1 & \alpha_{s-1} & \cdots & \alpha_0 & & & & \\ & 1 & \alpha_{s-1} & \cdots & \alpha_0 & & & \\ & & \ddots & \ddots & \ddots & \ddots & & \\ & & & 1 & \alpha_{s-1} & \cdots & \alpha_0 \end{pmatrix}.$$

As $(h_1, h_0) = 1$, it follows from Lemma 2.2.16 that $res(h_1, h_0) \in U(R)$. That is, $det(A) \in U(R)$, and so we can find $y_i, z_j \in R$ such that $(*)$ and $(**)$ hold. In other words, we have H_0 and H_1 such that $H(g,t) = H_0H_1$. Obviously, $H_0(g,0) = y_0 + (\alpha_0 - y_0)g$. As $y_0 + (\alpha_0 - y_0) = \alpha_0 = h_0(0) \in U(R)$, it follows by Lemma 2.2.20 that $H_0(g,0) \in U(RG)$, i.e., $H_0 \in \mathbb{S}_0$. Further, $H_1(g,1) = 1 + \sum\limits_{i=1}^{s-1} \big(z_i + (\beta_i - z_i)g\big) = 1 + \sum\limits_{i=1}^{s-1} z_i + \Big(\sum\limits_{i=1}^{s-1} (\beta_i - z_i)\Big)g$. It is easy to check that $1 + \sum\limits_{i=1}^{s-1} z_i + \sum\limits_{i=1}^{s-1} (\beta_i - z_i) = 1 + \sum\limits_{i=1}^{s-1} \beta_i = h_1(1) \in U(R)$. In view of Lemma 2.2.20, $H_1 \in \mathbb{S}_1$. Clearly, $\varphi(g) := res(H_0, H_1) \in RG$. As $\varphi(1) = res\big(H_0(1,t), H_1(1,t)\big) = res(h_0, h_1) \in U(R)$. By using Lemma 2.2.20 again, $\varphi(g) \in U(RG)$, i.e., $res(H_0, H_1) \in U(RG)$. In light of Lemma 2.2.16, $(H_0, H_1) = 1$. Therefore, $A(g) \in M_n\big(RG\big)$ is strongly clean, as required.

(2) $\Rightarrow$ (1) Let $\omega : RG \to R, a + bg \mapsto a + b$. Then we get a corresponding ring morphism $\mu : M_n(RG) \to M_n(R), \big(a_{ij}(g)\big) \mapsto \big(\omega(a_{ij}(g))\big)$. As $A(g)$ is strongly clean, we can find an idempotent $E \in M_n(RG)$ such that $A(g) - E \in GL_n(RG)$ and $EA = AE$. Applying μ, we get $A(1) - \mu(E) \in GL_n(R)$, where $\mu(E)A(1) = A(1)\mu(E)$ and $\mu(E) \in M_n(R)$ is an idempotent, hence the result. □

2.3 Explicit Criteria

The aim of this section is to develop characterizations for $n \times n$ matrix rings over commutative local rings to be strongly clean for $n \leq 5$.

Theorem 2.3.1. *Let R be a commutative local ring and let $h \in R[t]$ be a monic polynomial of degree n. If $n = 2, 3$, then the following are equivalent:*

(1) *Every* $\varphi \in M_n(R)$ *with* $\chi(\varphi) = h$ *is strongly clean.*
(2) *The companion matrix* C_h *of* h *is strongly clean.*
(3) *There exists a factorization* $h = h_0h_1$ *such that* $h_0 \in \mathbb{S}_0$ *and* $h_1 \in \mathbb{S}_1$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) These are clear from Theorem 2.2.9.

(3) $\Rightarrow$ (1) For every $\varphi \in M_n(R)$ with $\chi(\varphi) = h$, assume that $h = h_0h_1$ such that $h_0 \in \mathbb{S}_0$ and $h_1 \in \mathbb{S}_1$.

Case I. $n = 2$.

(i) $deg(h_0) = 0$ and $deg(h_1) = 2$. Write $h_0 = 1$ and $h_1 = t^2 + at + b$. Then $h = h_0h_1, h_0 \in \mathbb{S}_0$ and $h_1 \in \mathbb{S}_1$, i.e., h has a trivial factorization. Obviously, $(h_0, h_1) = 1$. According to Theorem 2.2.9, φ is strongly clean.

(ii) $deg(h_0) = 1$ and $deg(h_1) = 1$. Write $h_0 = t + a$ and $h_1 = t + b$. As $h_0 \in \mathbb{S}_0$ and $h_1 \in \mathbb{S}_1$, we see that $a, 1 + b \in U(R)$. If $b - a \in J(R)$, then $b \in U(R)$; hence, $ab \in U(R)$. Let $h_0' = t^2 + (a + b)t + ab$ and $h_1' = 1$. Then $h = h_0'h_1'$ where $h_0' \in \mathbb{S}_0$ and $h_1' \in \mathbb{S}_1$. In addition, $(h_0', h_1') = 1$. If $b - a \in U(R)$, then $(t + a) - (t + b) = a - b \in U(R)$; hence, $(t + a)(a - b)^{-1} - (t + b)(a - b)^{-1} = 1$. This implies that $(h_0, h_1) = 1$. According to Theorem 2.2.9, φ is strongly clean.

(iii) $deg(h_0) = 2$ and $deg(h_1) = 0$. Write $h_0 = t^2 + at + b$ and $h_1 = 1$. Then h has a trivial factorization h_0h_1. Clearly, $(h_0, h_1) = 1$. In view of Theorem 2.2.9, φ is strongly clean.

Case II. $n = 3$.

(i) $deg(h_0) = 0$ and $deg(h_1) = 3$. Write $h_0 = 1$ and $h_1 = t^3 + at^2 + bt + c$. Obviously, $(h_0, h_1) = 1$. By virtue of Theorem 2.2.9, φ is strongly clean.

(ii) $deg(h_0) = 1$ and $deg(h_1) = 2$. Write $h_0 = t + a$ and $h_1 = t^2 + bt + c$. As $h_0 \in \mathbb{S}_0$ and $h_1 \in \mathbb{S}_1$, we see that $a, 1 + b + c \in U(R)$. If $1 + a \in J(R)$, we choose $\delta = b - a$. Then $(t + a)(t + \delta) - (t^2 + bt + c) = a\delta - c = ab - a^2 - c = (1 + a)b + (1 + a)(1 - a) - (1 + b + c) \in U(R)$. This implies that $(t + a)(t + \delta)(a\delta - c)^{-1} - (t^2 + bt + c)(a\delta - c)^{-1} = 1$, i.e., $(h_0, h_1) = 1$. If $1 + a \in U(R)$, we choose $h_0' = 1$ and $h_1' = (t + a)(t^2 + bt + c)$. Then $h_0' \in \mathbb{S}_0$ and $h_1' \in \mathbb{S}_1$. In addition, $(h_0', h_1') = 1$. In light of Theorem 2.2.9, φ is strongly clean.

(iii) $deg(h_0) = 2$ and $deg(h_1) = 1$. Write $h_0 = t^2 + at + b$ and $h_1 = t + c$. Since $h_0 \in \mathbb{S}_0$ and $h_1 \in \mathbb{S}_1$, we have $b, 1 + c \in U(R)$. Set $g(u) = -h(1 - u)$. Then $g(u) = -\big((1 - u)^2 + a(1 - u) + b\big)(1 - u + c) = (u - 1 - c)\big(u^2 - (2 + a)u + 1 + a + b\big)$. Let $h_0' = u - 1 - c$ and $h_1' = u^2 - (2 + a)u + 1 + a + b$. Then $h_0'(0) = -(1 + c) \in U(R)$ and $h_1'(1) = b \in U(R)$. On the other hand, $g(I_3 - \varphi) = -h(\varphi) = 0$. In light of the proceeding discussion, we see that $I_3 - \varphi \in M_3(R)$ is strongly clean by Theorem 2.2.9, and so $\varphi \in M_3(R)$ is strongly clean.

(iv) $deg(h_0) = 3$ and $deg(h_1) = 0$. Write $h_0 = t^3 + at^2 + bt + c$ and $h_1 = 1$. Obviously, $(h_0, h_1) = 1$. According to Theorem 2.2.9, φ is strongly clean.

Therefore we conclude that every $\varphi \in M_n(R)$ with $\chi(\varphi) = h$ is strongly clean for $n = 2, 3$. □

Corollary 2.3.2. *Let R be a commutative local ring. Then $M_3(R)$ is strongly clean if and only if for any $A \in M_3(R)$,*

(1) $A \in GL_3(R)$, *or*
(2) $I_3 - A \in GL_3(R)$, *or*
(3) $\chi(A)$ *has a root in* $1 + J(R)$ *and* $tr(A) \in 2 + U(R)$, *or*
(4) $\chi(A)$ *has a root in* $J(R)$ *and* $tr(A) \in 1 + U(R)$.

Proof. Suppose that $A \in M_3(R)$ is strongly clean. It follows from Theorem 2.3.1 that there exist $h_0 \in \mathbb{S}_0$ and $h_1 \in \mathbb{S}_1$ such that $\chi(A) = h_0h_1$. Set $mid(A) = det(I_3 - A) - 1 + tr(A) + det(A)$.

Case I. $deg(h_0) = 3$ and $deg(h_1) = 0$. Then $h_0 = t^3 - tr(A)t^2 + mid(A)t - det(A)$ and $h_1 = 1$. As $h_0 \in \mathbb{S}_0$, we have $det(A) \in U(R)$, and so $A \in GL_3(R)$.

Case II. $deg(h_0) = 0$ and $deg(h_1) = 3$. Then $h_0 = 1$ and $h_1 = t^3 - tr(A)t^2 + mid(A)t - det(A)$. As $h_1 \in \mathbb{S}_1$, we see that $h_1(1) = 1 - tr(A) + mid(A) - det(A) \in U(R)$; hence, $det(I_3 - A) \in U(R)$. Therefore $I_3 - A \in GL_3(R)$.

Case III. $deg(h_0) = 1$ and $deg(h_1) = 2$. Then $h_0 = t - \alpha$ and $h_1 = t^2+bt+c$. Hence $\alpha, 1+b+c \in U(R)$. If $det(A) \in U(R)$ or $det(I_3-A) \in U(R)$, then $A \in GL_3(R)$ or $I_3-A \in GL_3(R)$. Hence, $A \in M_3(R)$ is strongly clean, and so we may assume that $det(A), det(I_3 - A) \in J(R)$. It is easy to verify that $\alpha - b = tr(A)$ and $c\alpha = det(A)$. Thus, $c = \alpha^{-1}det(A) \in J(R)$. As $1 + b + c \in U(R)$, we get $b \in -1 + U(R)$. Clearly, $det(I_3 - A) = (1 - \alpha)(1 + b + c)$, and so $1 - \alpha = det(I_3 - A)(1 + b + c)^{-1}$. We infer that $\alpha \in 1 + J(R)$. Therefore $tr(A) = \alpha - b \in 2 + U(R)$.

Case IV. $deg(h_0) = 2$ and $deg(h_1) = 1$. Then $h_0 = t^2 + bt + c$ and $h_1 = t - \alpha$. Hence $c, 1 - \alpha \in U(R)$. It is easy to verify that $\alpha - b = tr(A)$ and $c\alpha = det(A)$. As in the preceding discussion, we may assume that $det(A), det(I_3 - A) \in J(R)$. Let $B = I_3 - A$. Then B is strongly clean with $det(B), det(I_3 - B) \in J(R)$. It is easy to verify that $\chi(B) = det\big(tI_3 - (I_3 - A)\big) = -det\big((1-t)I_3 - A\big) = -\big((1-t)^2+b(1-t)+c\big)\big((1-t)-\alpha\big) = \big(t-(1-\alpha)\big)\big((1-t)^2+b(1-t)+c\big)$. Set $h_0' = t-(1-\alpha)$ and $h_1' = (1-t)^2+b(1-t)+c$.

One easily checks that $h_0' \in \mathbb{S}_0$ and $h_1' \in \mathbb{S}_1$ such that $\chi(B) = h_0'h_1'$. By the consideration in Case III, we see that $tr(B) \in 2 + U(R)$. This implies that $tr(A) \in 1 + U(R)$. In addition, $\chi(B)$ has a root $\beta \in 1 + J(R)$. Therefore $\chi(A)$ has a root $1 - \beta \in J(R)$. Hence (4) is satisfied.

We now show the converse. Let $A \in M_3(R)$. If (1) or (2) holds, then A is strongly clean.

If (4) holds, then $\chi(A)$ has a root $\alpha \in J(R)$. Write $\chi(A) = t^3 - tr(A)t^2 + mid(A)t - det(A)$. Then $\alpha^3 - tr(A)\alpha^2 + mid(A)\alpha - det(A) = 0$. Hence, $\chi(A) = (t^3 - \alpha^3) - tr(A)(t^2 - \alpha^2) + mid(A)(t - \alpha)$. Thus, we see that $\chi(A) = (t - \alpha)(t^2 + bt + c)$ where $b, c \in R$ and $\alpha \in J(R)$. If $det(A) \in U(R)$ or $det(I_3 - A) \in U(R)$, then $A \in M_3(R)$ is strongly clean. Hence we may assume that $det(A), det(I_3 - A) \in J(R)$. Obviously, $det(I_3 - A) = (1 - \alpha)(1 + b + c)$, and so $1 + b + c = (1 - \alpha)^{-1}det(I_3 - A) \in J(R)$. Clearly, $\alpha - b = tr(A)$, and so $b = \alpha - tr(A) \in -1 + U(R)$. It is easy to see that $\chi(I_3 - A) = det\big(tI_3 - (I_3 - A)\big) = -det\big((1-t)I_3 - A\big) = -\big((1-t) - \alpha\big)\big((1-t)^2 + b(1-t) + c\big)$. Set $h_0 = t - 1 + \alpha$ and $h_1 = t^2 - (b+2)t + 1 + b + c$. Then $h_0 \in \mathbb{S}_0$ and $h_1 \in \mathbb{S}_1$. According to Theorem 2.3.1, $A \in M_3(R)$ is strongly clean.

If (3) holds, then $\chi(A) = (t - \alpha)(t^2 + bt + c)$ where $\alpha \in 1 + J(R)$ and $tr(A) \in 2 + U(R)$. We may assume that $det(A), det(I_3 - A) \in J(R)$. Let $B = I_3 - A$. Then $\chi(B)$ has a root in $J(R)$ and that $tr(B) = 3 - tr(A) \in 1 + U(R)$. Additionally, $det(B), det(I_3 - B) \in J(R)$. By the preceding consideration, B is strongly clean, and then so is A. Therefore $A \in M_3(R)$ is strongly clean, as required. □

A natural problem asks if Theorem 2.3.1 can be extended to $n \times n$ matrices with n more than 3. The following counterexample gives a negative answer (cf. [241, Example 3.1.7]).

Example 2.3.3. Let $R = \mathbb{Z}_{(3)}$, and let $h(t) = t^4 + t^3 + 5t^2 + 2t + 6 \in R[t]$. Obviously, $h(0) = 6, h(1) = 15$. Hence, $h(0), h(1) \in J(R)$. This implies that h has no trivial factorizations. Set $h_0 = (t - 1)(t + 1) + 3$ and $h_1 = t(t + 1) + 3$. As $h_0(0) = 2, h_1(1) = 5 \in U(R)$, we see that $h = h_0h_1$ with $h_0 \in \mathbb{S}_0$ and $h_1 \in \mathbb{S}_1$. But in $\big(R/J(R)\big)[t]$, $gcd(\overline{h_0}, \overline{h_1}) = t + \overline{1}$. Accordingly, $(h_0, h_1) \neq (1)$.

Assume that $h = k_0k_1, k_0 \in \mathbb{S}_0, k_1 \in \mathbb{S}_1$ and $(k_0, k_1) = 1$. Since h has no trivial factorizations, we see that $k_0 \neq 1, k_1 \neq 1$. In $\mathbb{Q}$, $(t-1)(t+1)+3$ and $t(t + 1) + 3$ are irreducible. Hence, $h = h_0h_1$ is the only factorization

of h into monic polynomials over $\mathbb{Q}$. But the roots of h in R is less than in $\mathbb{Q}$. Thus, the preceding factorization of h is the only one in $R[t]$, and then we deduce that $k_0 = h_0, k_1 = h_1$. This gives a contradiction. Therefore we conclude that C_h is not strongly clean by Theorem 2.2.9.

Let R be a commutative ring, and let $f_1, \ldots, f_m; g_1, \ldots, g_n \in R[t]$. If each $(f_i, g_j) = 1$ for any i, j, then $(f_1 \cdots f_m; g_1 \cdots g_n) = 1$. We frequently use this fact without any remark. From the preceding example, Theorem 2.3.1 can not be directly generalized to the case n more than 3. Now we consider strongly clean $n \times n$ matrices over a commutative local ring R for $n = 4, 5$.

Theorem 2.3.4. *Let R be a commutative local ring and let $h \in R[t]$ be a monic polynomial of degree 4. If h has a root in $J(R)$, then the following are equivalent:*

(1) *Every $\varphi \in M_n(R)$ with $\chi(\varphi) = h$ is strongly clean.*
(2) *The companion matrix C_h of h is strongly clean.*
(3) *There exists a factorization $h = h_0 h_1$ such that $h_0 \in \mathbb{S}_0$ and $h_1 \in \mathbb{S}_1$.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) These are obvious by Theorem 2.2.9.

(3) $\Rightarrow$ (1) For every $\varphi \in M_n(R)$ with $\chi(\varphi) = h$, we may assume that $h = h_0 h_1$ such that $h_0 \in \mathbb{S}_0$ and $h_1 \in \mathbb{S}_1$.

(i) $deg(h_0) = 0$ and $deg(h_1) = 4$. Write $h_0 = 1$ and $h_1 = t^4 + at^3 + bt^2 + ct + d$. Obviously, $(h_0, h_1) = 1$. According to Theorem 2.2.9, φ is strongly clean.

(ii) $deg(h_0) = 1$ and $deg(h_1) = 3$. Write $h_0 = t + a$ and $h_1 = t^3 + bt^2 + ct + d$. As $h_0 \in \mathbb{S}_0$ and $h_1 \in \mathbb{S}_1$, we see that $a, 1 + b + c + d \in U(R)$. If $1 + a \in J(R)$, we choose $\delta = b - a$ and $\gamma = c - a\delta$. Then $(t+a)(t^2+\delta t+\gamma)-(t^3+bt^2+ct+d) = a\gamma-d = ac-a^2b+a^3-d = (1+a)c+(1+a)(1-a)b+(1+a)(1-a+a^2)-(1+b+c+d) \in U(R)$. This implies that $(t+a)(t^2+\delta t+\gamma)(a\gamma-d)^{-1}-(t^3+bt^2+ct+d)(a\gamma-d)^{-1} = 1$, i.e., $(h_0, h_1) = 1$. If $1 + a \in U(R)$, we choose $h_0' = 1$ and $h_1' = (t + a)(t^3 + bt^2 + ct + d)$. Then $h_0' \in \mathbb{S}_0$ and $h_1' \in \mathbb{S}_1$. In addition, $(h_0', h_1') = 1$. According to Theorem 2.2.9, φ is strongly clean.

(iii) $deg(h_0) = 2$ and $deg(h_1) = 2$. Write $h_0 = t^2 + at + b$ and $h_1 = t^2 + ct + d$. Since $h_0 \in \mathbb{S}_0$ and $h_1 \in \mathbb{S}_1$, we see that $b, 1 + c + d \in U(R)$. If $1+a+b \in U(R)$, we choose $h_0' = 1$ and $h_1' = (t^2+at+b)(t^2+ct+d)$. Then $h_0' \in \mathbb{S}_0$ and $h_1' \in \mathbb{S}_1$. Further, $(h_0', h_1') = 1$. According to Theorem 2.2.9, φ is strongly clean. If $d \in U(R)$, we choose $h_0' = (t^2 + at + b)(t^2 + ct + d)$

and $h_1' = 1$. Then $h_0' \in \mathbb{S}_0$ and $h_1' \in \mathbb{S}_1$, and that $(h_0', h_1') = 1$. By using Theorem 2.2.9, φ is strongly clean. Thus, we may assume that $b, 1+c+d \in U(R)$, $1+a+b \in J(R)$ and $d \in J(R)$.

Since h has a root $\alpha \in J(R)$. Write $h(t) = (t-\alpha)(t^3+\gamma t^2+\delta t+\lambda)$. As $b \in U(R)$, we see that $\alpha^2 + a\alpha + b \in U(R)$. It follows from $h(\alpha) = 0$ that $\alpha^2 + c\alpha + d = 0$. This shows that $t^2 + ct + d = (t-\alpha)(t+\alpha+c)$. Hence $h = (t^2+at+b)(t+\alpha+c)(t-\alpha)$. Assume that $c \in U(R)$. Choose $h_0' = (t^2+at+b)(t+\alpha+c)$ and $h_1' = t-\alpha$. Then $h_0' \in \mathbb{S}_0$ and $h_1' \in \mathbb{S}_1$. Clearly, $(t+\alpha+c, t-\alpha) = 1$. It is easy to verify that $(t^2+at+b)-(t+\alpha-a)(t-\alpha) = b-\alpha(a-\alpha) \in U(R)$, as see that $(t^2+at+b, t-\alpha) = 1$. Hence $(h_0', h_1') = 1$. Assume that $c \in J(R)$. Choose $h_0' = t^2+at+b$ and $h_1' = (t+\alpha+c)(t-\alpha)$. Then $h_0' \in \mathbb{S}_0$ and $h_1' \in \mathbb{S}_1$. Clearly, $(t^2+at+b, t-\alpha) = 1$. Moreover, $(t^2+at+b)-(t+a-\alpha-c)(t+\alpha+c) = b-(a-\alpha-c)(\alpha+c) \in U(R)$, and so $(t^2+at+b, t+\alpha+c) = 1$. Hence $(h_0', h_1') = 1$.

(iv) $deg(h_0) = 3$ and $deg(h_1) = 1$. Write $h_0 = t^3+at^2+bt+c$ and $h_1 = t+d$. As $h_0 \in \mathbb{S}_0$ and $h_1 \in \mathbb{S}_1$, we see that $c, 1+d \in U(R)$. Set $g(u) = h(1-u)$. Then $g(u) = \big((1-u)^3+a(1-u)^2+b(1-u)+c\big)(1-u+d) = (u-1-d)\big(u^3-(3+a)u^2+(3+2a+b)u-(1+a+b+c)\big)$. Let $h_0' = u-1-d$ and $h_1' = u^3-(3+a)u^2+(3+2a+b)u-(1+a+b+c)$. Then $h_0'(0) = -(1+d) \in U(R)$ and $h_1'(1) = -c \in U(R)$. On the other hand, $g(I_4-\varphi) = h(\varphi) = 0$. In light of the proceeding discussion, we see that $I_4-\varphi \in M_4(R)$, and so $\varphi \in M_4(R)$ is strongly clean.

(v) $deg(h_0) = 4$ and $deg(h_1) = 0$. Then $h_1 = 1$, and so $(h_0, h_1) = 1$. Therefore $\varphi \in M_4(R)$ is strongly clean by Theorem 2.2.9

Therefore we conclude that φ is strongly clean. □

Example 2.3.5. Let $A = \begin{pmatrix} 0 & -1 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{pmatrix} \in M_4\big(\mathbb{Z}_{(3)}\big)$. Then the characteristic polynomial $\chi(A) = (t^2+t+1)(t^2+t+2)$. Set $h_0 = t^2+t+1$ and $h_1 = t^2+t+2$. Then $h = h_0h_1, h_0 \in \mathbb{S}_0, h_1 \in \mathbb{S}_1$ and $(h_0, h_1) = 1$. In view of Theorem 2.2.9, $A \in M_4(\mathbb{Z}_{(3)})$ is strongly clean. But $h \in \mathbb{Z}_{(3)}[t]$ has no any root in $J\big(\mathbb{Z}_{(3)}\big)$.

Theorem 2.3.6. *Let R be a commutative local ring and let $h \in R[t]$ be a monic polynomial of degree 5. If h has a root in $J(R)$ and a root in $1+J(R)$, then the following are equivalent:*

(1) *Every* $\varphi \in M_n(R)$ *with* $\chi(\varphi) = h$ *is strongly clean.*
(2) *The companion matrix* C_h *of* h *is strongly clean.*
(3) *There exists a factorization* $h = h_0h_1$ *such that* $h_0 \in \mathbb{S}_0$ *and* $h_1 \in \mathbb{S}_1$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) These are obvious.

(3) $\Rightarrow$ (1) For every $\varphi \in M_n(R)$ with $\chi(\varphi) = h$, we may assume that $h = h_0h_1$ such that $h_0 \in \mathbb{S}_0$ and $h_1 \in \mathbb{S}_1$.

(i) $deg(h_0) = 0$ and $deg(h_1) = 5$. Write $h_0 = 1$ and $h_1 = t^5 + at^4 + bt^3 + ct^2 + dt + e$. Obviously, $(h_0, h_1) = 1$. According to Theorem 2.2.9, φ is strongly clean.

(ii) $deg(h_0) = 1$ and $deg(h_1) = 4$. Write $h_0 = t + a$ and $h_1 = t^4 + bt^3 + ct^2 + dt + e$. As $h_0 \in \mathbb{S}_0$ and $h_1 \in \mathbb{S}_1$, we have $a, 1 + b + c + d + e \in U(R)$. Choose $\alpha = b - a, \beta = c - a\alpha, \gamma = d - a\beta$. If $1 + a \in J(R)$, then $(t + a)(t^3 + \alpha t^2 + \beta t + \gamma) - (t^4 + bt^3 + ct^2 + dt + e) = a\gamma - e = ad - a^2c + a^3b - a^4 - e = (1 + a)d + (1 + a)(1 - a)c + (1 + a)(1 - a + a^2)b + (1 + a)(1 - a)(1 + a^2) - (1 + b + c + d + e) \in U(R)$; hence, $(t + a)(t^3 + \alpha t^2 + \beta t + \gamma)(a\gamma - e)^{-1} - (t^4 + bt^3 + ct^2 + dt + e)(a\gamma - e)^{-1} = 1$, i.e., $(h_0, h_1) = 1$. If $1 + a \in U(R)$, we choose $h_0' = 1$ and $h_1' = (t + a)(t^4 + bt^3 + ct^2 + dt + e)$. Then $h_0' \in \mathbb{S}_0$ and $h_1' \in \mathbb{S}_1$. In addition, $(h_0', h_1') = 1$. According to Theorem 2.2.9, φ is strongly clean.

(iii) $deg(h_0) = 2$ and $deg(h_1) = 3$. Write $h_0 = t^2 + at + b$ and $h_1 = t^3 + ct^2 + dt + e$. Since $h_0 \in \mathbb{S}_0$ and $h_1 \in \mathbb{S}_1$, we see that $b, 1 + c + d + e \in U(R)$. If $1 + a + b \in U(R)$, we choose $h_0' = 1$ and $h_1' = (t^2 + at + b)(t^3 + ct^2 + dt + e)$. Then $h_0' \in \mathbb{S}_0$ and $h_1' \in \mathbb{S}_1$. In addition, $(h_0', h_1') = 1$. By Theorem 2.2.9, φ is strongly clean. If $e \in U(R)$, we choose $h_0' = (t^2 + at + b)(t^3 + ct^2 + dt + e)$ and $h_1' = 1$. Then $h_0' \in \mathbb{S}_0$ and $h_1' \in \mathbb{S}_1$. In addition, $(h_0', h_1') = 1$. According to Theorem 2.2.9, φ is strongly clean. Thus, we may assume that $b, 1 + c + d + e \in U(R)$ and $1 + a + b, e \in J(R)$.

By hypothesis, h has a root $\alpha \in J(R)$. As $\alpha^2 + a\alpha + b \in U(R)$, we see that $\alpha^3 + c\alpha^2 + d\alpha + e = 0$. This shows that $h = (t^2 + at + b)(t - \alpha)(t^2 + \gamma t + \delta)$ for some $\gamma, \delta \in R$. (a) Assume that $\delta \in U(R)$. Choose $h_0' = (t^2 + at + b)(t^2 + \gamma t + \delta)$ and $h_1' = t - \alpha$. Then $h_0' \in \mathbb{S}_0$ and $h_1' \in \mathbb{S}_1$. It is easy to verify that $\big(b + (a - \alpha)\alpha\big)^{-1}(t^2 + at + b) - \big(b + (a - \alpha)\alpha\big)^{-1}(t + a - \alpha)(t - \alpha) = 1$, and so $(t^2 + at + b, t - \alpha) = 1$. Likewise, $(t^2 + \gamma t + \delta, t - \alpha) = 1$. Accordingly, $(h_0', h_1') = 1$. (b) Assume that $\delta \in J(R)$. By hypothesis, h has a root $\beta \in 1 + J(R)$. As $\beta^3 + c\beta^2 + d\beta + e \in 1 + c + d + e + J(R) \subseteq U(R)$, we see that $\beta^2 + a\beta + b = 0$. Write $t^2 + at + b = (t - \beta)(t - \eta)$. Case I. $\eta \in 1 + J(R)$. Choose $h_0' = (t - \beta)(t - \eta)$ and $h_1' = h_1 = (t - \alpha)(t^2 + \gamma t + \delta)$. Then $h_0' \in \mathbb{S}_0$ and $h_1' \in \mathbb{S}_1$. Obviously, $\overline{h_0'} = \overline{(t - 1)^2}$ and $\overline{h_1'} = \overline{t^2(t + \gamma)}$. Clearly,

$1+\gamma \in U(R)$. One easily checks that $(\overline{t-1}, \overline{t}) = \overline{1}$ and $(\overline{t-1}, \overline{t+\gamma}) = \overline{1}$, and so $(h_0, h_1) = 1$. We infer that $(h_0', h_1') = 1$. Case II. $\eta \in 1+U(R)$. Choose $h_0' = t-\beta$ and $h_1' = (t-\eta)(t-\alpha)(t^2+\gamma t+\delta)$. Then $h_0' \in \mathbb{S}_0$ and $h_1' \in \mathbb{S}_1$. Clearly, $\overline{h_0'} = \overline{t-1}$ and $\overline{h_1'} = \overline{t^2(t-\eta)(t+\gamma)}$, and so $(h_0, h_1) = 1$. Accordingly, $(h_0', h_1') = 1$.

(iv) $deg(h_0) = 3$ and $deg(h_1) = 2$. Write $h_0 = t^3 + at^2 + bt + c$ and $h_1 = t^2 + dt + e$. As $h_0 \in \mathbb{S}_0$ and $h_1 \in \mathbb{S}_1$, we see that $c, 1+d+e \in U(R)$. Set $g(u) = -h(1-u)$. Then $g(u) = \big((1-u)^2 + d(1-u) + e\big)\big(- (1-u)^3 - a(1-u)^2 - b(1-u) - c\big)$. Let $h_0' = (1-u)^2 + d(1-u) + e$ and $h_1' = -(1-u)^3 - a(1-u)^2 - b(1-u) - c$. Then $h_0'(0) = 1+d+e \in U(R)$ and $h_1'(1) = -c \in U(R)$. On the other hand, $g(I_5 - \varphi) = -h(\varphi) = 0$. Clearly, g has a root in $J(R)$ and a root in $1+J(R)$. By the proceeding discussion, we see that $I_5 - \varphi \in M_5(R)$, and so $\varphi \in M_5(R)$ is strongly clean.

(v) $deg(h_0) = 4$ and $deg(h_1) = 1$. Write $h_0 = t^4 + at^3 + bt^2 + ct + d$ and $h_1 = t + e$. As $h_0 \in \mathbb{S}_0$ and $h_1 \in \mathbb{S}_1$, we see that $d, 1+e \in U(R)$. Set $g(u) = -h(1-u)$. Then $g(u) = \big(- (1-u) - e\big)\big((1-u)^4 + a(1-u)^3 + b(1-u)^2 + c(1-u) + d\big)$. Let $h_0' = -(1-u) - e$ and $h_1' = (1-u)^4 + a(1-u)^3 + b(1-u)^2 + c(1-u) + d$. Then $h_0'(0) = -1 - e \in U(R)$ and $h_1'(1) = d \in U(R)$. On the other hand, $g(I_5 - \varphi) = -h(\varphi) = 0$. In light of the proceeding discussion, we see that $I_5 - \varphi \in M_5(R)$, and so $\varphi \in M_5(R)$ is strongly clean.

(vi) $deg(h_0) = 5$ and $deg(h_1) = 0$. Then $h_1 = 1$, and so $(h_0, h_1) = 1$. Therefore $\varphi \in M_5(R)$ is strongly clean by Theorem 2.2.9.

Consequently, we conclude that every $\varphi \in M_n(R)$ with $\chi(\varphi) = h$ is strongly clean for $n = 5$. □

Lemma 2.3.7. *Let R be a commutative local ring, and let $n \in \mathbb{N}$. The following are equivalent:*

(1) *For every monic polynomial h of degree n, there exist $h_0 \in \mathbb{S}_0, h_1 \in \mathbb{S}_1$ such that $h = h_0 h_1$.*
(2) *For every monic polynomial h of degree n and every $x_0, \ldots, x_m \in R (m \in \mathbb{N})$ such that $x_i - x_j \in U(R)(i \neq j)$, there exist $h_{x_i} \in \bigcap_{j \neq i} \mathbb{S}_{x_j} (0 \leq i \leq m)$ such that $h = \prod_{i=0}^{m} h_{x_i}$.*

Proof. (1) $\Rightarrow$ (2) Let $h \in R[t]$ be a monic polynomial of degree n, and let $x_0, \ldots, x_m \in R (m \in \mathbb{N})$ such that $x_i - x_j \in U(R)(i \neq j)$. Choose $H(t) = (x_1 - x_0)^{-n} h\big((x_1 - x_0)t + x_0\big)$. Then $H(t) \in R[t]$ is a monic polynomial of degree n. By hypothesis, there exist $H_0 \in \mathbb{S}_0, H_1 \in \mathbb{S}_1$ such

that $H(t) = H_0H_1$. Set $h_{x_i}(t) = (x_1-x_0)^{deg(H_i)}H_i\big((x_1-x_0)^{-1}(t-x_0)\big), i = 0, 1$. It is easy to verify that $h_{x_i} \in \mathbb{S}_{x_i}$ such that $h = h_0h_1$. Accordingly, the result holds for $m = 1$. Suppose that the result holds for $m = k(k \geq 1)$. Consider $m = k + 1$. By assumption, there exist $g_{x_0} \in \mathbb{S}_{x_0}, g_{x_1} \in \mathbb{S}_{x_1}$ such that $h = g_{x_0}g_{x_1}$.

By hypothesis, we can find some $\alpha_{x_i}, \beta_{x_i} \in \bigcap_{j\neq i} \mathbb{S}_{x_j}$ such that $g_{x_0} = \alpha_{x_1}\alpha_{x_2}\cdots\alpha_{x_m}$ and $g_{x_1} = \beta_{x_0}\alpha_{x_2}\cdots\beta_{x_m}$. Thus, $h = \beta_{x_0}\alpha_{x_1}\prod_{i=2}^{m}\big(\alpha_{x_i}\beta_{x_i}\big)$. It is easy to verify that

$$\beta_{x_0} \in S_{x_1} \cap \Big(\bigcap_{j\neq 0,1} S_{x_j}\Big) = \bigcap_{j\neq 0} S_{x_j}, \alpha_{x_1} \in S_{x_0} \cap \Big(\bigcap_{j\neq 0,1} S_{x_j}\Big) = \bigcap_{j\neq 1} S_{x_j},$$

$$\alpha_{x_i} \in S_{x_0} \cap \Big(\bigcap_{j\neq 0,i} S_{x_j}\Big) = \bigcap_{j\neq i} S_{x_j}, \beta_{x_i} \in S_{x_1} \cap \Big(\bigcap_{j\neq 1,i} S_{x_j}\Big) = \bigcap_{j\neq i} S_{x_j} (i \geq 2)$$

By induction, the result follows.

(2) $\Rightarrow$ (1) Choose $x_0 = 1, x_1 = 0$. Then the result follows. □

Contrast to Theorem 2.3.1, Theorem 2.3.4 and Theorem 2.2.6, we derive the following result which is due to Dorsey [241, Proposition 3.2.12].

Theorem 2.3.8. *Let R be a commutative local ring, and let $n \leq 5$. The following are equivalent:*

(1) *$M_n(R)$ is strongly clean.*
(2) *For every monic polynomial h of degree n, there exist $h_0 \in \mathbb{S}_0, h_1 \in \mathbb{S}_1$ such that $h = h_0h_1$.*

Proof. (1) $\Rightarrow$ (2) is trivial from Theorem 2.2.9.

(2) $\Rightarrow$ (1) Let h be a monic polynomial of degree n. Then there exist $h_0 \in \mathbb{S}_0, h_1 \in \mathbb{S}_1$ such that $h = h_0h_1$. If $h(0) \in U(R)$, then h has the trivial factorization $h = g_0g_1$ such that $(g_0, g_1) = 1$, where $g_0 = h \in \mathbb{S}_0$ and $g_1 = 1 \in \mathbb{S}_1$. Thus, we may assume that $h(0) \in J(R)$. Likewise, we assume that $h(1) \in J(R)$. Write $\overline{h} = H(t)\prod_{i=0}^{m}(t-\overline{r_i})^{k_i}\big(m \geq 2, k_i \geq 1, \overline{r_i} \neq \overline{r_j}(i \neq j)\big)$. In addition, we may assume that $r_0 = 0, r_1 = 1$ and $H(t)$ has no any roots in $\overline{R}$. In view of Lemma 2.3.7, there exist $g_{r_i} \in \bigcap_{j\neq i} \mathbb{S}_{r_j} (0 \leq i \leq m)$ such that $h = \prod_{i=0}^{m} g_{r_i}$. Choose $f_0 = g_{r_1}$ and $f_1 = g_{r_0}\prod_{i=2}^{m} g_{r_i}$. Then $h = f_0f_1, f_0 \in \mathbb{S}_0$ and $f_1 \in \mathbb{S}_1$. Set $\overline{R} = R/J(R)$. Write $\overline{f_0} = F(t)(t-\overline{1})^{k_1}$ and $\overline{f_1} = G(t)t^{k_0}\prod_{i=2}^{m}(t-\overline{r_i})^{k_i}$, where $F(t)$ and $G(t)$ have no any roots in $\overline{R}$.

If $(f_0, f_1) \neq 1$, then $F(t)$ and $G(t)$ has a common factor of positive degree. Thus, $deg(f) = deg(f_0) + deg(f_1) \geq 2 + 1 + 2 + 1 = 6$, a contradiction. Therefore $(f_0, f_1) = 1$, as required. □

2.4 Projective-free Rings

A commutative ring R is said to be projective-free provided that every finitely generated projective R-module is free. The class of projective-free rings is very large. The goal of this section is to study elementary properties of projective-free rings and investigate strongly clean matrices over such rings.

Proposition 2.4.1. *Let I be an ideal of a commutative ring R, and let $I \subseteq J(R)$. If R/I is projective-free, then so is R.*

Proof. Let P be a finitely generated projective R-module. Then $\big(R/I\big) \bigotimes\limits_{R} P$ is a finitely generated projective R/I-module, and so it is free. Write $\big(R/I\big) \bigotimes\limits_{R} P \cong n\big(R/I\big)$. As in the proof of Lemma 1.2.3, $P/IP \cong nR/I(nR)$. In light of Lemma 1.2.2, we obtain that $P \cong nR$. This establishes the claim. □

Corollary 2.4.2. *Every commutative local ring is projective-free.*

Proof. Let R be a commutative local ring. Then $R/J(R)$ is a filed, and so it is projective-free. By Proposition 2.4.1, R is projective-free. □

Corollary 2.4.3. *Let R be a commutative ring. Then $R[[x]]$ is projective-free if and only if so is R.*

Proof. Suppose $R[[x]]$ is projective-free. Construct ring morphisms

$$\varphi : R[[x]] \to R, f(x) \mapsto f(0);$$
$$\phi : R \to R[[x]], r \mapsto r.$$

Then $\varphi\phi = 1_R$. Let P be a finitely generated projective R-module. Then $R[[x]] \bigotimes\limits_{\phi} P$ is a finitely generated projective $R[[x]]$-module. Thus, $R[[x]] \bigotimes\limits_{\phi} P \cong n\big(R[[x]]\big)$ for some $n \geq 0$. Accordingly, $P \cong R \bigotimes\limits_{\varphi} \Big(R[[x]] \bigotimes\limits_{\phi} P\Big) \cong nR$, as required.

Conversely, assume that R is projective-free. One easily checks that $Ker\varphi = \{ f(x) \in R[[x]] \big| f(0) = 0 \} \subseteq J\big(R[[x]]\big)$. On the other hand, $R[[x]]/Ker\varphi \cong R$, and so $R[[x]]/Ker\varphi$ is projective-free. By Proposition 2.4.1, $R[[x]]$ is projective-free. □

Corollary 2.4.4. *Let R be a commutative ring, and let $n \in \mathbb{N}$. Then $R[[x]]/\big(x^n\big)$ is projective-free if and only if so is R.*

Proof. Clearly, $R[[x]]/\big(x^n\big) = \left\{ \sum_{i=0}^{n-1} a_i x^i \mid a_i \in R, x^n = 0 \right\}$. Construct ring morphisms

$$\varphi : R[[x]]/\big(x^n\big) \to R, f(x) \mapsto f(0);$$
$$\phi : R \to R[[x]]/\big(x^n\big), r \mapsto r.$$

Then $\varphi\phi = 1_R, Ker\varphi = \{ f(x) \in R[[x]] \big| f(0) = 0 \} \subseteq J\big(R[[x]]\big)$, and that $R[[x]]/(x^n)/Ker\varphi \cong R$. Analogously to Corollary 2.4.3, we complete the proof. □

By induction, we prove that a commutative ring R is projective-free if and only if so is $R[[x_1, \ldots, x_n]]$, if and only if so is $R[[x_1, \ldots, x_n]]/\big(x_1^{s_1}, \ldots, x_n^{s_n}\big)(s_1, \ldots, s_n \in \mathbb{N})$.

In the sequel, we use ω to denote the augmentation of RG. That is, $\omega : RG \to R, \sum\limits_{g \in G} r_g g \mapsto \sum\limits_{g \in G} r_g$. Clearly, ω is a surjective ring morphism. Given a prime p, a p-group is a group in which each element has a power of p as its order. That is, for each element g of the group, there exists some $n \in \mathbb{N}$ such that g^{p^n} is equal to the identity element.

Proposition 2.4.5. *Let R be a commutative ring with $char(R) = p$ (p is prime), and let G be a p-group. Then RG is is projective-free if and only if so is R.*

Proof. Construct ring morphisms

$$\varphi : R \to RG$$

$$r \mapsto r \cdot 1_G;$$

$$\omega : RG \to R$$

$$\sum_{g \in G} r_g g \mapsto \sum_{g \in G} r_g.$$

Suppose RG is projective-free. Let P be a finitely generated projective R-module. Then $P \bigotimes_{\varphi} RG$ is a finitely generated projective RG-module; hence, it is free. Write $P \bigotimes_{\varphi} RG \cong n(RG)$. Then

$$\left(P \bigotimes_{\varphi} RG\right) \bigotimes_{\omega} R \cong n(RG) \bigotimes_{\omega} R.$$

As $\omega\varphi = 1$, we get $P \cong nR$ is free. Therefore R is projective-free.

Conversely, assume that R is projective-free. Obviously, $Ker(\omega) = \left\{ \sum\limits_g r_g g \mid \sum\limits_g r_g = 0 \right\}$ is an ideal of RG. If $\sum\limits_g r_g g \in Ker(\omega)$, then $\sum\limits_g r_g g = r_1 g_1 + r_2 g_2 + \cdots + r_m g_m = r_1(g_1 - 1) + \cdots + r_m(g_m - 1)$. As G is a p-group, we have $g_i^{p^{s_i}} = 1$, where p_i is prime, $s_i \in \mathbb{N}$. Since $charR = p$, we easily check that $(g_i - 1)^{p^{s_i+1}} = g_i^{p^{s_i+1}} - 1 = 0$ in R. Let $n = \sum\limits_{i=1}^{m} s_i + 1$, Then $\left(\sum\limits_g r_g g\right)^{mp^n} = \sum\limits_{t_1+t_2+\cdots+t_m=mp^n} c_{t_1 t_2 \cdots t_m} (g_1 - 1)^{t_1} (g_2 - 1)^{t_2} \cdots (g_m - 1)^{t_m}$. As $t_1 + t_2 + \cdots + t_m = mp^n$, we can find some $t_i \geq p^{s_i+1}$, and so $(g_i - 1)^{t_i} = (g_i - 1)^{p^{s_i+1}} (g_i - 1)^{t_i - p^{s_i+1}} = 0$, i.e., $\sum\limits_g r_g g$ is nilpotent. This shows that $R \cong RG/Ker(\omega)$ with $Ker(\omega) \subseteq J(RG)$. In light of Proposition 2.4.1, RG is projective-free. □

Lemma 2.4.6. *Let $e, f \in R$ be idempotents. If $Re \cong Rf$ and $R(1-e) \cong R(1-f)$, then $e = ufu^{-1}$ for a $u \in U(R)$, i.e., e is similar to f.*

Proof. Let $\varphi : Re \cong Rf$ and $\phi : R(1-e) \cong R(1-f)$. Construct a ring morphism $u : R = Re \oplus R(1-e) \to Rf \oplus R(1-f), re + r(1-e) \mapsto (re)\varphi + \big(r(1-e)\big)\phi$. Then $u \in Aut(R)$. Regarding u as an invertible element in R, we can find $s, t \in R$ such that $eu = sf$ and $(1-e)u = t(1-f)$. Thus, $u = eu + (1-e)u = sf + t(1-f)$, and so $uf = \big(sf + t(1-f)\big)f = sf = eu$. Therefore $e = ufu^{-1}$. □

Theorem 2.4.7. *Let R be a commutative ring. Then the following are equivalent:*

(1) *R is projective-free.*
(2) *For any idempotent $E \in M_n(R)$, there exists $U \in GL_n(R)$ such that $UEU^{-1} = \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}$ for some m.*

Proof. (1) $\Rightarrow$ (2) Let $E \in M_n(R)$ be an idempotent matrix. Then $nR = (nR)E \oplus (nR)\big(I_n - E\big)$. Hence $(nR)E$ and $(nR)\big(I_n - E\big)$ are finitely

generated projective R-modules. Write $(nR)E \cong mR$ and $(nR)\big(I_n - E\big) = (n-m)R$. Then

$$\eta : (nR)E \cong (nR)\begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \mu : (nR)\big(I_n - E\big) = (nR)\begin{pmatrix} 0 & 0 \\ 0 & I_{n-m} \end{pmatrix}.$$

Construct a ring morphism

$$\varphi : M_n(R)E \to M_n(R)\begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} E \mapsto \begin{pmatrix} (\alpha_1 E)\eta \\ \vdots \\ (\alpha_n E)\eta \end{pmatrix}.$$

Then $\varphi : M_n(R)E \cong M_n(R)\begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}$. Likewise, $M_n(R)\big(I_n - E\big) \cong M_n(R)\begin{pmatrix} 0 & 0 \\ 0 & I_{n-m} \end{pmatrix}$. According to Lemma 2.4.6, A is similar to $\begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}$.

(2) $\Rightarrow$ (1) Let P be a finitely generated projective R-module. Write $P \oplus Q \cong nR$. Let $\sigma : nR \to P$ be the projection onto P. Then $\sigma = \sigma^2 \in End(nR)$. Then $P = Im(\sigma)$. Regarding σ as a matrix E in $M_n(R)$, we see that $P \cong (nR)E$. By hypothesis, there exists $U \in GL_n(R)$ such that $UEU^{-1} = \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}$. Thus, $P \cong (nR)U\begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}U^{-1} \cong mR$, as desired. □

A ring R is said to be connected if it has only trivial idempotents, i.e, $e = e^2 \in R \Longrightarrow e = 0$ or 1.

Corollary 2.4.8. *A commutative ring R is projective-free if and only if*

(1) *R is connected;*
(2) *Every idempotent matrix over R admits a diagonal reduction.*

Proof. $\Longrightarrow$ Let $0 \neq e \in R$ be an idempotent. Then Re is a finitely generated projective R-module, and so Re is free. Thus, $Re \cong R$. This implies that $e = 1$, and so R is connected. In view of Theorem 2.4.7, every idempotent matrix over R admits a diagonal reduction.

$\Longleftarrow$ Let $E = E^2 \in M_n(R)$. The there exists $U \in GL_n(R)$ such that $UEU^{-1} = diag(r_1, \ldots, r_n)$. Hence, $r_1, \ldots, r_n \in R$ are idempotents. As R is connected, $r_1, \ldots, r_n$ are 0 or 1. We infer that E is similar to $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$. Accordingly, the result follows by Theorem 2.4.7. □

Corollary 2.4.9. *A projective-free ring R is clean if and only if it is local.*

Proof. Suppose R is a clean projective-free ring. Then R is connected, by Corollary 2.4.8. Thus, for any $a \in R$, $a \in U(R)$ or $a-1 \in U(R)$. Hence, R is local. The converse is obvious. □

Lemma 2.4.10. *Let R be a ring, and let $e, f \in R$ be idempotents. Then the following are equivalent:*

(1) *There exists $u \in U(R)$ such that $ufu^{-1} = e$.*
(2) *There exist $v, w \in U(R)$ such that $vfw = e$.*

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1) Set $y = vfv^{-1}$, and $x = 1 - e + y$. Then $ye = e, y = ew^{-1}v^{-1}$ and $x^{-1} = 1 + e - y$. Since $evf = eyv = yv = vf$, we see that $xvf = (1 - e + y)vf = vf$. Moreover, $exv = yv = vf$, and so $exv = xvf$. Let $u = xv$. Then $ufu^{-1} = e$. □

Let R be a commutative ring, and let $A \in M_n(R)$. Then a column vector α in $M_{n\times 1}(R)$ is a characteristic vector with characteristic value $\lambda \in R$ if (a) there is an invertible matrix in $M_n(R)$ whose first column is α; (b) $A\alpha = \lambda\alpha$.

Theorem 2.4.11. *Let R be a commutative ring. Then the following are equivalent:*

(1) *R is projective-free.*
(2) *Every nonzero idempotent matrix over R has a characteristic vector with characteristic value 1.*

Proof. (1) $\Rightarrow$ (2) Let $0 \neq A = A^2 \in M_n(R)$. In view of Theorem 2.4.7, there exists $U \in GL_n(R)$ such that $UAU^{-1} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ $(r > 0)$; hence,

$$AU^{-1}(1, 0, \ldots, 0)^T = U^{-1} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} (1, 0, \ldots, 0)^T.$$

Set $\alpha = U^{-1}(1, 0, \ldots, 0)^T$. Then $A\alpha = \alpha$. On the other hand, α is the first column of the invertible matrix U^{-1}. Thus, A has a characteristic vector α with characteristic value 1.

(2) $\Rightarrow$ (1) Let $0 \neq A = A^2 \in M_n(R)$. Then A has a characteristic vector α with characteristic value 1. Write $Q_1 = (\alpha, \alpha_2, \ldots, \alpha_n)^{-1} \in GL_n(R)$. Then

$AQ_1^{-1} = (\alpha, *, \ldots, *)$, and so

$$Q_1 A Q_1^{-1} = \begin{pmatrix} 1 & b_{21} & \cdots & b_{n1} \\ 0 & b_{22} & \cdots & b_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & b_{2n} & \cdots & b_{nn} \end{pmatrix}.$$

It is easy to check that $B := \begin{pmatrix} b_{22} & \cdots & b_{n2} \\ \vdots & \vdots & \vdots \\ b_{2n} & \cdots & b_{nn} \end{pmatrix} \in M_{n-1}(R)$ is an idempotent.

Moreover, $Q_1 A P_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & b_{22} & \cdots & b_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & b_{2n} & \cdots & b_{nn} \end{pmatrix}$, where

$$P_1 = Q_1^{-1} \begin{pmatrix} 1 & -b_{21} & \cdots & -b_{n1} \\ 0 & 1 & \cdots & b_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

If $B = 0$, then the result follows. If not, there exist $Q_2, P_2 \in GL_{n-1}(R)$ such that

$$Q_2 \begin{pmatrix} b_{22} & \cdots & b_{n2} \\ \vdots & \vdots & \vdots \\ b_{2n} & \cdots & b_{nn} \end{pmatrix} P_2 = \begin{pmatrix} 1 & & \\ & * \cdots * & \\ & \vdots \ddots \vdots & \\ & * \cdots * & \end{pmatrix}.$$

Hence, we get

$$\begin{pmatrix} 1 & \\ & Q_2 \end{pmatrix} Q_1 A P_1 \begin{pmatrix} 1 & \\ & P_2 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & * \cdots * & \\ & & \vdots \ddots \vdots & \\ & & * \cdots * & \end{pmatrix}.$$

By iteration of this process, we can find some $U, V \in GL_n(R)$ such that $UAV = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$. This establishes the claim by Lemma 2.4.10 and Theorem 2.4.7. □

A commutative ring is a Bézout ring if every finitely generated ideal is principal. Any principal ideal domain is a Bézout domain. Examples of Bézout domains that are not principal ideal domains include the valuation ring, the ring of entire functions (functions holomorphic on the whole complex plane) and the ring $\overline{\mathbb{Z}}$ of all algebraic integers.

Theorem 2.4.12. *Every Bézout domain is projective-free.*

Proof. Let R be a Bézout domain. Then R is connected. Given $a_1R + \cdots + a_nR = dR$, we will prove that there exists a matrix with the first row $a_1, \ldots, a_n$ and the determinant d. If $n = 1$, it is obvious. Assume that the result holds for $n = k$. Given $a_1R + \cdots + a_kR + a_{k+1}R = dR$, there is a $c \in R$ such that $a_1R + \cdots + a_kR = cR$. If $c = 0$, then $a_1 = \cdots = a_k = 0$ and $a_{k+1}R = dR$. Write $a_{k+1}r = d$. Then

$$\begin{vmatrix} a_1 & a_2 & \cdots & a_k & a_{k+1} \\ (-1)^k r & 0 & \cdots & a_k & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & 1 & 0 \end{vmatrix} = d.$$

Assume now that $c \neq 0$. As $cR + a_{k+1}R = dR$, we have $p, q \in R$ such that $pc - qa_{k+1} = d$. By hypothesis, there exists a matrix B with first row $a_1, \ldots, a_k$ and $det(B) = c$. Further, we can find $x_1, \ldots, x_k \in R$ such that $a_1 = cx_1, \ldots, a_k = cx_k$. Write

$$A = \begin{pmatrix} & & & a_{k+1} \\ & B & & 0 \\ & & & \vdots \\ & & & 0 \\ x_1q & \cdots & x_kq & p \end{pmatrix}.$$

Then

$$cdet(A) = \begin{vmatrix} & & & a_{k+1} \\ & B & & 0 \\ & & & \vdots \\ & & & 0 \\ a_1q & \cdots & a_kq & pc \end{vmatrix} = \begin{vmatrix} & & & a_{k+1} \\ & B & & 0 \\ & & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{vmatrix} = cd,$$

and so $det(A) = d$.

For any $0 \neq A = A^2 \in M_n(R)$, we may assume that there is a nonzero entry of the first row of A. Write $A = (\alpha, \alpha_2, \ldots, \alpha_n)$. Then $A\alpha = \alpha$. Write $\alpha = (e_1, \ldots, e_n)^T$. Then we have some $0 \neq \delta \in R$ such that $e_1R + \cdots + e_nR = \delta R$; hence, $e_1 = \delta x_1, \ldots, e_n = \delta x_n$ for $x_1, \ldots, x_n \in R$. Moreover, there exist $y_1, \ldots, y_n \in R$ such that $e_1y_1 + \cdots + e_ny_n = \delta$, and so $\delta\left(\sum_{i=1}^{n} x_iy_i\right) = \delta$. Thus, $x_1R + \cdots + x_nR = R$. By the preceding discussion, there exists a matrix with the first row $x_1, \ldots, x_n$ and the determinant 1. This implies that $\beta := (x_1, \ldots, x_n)^T$ is the first column of an invertible matrix. Then $A\alpha = \delta A\beta = \alpha = \delta\beta$, and then $A\beta = \beta$. In other words, A has a characteristic vector with characteristic value 1. This establishes the claim by Theorem 2.4.11. □

Corollary 2.4.13. *Every principal ideal domain is projective-free.*

Proof. It is clear from Theorem 2.4.12. □

Let $R = \mathbb{Z}\left[\sqrt{-3}\right]$. Then R is projective-free. In this case, R is not a principal domain, even it is not a Dedekind domain [474, Example 4.1]. If R is a principal ideal domain, it follows from the Qullen-Suslin's Theorem that the polynomial ring $R[x_1, \ldots, x_n]$ is projective-free [456, Theorem 4.63]. Lequain and Simis extended Qullen-Suslin's Theorem and proved that if R is a Bézout domain then $R[x_1, \ldots, x_n]$ is projective-free [388, Theorem b]. Furthermore, we have

Proposition 2.4.14. *Let R be a Bézout domain with $char(R) \neq 0$, and let G be a finitely generated abelian group. Then the following are equivalent:*

(1) *RG is projective-free.*
(2) *If G has an element of order p^s(p is prime , $s \in \mathbb{N}$), then $p \notin U(R)$.*

Proof. (1) $\Rightarrow$ (2) Since RG is projective-free, it is connected, by Corollary 2.4.8. Assume that there is an element $g \in G$ of order p^s, where p is prime, $s \in \mathbb{N}$ and $p \in U(R)$. Choose $e = \sum_{i=0}^{p^s-1} p^{-s} \cdot g^i$. Then $e \neq 0 \cdot 1, 1_R \cdot 1$. Observing that $(1-g)e = p^{-s}(1-g)\left(1+g+\cdots+g^{p^s-1}\right) = p^{-s}\left(1-g^{p^s}\right) = 0$, we get $ge = e$; hence, $e^2 = p^{-s}\left(1+g+\cdots+g^{p^s-1}\right)e = e$. Thus, $e \in RG$ is a nontrivial idempotent, which is impossible. If G has an element of order p^s(p is prime , $s \in \mathbb{N}$), we conclude that $p \notin U(R)$.

(2) $\Rightarrow$ (1) Write $G = G_0 \oplus Tor(G)$, where G_0 is torsion-free. Since R is a Bézout domain, by [388, Theorem b], RG_0 is projective-free. As

$char(R) \neq 0$ and R is an integral domain, $char(R) = q$ for a prime q. If G has an element of order $p^s (p$ is prime, $s \in \mathbb{N})$, then $p \notin U(R)$. If $p \neq q$, then $(p^s, q) = 1$. Hence, we can find some $k, l \in \mathbb{Z}$ such that $kp^s + lq = 1$. This shows that $(k \cdot 1_R)(p^s \cdot 1_R) = 1_R$. Hence, $p \in U(R)$, an absurd. Thus, $p = q$. This implies that $Tor(G) = 0$ or a q-group. If $Tor(G) = 0$, then $RG \cong RG_0$ is projective-free. If $Tor(G)$ is a q-group, it follows from $char(RG_0) = q$ and Proposition 2.4.5 that $RG \cong (RG_0)Tor(G)$ is projective-free. □

We now turn to determine when a matrix over projective-free rings is strongly clean.

Theorem 2.4.15. *Let R be a projective-free ring. Then $A \in M_n(R)$ is strongly clean if and only if either $A \in GL_n(R)$, or $I_n - A \in GL_n(R)$, or A is similar to a matrix $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $\alpha \in GL_m(R), \beta \in I_{n-m} + GL_{n-m}(R)$.*

Proof. If either A or $I_n - A$ is invertible, then A is strongly clean. For any $\alpha \in GL_m(R), \beta \in I_{n-m} + GL_{n-m}(R)$, we show that $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-m} \end{pmatrix} + \begin{pmatrix} \alpha & 0 \\ 0 & \beta - I_{n-m} \end{pmatrix}$ is strongly clean, as needed.

Conversely, assume that $A \in M_n(R)$ is strongly clean. Then there exists an idempotent $E \in M_n(R)$ and an invertible $U \in M_n(R)$ such that $A = E + U$ with $EU = UE$. Suppose that $A, I_n - A \notin GL_n(R)$. Since R is projective-free, by virtue of Theorem 2.4.7, we have $V \in GL_n(R)$ such that $VEV^{-1} = diag(0, I_{n-m})$. Therefore $VAV^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-m} \end{pmatrix} + VUV^{-1}$. Set $W = \begin{pmatrix} \alpha & \gamma \\ \delta & \beta - I_{n-m} \end{pmatrix} := VUV^{-1}$. It follows from $EU = UE$ that $\begin{pmatrix} 0 & 0 \\ 0 & I_{n-m} \end{pmatrix} W = W \begin{pmatrix} 0 & 0 \\ 0 & I_{n-m} \end{pmatrix}$. Hence, $\gamma = 0$ and $\delta = 0$ and $\alpha \in GL_m(R), \beta \in I_{n-m} + GL_{n-m}(R)$. Therefore A is similar to $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, as desired. □

Corollary 2.4.16. *Let R be a Bézout domain. Then $A \in M_2(R)$ is strongly clean if and only if either $A \in GL_2(R)$, or $I_2 - A \in GL_2(R)$, or A is similar to a matrix $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $\alpha \in U(R), \beta \in 1 + U(R)$.*

Proof. In view of Theorem 2.4.12, R is projective-free. Therefore we complete the proof by Theorem 2.4.15. □

Example 2.4.17. Let $A \in M_2(\mathbb{Z})$. Then A is strongly clean if and only if $A \in GL_2(\mathbb{Z})$, or $I_2 - A \in GL_2(\mathbb{Z})$, or A is similar to one of the matrices in the set $\left\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\right\}$.

Proof. Clearly, $\mathbb{Z}$ is a principal ideal domain with units ± 1. Thus the proof is true from Corollary 2.4.16. □

Notes. Theorem 2.1.2 and Theorem 2.1.5 were proved by Diesl and Dorsey [236, Theorem 4.4 and Theorem 4.6]. Theorem 2.2.3 is due to Yang [546, Theorem 3.1.1]. Theorem 2.2.9 was proved by Diesl and Dorsey [236, Theorem 4.4]. Theorem 2.2.13 was proved by Borooah et al. [78, Corollary 21]. Theorem 2.4.15 is essentially due to Yang and Zhou [549, Lemma 2].

This page intentionally left blank

Chapter 3

Triangular and Generalized Matrix Rings

We use $T_n(R)$ to denote the ring of all $n \times n$ upper triangular matrices over the ring R. The main purpose of this chapter is to study strongly clean properties of such rings. In Section 3.1 and 3.2, we investigate sufficient and necessary conditions on a ring R under which $T_n(R)$ is strong clean. In Section 3.3 we discuss triangular matrix rings over commutative rings and describe relatively strong cleanness with respect to full matrix rings. Finally, in the last section, we determine the strong cleanness for generalized matrix rings and enrich the supply of strongly clean rings.

3.1 Sufficient Conditions

We begin with an elementary result due to [236, Lemma 2.2], which will be used repeatedly.

Lemma 3.1.1. *Let R be a ring, $n \geq 2$ and $A, E \in T_n(R)$. Suppose that for all $(i,j) \neq (1,n)$, $(E^2)_{ij} = E_{ij}$ and $\big(AE - EA\big)_{ij} = 0$. Write*

$$A = \begin{pmatrix} a & \alpha & c \\ & B & \beta \\ & & b \end{pmatrix} \text{ and } E = \begin{pmatrix} e & \gamma & z \\ & F & \delta \\ & & f \end{pmatrix},$$

where $B, F \in T_{n-2}(R), a, b, c, e, f, z \in R, \alpha, \gamma \in M_{1,n-2}(R)$ and $\beta, \delta \in M_{n-2,1}(R)$. Then the following are equivalent:

(1) $AE = EA$ *and* $E^2 = E$.
(2)

$$\begin{aligned} ezf &= -e\gamma\delta f; \\ (1-e)z(1-f) &= (1-e)\gamma\delta(1-f); \\ (l_a - r_b)(l_e - r_f)^2(z) &= (l_e - r_f)(c) + (l_e - r_f)^2(\gamma\beta - \alpha\delta). \end{aligned}$$

Proof. $\Longrightarrow$ Obviously, $AE = EA$ if and only if $(AE)_{1,n} = (EA)_{1,n}$, if and only if $az + \alpha\delta + cf = ec + \gamma\beta + zb$. This is equivalent to $(l_a - r_b)(z) = (l_e - r_f)(c) + \gamma\beta - \alpha\delta$. Clearly,

$$(l_e - r_f)^2(l_a - r_b) = (l_a - r_b)(l_e - r_f)^2 \text{ and } (l_e - r_f)^3 = l_e - r_f.$$

Thus, we get

$$\begin{aligned} (l_a - r_b)(l_e - r_f)^2(z) &= (l_e - r_f)^2(l_a - r_b)(z) \\ &= (l_e - r_f)(c) + (l_e - r_f)^2\big(\gamma\beta - \alpha\delta\big). \end{aligned}$$

On the other hand, $E = E^2$ if and only if $(E)_{1,n} = (E^2)_{1,n}$, if and only if $ez + \gamma\delta + zf = z$. Let $Q = ez + \gamma\delta + zf - z$. Set $e' = 1 - e$ and $f' = 1 - f$. Then we easily check that

$$l_e r_f + l_e r_{f'} + l_{e'} r_f + l_{e'} r_{f'} = 1 \text{ and } l_e r_{f'} + l_{e'} r_f = (l_e - r_f)^2.$$

Hence, $l_e r_f(Q) = ezf + e\gamma\delta f = 0$, and so $ezf = -e\gamma\delta f$. Likewise, $e'zf' = e'\gamma\delta f'$ from $l_{e'} r_{f'}(Q) = 0$.

$\Longleftarrow$ As $(AE - EA)_{ij} = 0$ for $(i,j) \neq (1,n)$, we get

$$\begin{aligned} ae &= ea, \\ bf &= fb, \\ BF &= FB, \\ \alpha\gamma + \alpha F &= e\alpha + \gamma B, \\ B\delta + \beta f &= F\beta + \delta b. \end{aligned}$$

As $(E^2)_{ij} = (E)_{ij}$ for $(i,j) \neq (1,n)$, we get

$$\begin{aligned} e^2 &= e, \\ f^2 &= f, \\ F^2 &= F, \\ e\gamma + \gamma F &= \gamma, \\ F\delta + \delta f &= \delta. \end{aligned}$$

Set $P = ez + \gamma\delta + zf - z, e' = 1 - e$ and $f' = 1 - f$. By hypothesis, we have $l_e r_f(P) = l_{e'} r_{f'}(P) = 0$. Furthermore, $l_e r_{f'}(P) = (e\gamma)\delta f' = (\gamma - \gamma F)\delta f' = \gamma\delta f' - \gamma(F\delta)f' = \gamma\delta f' - \gamma(\delta - \delta f)f' = 0$. Similarly, $l_{e'} r_f(P) = 0$. We infer that $P = 0$, and so $ez + \gamma\delta + zf = z$. This implies that $E = E^2$.

Set $Q = (l_a - r_b)(z) - (l_e - r_f)(c) - \gamma\beta + \alpha\delta$. We will suffice to prove that $l_e r_f(Q), (l_e - r_f)^2(Q)$ and $l_{e'} r_{f'}(Q)$ are all zero.

Step 1. Clearly, $l_e r_f(Q) = (l_a - r_b)(ezf) + e\alpha\delta f - e\gamma\beta f = (-ae\gamma\delta f + e\alpha\delta f) - (-e\gamma\delta fb + e\gamma\beta f)$. As $ae = ea, \alpha\gamma + \alpha F = e\alpha + \gamma B$ and $F\delta f = 0$, we get $-ae\gamma\delta f + e\alpha\delta f = e(-a\gamma + e\alpha)\delta f = e(\alpha F - \gamma B)\delta f = -e\gamma B\delta f$. Analogously, we check that $-e\gamma\delta fb + e\gamma\beta f = -e\gamma B\delta f$. Therefore $l_e r_f(Q) = 0$.

Step 2. By hypothesis, $(l_a - r_b)(l_e - r_f)^2(z) = (l_e - r_f)(c) + (l_e - r_f)^2(\gamma\beta - \alpha\delta)$. Since $(l_e - r_f)^2(l_a - r_b) = (l_a - r_b)(l_e - r_f)^2$ and $(l_e - r_f)^3 = l_e - r_f$, we get $(l_e - r_f)^2(Q) = 0$.

Step 3. As in the proof in Step 2, we easily check that $l_{e'} r_{f'}(Q) = 0$.

Therefore $Q = 0$, and then $AE = EA$, the result follows. □

As a further consequence of Lemma 3.1.1, we have

Lemma 3.1.2. *Let A and E be as in Lemma 3.1.1. Then*

(1) *If $e = f = 1$, then $E = E^2$ if and only if $z = -\gamma\delta$, and in this case, $AE = EA$.*
(2) *If $e = f = 0$, then $E = E^2$ if and only if $z = \gamma\delta$, and in this case, $AE = EA$.*
(3) *If $e = 1$ and $f = 0$, then $E = E^2$ and $AE = EA$ if and only if $az - zb = \gamma\beta - \alpha\delta + c$.*
(4) *If $e = 0$ and $f = 1$, then $E = E^2$ and $AE = EA$ if and only if $az - zb = \gamma\beta - \alpha\delta - c$.*

The following observation is crucial (see [233, Lemma 2.1.6] and [241, Lemma 4.1.8]).

Lemma 3.1.3. *Let R be a ring and suppose that $A = (a_{ij}) \in T_n(R)$. Then for any set $\{e_{ii}\}$ of idempotents 0 and 1 in R such that $e_{ii} = e_{jj}$ whenever $l_{a_{ii}} - r_{a_{jj}}$ is not surjective, there exists an idempotent $E \in T_n(R)$ such that $AE = EA$ and $E_{ii} = e_{ii}$ for every $i \in \{1, \ldots, n\}$.*

Proof. If $n = 1$, the result is true. Write the idempotent E as (e_{ij}). Suppose that for some $k \geq 0$, we have defined e_{ij} for all $i \leq j \leq i + k$ so that for all $i = 1, \ldots, n - k$, $\begin{pmatrix} a_{ii} & \cdots & a_{i,i+k} \\ & \ddots & \vdots \\ & & a_{i+k,i+k} \end{pmatrix}$ commutes with $\begin{pmatrix} e_{ii} & \cdots & e_{i,i+k} \\ & \ddots & \vdots \\ & & e_{i+k,i+k} \end{pmatrix}$.

By induction, for each $i = 1, \ldots, n-k-1$, applying Lemma 3.1.2 to $A' = \begin{pmatrix} a_{ii} & \cdots & a_{i,i+k+1} \\ & \ddots & \vdots \\ & & a_{i+k+1,i+k+1} \end{pmatrix}$ and $\begin{pmatrix} e_{ii} & \cdots & e_{i,i+k+1} \\ & \ddots & \vdots \\ & & e_{i+k+1,i+k+1} \end{pmatrix}$.

If $e_{ii} = e_{i+k+1,i+k+1}$, it follows by Lemma 3.1.2 that we can define $e_{i,i+k}$ so that A' commutes with E'. If $e_{ii} \neq e_{i+k+1,i+k+1}$, by hypothesis, we see that $l_{a_{ii}} - r_{a_{i+k+1,i+k+1}}$ is surjective. By using Lemma 3.1.2 again, we can define $e_{i,i+k+1}$ so that A' commutes with E'. The inductive step is completed.

Choose $k = n-1$, we prove that A commutes with E and $E_{ii} = e_{ii}$, as asserted. □

Following Borooah et al., we use $Bl(X)$ to denote the set $\{a \in R \mid l_a - r_b, l_b - r_a$ are both surjective for all $b \in X\}$.

Theorem 3.1.4. *Let R be a ring, and let $A = (a_{ij}) \in T_n(R)$. Suppose $\{a_{ii}\} \subseteq X \cup Y$ where $X \subseteq U(R), Y \subseteq \big(1+U(R)\big)\backslash U(R)$. If $Y\backslash X \subseteq Bl(X)$, then $A \in T_n(R)$ is strongly clean.*

Proof. Choose $e_{ii} = 0$ whenever $a_{ii} \in X$, and $e_{ii} = 1$ whenever $a_{ii} \in Y\backslash X$. Thus we get a set $\{e_{ii}\}$ of idempotents 0 and 1 of R. Whenever $l_{a_{ii}} - r_{a_{jj}}$ is not surjective, we see that $a_{ii}, a_{jj} \in X$ or $a_{ii}, a_{jj} \in Y\backslash X$. This implies that $e_{ii} = e_{jj}$. In light of Lemma 3.1.3, there exists an idempotent $E \in T_n(R)$ such that $AE = EA$ and $E_{ii} = e_{ii}$ for every $i \in \{1, \ldots, n\}$. Since each $(A-E)_{ii} \in U(R)$, we easily checks that $A - E \in T_n(R)$ is invertible, as required. □

Following Diesl, a ring R is bleached provided that for any $a \in U(R), b \in J(R)$, $l_a - r_b$ and $l_b - r_a$ are both surjective. Thus, a local ring R is bleached if and only if $J(R) \subseteq Bl\big(U(R)\big)$.

Corollary 3.1.5. *Let R be a bleached local ring. Then $T_n(R)$ is strongly clean.*

Proof. Choose $X = U(R)$ and $Y = J(R) \subseteq \big(1 + U(R)\big)\backslash U(R)$. Then $Y\backslash X \subseteq Bl(X)$, and therefore we complete the proof by Theorem 3.1.4. □

Corollary 3.1.6. *Let R be a commutative local ring. Then $T_n(R)$ is strongly clean.*

Proof. It is obvious as every commutative local ring is bleached. □

Proposition 3.1.7. *Let R be a ring, and let $A = (a_{ij}) \in T_n(R)$. If $\{a_{ii}\} \subseteq U(R) \cup N(R)$, then $A \in T_n(R)$ is strongly clean.*

Proof. Choose $X = U(R)$ and $Y = N(R)$. Then $\{a_{ii}\} \subseteq X \cup Y$ where $X \subseteq U(R), Y \subseteq \big(1 + U(R)\big)\backslash U(R)$. Let $a \in X$ and $b \in Y$. Say $b^n = 0$. Choose $\varphi = l_{a^{-1}} + l_{a^{-2}}r_b + \cdots + l_{a^{-n}}r_{b^{n-1}} : R \to R$. Then $\big(l_a - r_b\big)\varphi = \varphi\big(l_a - r_b\big) = 1_R$. Thus, $l_a - r_b$ is surjective. Likewise, $l_b - r_a$ is surjective. Therefore $Y\backslash X \subseteq Bl(X)$. According to Theorem 3.1.4, we establish the result. □

Corollary 3.1.8. *Let R be a local ring. If $J(R)$ is nil, then $T_n(R)$ is strongly clean.*

Proof. As R is local, we see that $R = U(R) \cup J(R)$, and so $R = U(R) \cup N(R)$. Therefore the result follows by Proposition 3.1.7. □

Example 3.1.9. Let $A = (a_{ij}) \in T_n(\mathbb{Z})$. If each a_{ii} is $-1, 0$ or 1, then $A \in T_n(\mathbb{Z})$ is strongly clean.

Proof. Obviously, $\{a_{ii}\} = \{-1, 0, 1\} \subseteq U(R) \cup N(R)$. Hence the result follows by Proposition 3.1.7. □

We say that idempotents $e_1, \ldots, e_n \in R$ are orthogonal if $e_ie_j = 0 (i \neq j)$. A finite orthogonal set of idempotents $e_1, \ldots, e_n$ in a ring R is said to be complete in case $e_1 + \cdots + e_n = 1$.

Lemma 3.1.10. *Let I be an ideal of a ring R with $I \subseteq J(R)$. Then the following are equivalent:*

(1) *Idempotents lift modulo I.*
(2) *Every complete finite orthogonal set of idempotents in R/I lifts to a complete orthogonal set of idempotents in R.*

Proof. (1) $\Rightarrow$ (2) We claim that n numbers of orthogonal set of idempotents in R/I lifts to an orthogonal set of idempotents in R for any $n \in \mathbb{N}$. This is clear for $n = 1$. Suppose this holds for $n (n \in \mathbb{N})$. Let $\{u_1, \ldots, u_n, u_{n+1}\}$ be a orthogonal set of idempotents in R/I. Then we have orthogonal idempotents $e_1, \ldots, e_n \in R$ such that each $\overline{e_i} = u_i$. Set $\alpha = e_1 + \cdots + e_n$. Let β be the idempotent lifting u_{n+1}. Then $\overline{\alpha}$ and $\overline{\beta}$ are orthogonal; hence, $1 - \beta\alpha \in U(R)$. Set $e_{n+1} = (1-\alpha)\big(1-\beta\alpha\big)^{-1}\beta\big(1-\beta\alpha\big)$. Then $e_{n+1}^2 = e_{n+1}, \overline{e_{n+1}} = \overline{\beta}$ and that α and e_{n+1} orthogonal. This implies that $e_1, \ldots, e_n, e_{n+1} \in R$ are orthogonal and $\overline{e_{n+1}} = u_{n+1}$. By induction, the claim is true.

Write $1 = u_1 + \cdots + u_m$ with orthogonal idempotents $u_i \in R/J(R)$. Then we have orthogonal idempotents e_i such that each $\overline{e_i} = u_i$. Hence, $1 - (e_1 + \cdots + e_m) \in J(R)$, and so $e_1 + \cdots + e_m \in U(R)$. We infer that $1 = e_1 + \cdots + e_m$, as required.

(2) $\Rightarrow$ (1) is trivial. □

Lemma 3.1.11. *Let $e \in R$ be an idempotent. Then the following are equivalent:*

(1) *$eR/eJ(R)$ is simple.*
(2) *eRe is a local ring.*

Proof. (1) $\Rightarrow$ (2) Let $\pi : eR \to eR/eJ(R)$ be the natural epimorphism. Then for any $\varphi : eR \to eR$, there exists a unique $\varphi^* : eR/eJ(R) \to eR/eJ(R)$ such that the diagram

$$\begin{array}{ccc} eR & \overset{\pi}{\twoheadrightarrow} & eR/eJ(R) \\ \varphi \downarrow & & \downarrow \varphi^* \\ eR & \overset{\pi}{\twoheadrightarrow} & eR/eJ(R) \end{array}$$

commutes, i.e., $\varphi^*\pi = \pi\varphi$. Construct a map $\phi : End_R(eR) \to End_R\big(eR/eJ(R)\big)$ given by $\varphi \mapsto \varphi^*$. As eR is projective, ϕ is epimorphic. Hence, $End_R(eR)/Ker(\phi) \cong End_R\big(eR/eJ(R)\big)$. But $Ker(\phi) = Hom_R\big(eR, eJ(R)\big) = J\big(End_R(eR)\big)$. Thus,

$$End_R\big(eR/eJ(R)\big) \cong End_R(eR)/J\big(End_R(eR)\big) \cong eRe/J\big(eRe\big).$$

By hypothesis, $eR/eJ(R)$ is simple, and so $End_R\big(eR/eJ(R)\big)$ is a division ring by Schur's Lemma. Therefore eRe is local.

(2) $\Rightarrow$ (1) By hypothesis, $eRe/J(eRe) \cong \overline{e}S\overline{e}$ is a division ring, where $S = R/J(R)$. For any $0 \neq \overline{e}x \in \overline{e}S$, we see that $\overline{e}xS\overline{e}x \neq 0$ as $J(S) = 0$. Write $\overline{e}x\overline{y}\overline{e} \neq 0$ for some $\overline{y} \in S$. Then we have a $\overline{z} \in S$ such that $\overline{e}x\overline{y}\overline{e}\overline{z} = e$. This shows that $\overline{e}xR \subseteq \overline{e}S \subseteq \big(\overline{e}x\big)yezR \subseteq \overline{e}xR$; whence, $\overline{e}xR = \overline{e}S$. Accordingly, $eR/eJ(R) \cong \overline{e}S$ is a simple R-module, as asserted. □

A ring R is semilocal if $R/J(R)$ is artinian. As is well known, a commutative ring R is semilocal if and only if R has a finite number of maximal ideals. A ring R is called semipecfect if R is semilocal, and every idempotent lifts modulo $J(R)$. Examples of semiperfect rings include: left (right) perfect rings; local rings; right (left) artinian rings and finitely dimensional algebras over fields. The following result shows that semiperfect rings are generalized matrix rings over local rings.

Lemma 3.1.12. *Let R be a ring. Then R is semiperfect if and only if there exist orthogonal idempotents $e_1, \ldots, e_n \in R$ such that $1 = e_1 + \cdots + e_n$, and each $e_i R e_i$ is local.*

Proof. Suppose R is semipecfect. Then $R/J(R)$ is semisimple. By using the Wedderburn-Artin Theorem, $R/J(R)$ is the direct sum of finite numbers of ideals, each of which is isomorphic to a complete ring of matrices of finite order over a division ring. Thus, we have a decomposition $\overline{1} = u_1 + \cdots + u_n$ where $u_i \in R/J(R)$ are mutually orthogonal idempotents, and that each $u_i\big(R/J(R)\big)$ is a minimal right ideal of $R/J(R)$. In view of Lemma 3.1.10, there are orthogonal idempotents $e_i \in R$ such that $1 = e_1 + \cdots + e_n$ with each $\overline{e_i} = u_i$. Clearly, $e_i R e_i / J\big(e_i R e_i\big) \cong u_i\big(R/J(R)\big)u_i \cong End_{R/J(R)}\big(u_i(R/J(R))\big)$ is a division ring. Hence, $e_i R e_i$ is local, as desired.

Conversely, assume that there exist orthogonal idempotents $e_1, \ldots, e_n \in R$ such that $1 = e_1 + \cdots + e_n$, and each $e_i R e_i$ is local. Then $R \cong \bigoplus_{i=1}^{n} e_i R$ as right R-modules. In view of Lemma 3.1.11, $eR/e_i J(R)$ is simple. Therefore, $R/J(R) \cong R \bigotimes_R R/J(R) \cong \bigoplus_{i=1}^{n} e_i R \bigotimes_R R/J(R) \cong \bigoplus_{i=1}^{n} \big(e_i R/e_i J(R)\big)$. Hence, $R/J(R)$ is semisimple, i.e., R is semilocal. Since $R \cong e_1 R e_1 \times \cdots \times e_n R e_n$ and each $e_i R e_i$ is local, we easily check that idempotents lift modulo $J(R)$. Therefore R is semiperfect. □

It is convenient at this stage to include the following proposition (see [546, Corollary 4.1.4]).

Proposition 3.1.13. *Let R be a commutative ring, and let $n \in \mathbb{N}$. Then R is semiperfect if and only if*

(1) *R is semilocal.*
(2) *$T_n(R)$ is strongly clean.*

Proof. $\Longrightarrow$ Clearly, R is semilocal. It follows from Lemma 3.1.12 that there are orthogonal idempotents $e_1, \ldots, e_n \in R$ such that $1 = e_1 + \cdots + e_n$ with each $e_i R e_i$ is local. Since R is commutative, $T_n(R) \cong \bigoplus_{i=1}^{n} T_n\big(e_i R e_i\big)$. In light of Corollary 3.1.6, each $T_n\big(e_i R e_i\big)$ is strongly clean, as required.

$\Longleftarrow$ In view of Theorem 1.1.12, R is strongly clean; hence, it is an exchange ring. By virtue of [430, Corollary 2.4], every idempotent lifts modulo $J(R)$, and we therefore complete the proof. □

Let R be a local ring. Define the sets J_n recursively as follows. $J_0 = \{0\}, J_{n+1} = Bl\big(1 - J_n\big)\backslash Bl\big(J_n\big)$. Clearly, $J_1 = J(R)$. Let $A \in T_n(R)$. We use $\widehat{A}$ to denote the set $\{A_{11}, \ldots, A_{nn}\}$ of all diagonal entries of A.

Lemma 3.1.14. *Let R be a local ring, let $n \in \mathbb{N}$, and let $A \in T_n(R)$. Then the following hold:*

(1) *If $X \subseteq Y \subseteq R$, then $Bl(Y) \subseteq Bl(X)$.*
(2) *$J_n \cap \big(1 - J_n\big) = \emptyset$.*
(3) *If $J_{n-1} \subseteq J_n$ and $\widehat{A} \subseteq Bl\big(J_n\big) \cup Bl\big(1 - J_n\big)$, then $\widehat{A} \cap \big(1 - J_n\big) \subseteq Bl\big(J_n\big)$.*
(4) *If $J_0 \subseteq J_1 \subseteq \cdots \subseteq J_n$ and $\widehat{A} \subseteq Bl\big(J_n\big) \cup Bl\big(1 - J_n\big)$, then $\widehat{A} \subseteq Bl\big(J_i\big) \cup Bl\big(1 - J_i\big)$ for every $0 \leq i \leq n$.*

Proof. (1) For any $x \in Bl(Y)$ and $y \in X$, we have $y \in Y$, and so $l_x - r_y$ and $l_y - r_x$ are both surjective. Hence, $x \in Bl(X)$. Therefore, $Bl(Y) \subseteq Bl(X)$.

(2) Clearly, $J_n = Bl\big(1 - J_{n-1}\big)\backslash Bl\big(J_{n-1}\big)$. But $Bl\big(1 - J_{n-1}\big) = 1 - Bl\big(J_{n-1}\big)$. Hence, $1 - J_n = Bl\big(J_{n-1}\big)\backslash Bl\big(1 - J_{n-1}\big)$. Thus, $J_n \cap \big(1 - J_n\big) = \emptyset$.

(3) Since $J_{n-1} \subseteq J_n$, we get $1 - J_{n-1} \subseteq 1 - J_n$, and so $B\big(1 - J_n\big) \subseteq B\big(1 - J_{n-1}\big)$, by (1). But $1 - J_n = Bl\big(J_{n-1}\big)\backslash Bl\big(1 - J_{n-1}\big)$; hence, $\big(1 - J_n\big) \cap B\big(1 - J_n\big) = \emptyset$. It follows from $\widehat{A} \subseteq Bl\big(J_n\big) \cup Bl\big(1 - J_n\big)$ that $\widehat{A} \cap \big(1 - J_n\big) \subseteq Bl\big(J_n\big)$.

(4) Since $J_i \subseteq J_n$, we get $1 - J_i \subseteq 1 - J_n$. Hence, $Bl\big(J_n\big) \subseteq Bl\big(J_i\big)$ and $Bl\big(1 - J_n\big) \subseteq Bl\big(1 - J_i\big)$, by (1). It follows that

$$\widehat{A} \subseteq Bl\big(J_n\big) \cup Bl\big(1 - J_n\big) \subseteq Bl\big(J_i\big) \cup Bl\big(1 - J_i\big),$$

and therefore $\widehat{A} \subseteq Bl\big(J_i\big) \cup Bl\big(1 - J_i\big)$ for every $0 \leq i \leq n$. □

Lemma 3.1.15. *Let R be a local ring, $n \in \mathbb{N}$, and let $A \in T_n(R)$. If $\widehat{A} \subseteq Bl(J_{n-1}) \cup Bl(1 - J_{n-1})$, then $\widehat{A} \cap \big(R\backslash J_n\big) \subseteq Bl(J_{n-1})$.*

Proof. Suppose $\widehat{A} \subseteq Bl(J_{n-1}) \cup Bl(1 - J_{n-1})$. For any $a \in \widehat{A} \cap \big(R\backslash J_n\big)$, we see that $a \notin J_n$. Hence, $a \notin Bl\big(1 - J_{n-1}\big)$, or $a \in Bl\big(1 - J_{n-1}\big) \cap Bl\big(J_{n-1}\big)$. By hypothesis, we get $a \in Bl\big(J_{n-1}\big)$. Therefore $\widehat{A} \cap \big(R\backslash J_n\big) \subseteq Bl(J_{n-1})$, as asserted. □

Let $A \in T_n(R)$, and let $n \in \mathbb{N}$. Define the condition B_n on A as follows, depending on whether n is odd or even.

$$B_{2k+1}: \begin{cases} J_0 \subseteq J_1 \subseteq \cdots \subseteq J_k; \\ \widehat{A} \subseteq Bl\big(J_k\big) \cup Bl\big(1-J_k\big). \end{cases}$$

$$B_{2k}: \begin{cases} J_0 \subseteq J_1 \subseteq \cdots \subseteq J_k; \\ \widehat{A} \cap J_k \subseteq Bl\big(1-J_k\big); \\ \widehat{A} \subseteq Bl\big(J_{k-1}\big) \cup Bl\big(1-J_{k-1}\big). \end{cases}$$

We say that R satisfies the condition B_n provided that every $n \times n$ matrix over R satisfies the condition B_n. Following a similar route, we give a matrix version of [233, Theorem 2.2.7] and [241, Theorem 4.2.10].

Theorem 3.1.16. *Let R be a local ring, and let $A \in T_n(R)$. If A satisfies the condition B_n, then A is strongly clean.*

Proof. Let $A = (a_{ij}) \in T_n(R)$. The result is trivial for $n = 1$. Thus, we next consider $n \geq 2$.

(1) n is odd. Write $n = 2k + 1 (k \geq 1)$.

Case I. The set $R\backslash\big(J_k \cup (1-J_k)\big)$ contains two or more of the diagonal entries a_{ii}. In view of Lemma 3.1.14, $J_k, 1-J_k$ and $R\backslash\big(J_k \cup (1-J_k)\big)$ are mutually disjoint. Thus, each element not in $R\backslash\big(J_k \cup (1-J_k)\big)$ is contained in J_k or $1-J_k$. Let $K_i = J_i\backslash J_{i-1}$. Then $J_k = K_1 \sqcup K_2 \sqcup \cdots \sqcup K_k$ and $1-J_k = (1-K_1) \sqcup (1-K_2) \sqcup \cdots \sqcup (1-K_k)$. This implies that there is a least $1 \leq l \leq k$ such that $K_l = J_l\backslash J_{l-1}$ or $1-K_l = (1-J_l)\backslash(1-J_{l-1})$ contains no a_{ii}. Without loss of generality, we assume that K_l contains no a_{ii}. If $l = 1$, then each $a_{ii} \notin J(R)$, hence, $a_{ii} \in U(R)$. This implies that A is strongly clean in $T_n(R)$. In the next, we assume $l \geq 2$. As $a_{ii} \notin J_l\backslash J_{l-1}$, we see that $a_{ii} \in J_{l-1}$ or $a_{ii} \in R\backslash J_l$. In addition, $J_{l-1} \cap \big(R\backslash J_l\big) = \emptyset$. Choose $e_{ii} = 1$ if $a_{ii} \in J_{l-1}$ and $e_{ii} = 0$ if $a_{ii} \in R\backslash J_l$.

If $a_{ii} \in J_1$, then $a_{ii} \in J_{l-1}$, and so $a_{ii} - e_{ii} \in U(R)$. If $a_{ii} \in 1-J_1$, then $a_{ii} \in 1-J_{l-1}$. Hence, $a_{ii} \notin J_{l-1}$ as $J_{l-1} \cap \big(1-J_{l-1}\big) = \emptyset$. This implies that $a_{ii} \in R\backslash J_l$, and so $a_{ii} - e_{ii} \in U(R)$. If $a_{ii} \in R\backslash\big(J_1 \cup (1-J_1)\big)$, then $a_{ii} - 0, a_{ii} - 1 \in U(R)$. Thus, we always have $a_{ii} - e_{ii} \in U(R)$.

If $e_{ii} \neq e_{jj}$, then $a_{ii} \in J_{l-1}$ and $a_{jj} \in R\backslash J_l$ (or vice versa). It follows from Lemma 3.1.14 that $\widehat{A} \subseteq Bl\big(J_{l-1}\big) \cup Bl\big(1-J_{l-1}\big)$. According to Lemma 3.1.15, $a_{jj} \in Bl(J_{l-1})$, and so $a_{jj} \in Bl(a_{ii})$. Thus, we can find an idempotent $E \in T_n(R)$ such that $AE = EA$ with $E_{ii} = e_{ii}$, by Lemma 3.1.3. Further, $A - E \in U\big(T_n(R)\big)$. Therefore A is strongly clean.

Case II. The set $R\backslash\big(J_k\cup(1-J_k)\big)$ contains 1 or 0 of the diagonal entries A_{ii}. Choose $e_{ii}=1$ if $a_{ii}\in J_k$ and $e_{ii}=0$ if $a_{ii}\in 1-J_k$. If there exists i such that $a_{ii}\in R\backslash\big(J_k\cup(1-J_k)\big)$, then choose $e_{ii}=1$ if $a_{ii}\in Bl\big(1-J_k\big)$ and $e_{ii}=0$ otherwise.

If $a_{ii}\in J_1$, then $a_{ii}\in J_k$, and so $a_{ii}-e_{ii}\in U(R)$. If $a_{ii}\in 1-J_1$, then $a_{ii}\in 1-J_k$. Thus, $a_{ii}-e_{ii}\in U(R)$. If $a_{ii}\notin J_1\cup(1-J_1)$, we have $a_{ii}-0, a_{ii}-1\in U(R)$, and so $a_{ii}-e_{ii}\in U(R)$.

In view of Lemma 3.1.14, $\widehat{A}\cap\big(1-J_k\big)\subseteq Bl\big(J_k\big)$. If $e_{ii}\neq e_{jj}$, then

(i) $a_{ii}\in J_k$ and $a_{jj}\in 1-J_k$ or $a_{jj}\notin Bl\big(1-J_k\big)$. If $a_{jj}\in 1-J_k$, then $a_{jj}\in Bl\big(J_k\big)$. If $a_{jj}\notin Bl\big(1-J_k\big)$, then $a_{jj}\in Bl\big(J_k\big)$ as $\widehat{A}\subseteq Bl\big(J_k\big)\cup Bl\big(1-J_k\big)$. Therefore $a_{jj}\in Bl(a_{ii})$.

(ii) $a_{ii}\in 1-J_k$ and $a_{jj}\in J_k$ or $a_{jj}\in Bl\big(1-J_k\big)$. If $a_{jj}\in J_k$, then $a_{ii}\in Bl(a_{jj})$. If $a_{jj}\in Bl\big(1-J_k\big)$, then $a_{jj}\in Bl(a_{ii})$.

(iii) $a_{ii}\notin J_k\cup\big(1-J_k\big)$. Then $a_{jj}\in J_k$ or $a_{jj}\in 1-J_k$. As in the preceding discussion, $a_{jj}\in Bl(a_{ii})$.

In any case, we conclude that $l_{a_{ii}}-r_{a_{jj}}$ is surjective. In light of Lemma 3.1.3, we have an idempotent $E\in T_n(R)$ such that $AE=EA$ and $A-E\in U\big(T_n(R)\big)$, as required.

(2) n is even. Write $n=2k(k\geq 1)$.

Case I. $R/\big(J_k\cup(1-J_k)\big)$ contains 1 or more of the diagonal elements a_{ii}. As in the case $n=2k+1$, we have $J_k=K_1\sqcup K_2\sqcup\cdots\sqcup K_k$ and $1-J_k=(1-K_1)\sqcup(1-K_2)\sqcup\cdots\sqcup(1-K_k)$, where $K_i=J_i\backslash J_{i-1}$. There is a least $1\leq l\leq k$ such that $K_l=J_l\backslash J_{l-1}$ or $1-K_l=(1-J_l)\backslash(1-J_{l-1})$ contains no a_{ii}. Without loss of generality, we assume that K_l contains no a_{ii}. we assume $l\geq 2$, and that $a_{ii}\in J_{l-1}$ or $a_{ii}\in R\backslash J_l$. Choose $e_{ii}=1$ if $a_{ii}\in J_{l-1}$ and $e_{ii}=0$ if $a_{ii}\in R\backslash J_l$. Further, $a_{ii}-e_{ii}\in U(R)$.

If $e_{ii}\neq e_{jj}$, then $a_{ii}\in J_{l-1}$ and $a_{jj}\in R\backslash J_l$ (or vice versa). Applying Lemma 3.1.14, we get $\widehat{A}\subseteq Bl\big(J_{l-1}\big)\cup Bl\big(1-J_{l-1}\big)$. By using Lemma 3.1.15, $a_{jj}\in Bl(J_{l-1})$; hence, $a_{jj}\in Bl(a_{ii})$. Therefore we can find an idempotent $E\in T_n(R)$ such that $AE=EA, E_{ii}=e_{ii}$ and $A-E\in U\big(T_n(R)\big)$, as desired.

Case II. $R/\big(J_k\cup(1-J_k)\big)$ contains none of the diagonal elements a_{ii}. Choose $e_{ii}=1$ if $a_{ii}\in J_k$ and $e_{ii}=0$ if $a_{ii}\in 1-J_k$. If $a_{ii}\in J_1$, then $a_{ii}\in J_k$, and so $a_{ii}-e_{ii}\in U(R)$. If $a_{ii}\in 1-J_1$, then $a_{ii}\in 1-J_k$, and so $a_{ii}-e_{ii}\in U(R)$. If $a_{ii}\notin J_1(R)\cup\big(1-J_1(R)\big)$, we always have $a_{ii}-e_{ii}\in U(R)$. By hypothesis, $\widehat{A}\cap\big(1-J_k\big)\subseteq Bl\big(J_k\big)$. If $e_{ii}\neq e_{jj}$, then $a_{jj}\in Bl(a_{ii})$. In light of Lemma 3.1.3, we have an idempotent $E\in T_n(R)$ such that $AE=EA$ with $E_{ii}=e_{ii}$. Therefore $A-E\in U\big(T_n(R)\big)$, as required. □

Corollary 3.1.17. *Let R be a local ring. If R satisfies the condition B_n, then $T_n(R)$ is strongly clean.*

Proof. It is obvious from Theorem 3.1.16.

Corollary 3.1.18. *Let R be a local ring. If $Bl\big(J_n\big) \cup Bl\big(1 - J_n\big) = R$ for any $n \in \mathbb{N}$, then $T_n(R)$ is strongly clean.*

Proof. Clearly, $J_0 \subseteq J_1$. Suppose now that $J_{k-1} \subseteq J_k (k \in \mathbb{N})$. Since $Bl\big(J_k\big) \cup Bl\big(1 - J_k\big) = R$, as in the proof of Lemma 3.1.14, we see that $J_k \subseteq Bl\big(1 - J_k\big)$. Furthermore, $Bl\big(J_k\big) \subseteq Bl\big(J_{k-1}\big)$. For any $a \in J_k$, we see that $a \notin Bl\big(J_{k-1}\big)$, and then $a_k \notin Bl\big(J_k\big)$. Hence, $J_k \cap Bl\big(J_k\big) = \emptyset$. As $J_{k+1} = Bl\big(1 - J_k\big) \backslash Bl\big(J_k\big)$, we get $J_k \subseteq J_{k+1}$. By induction, we conclude that $J_0 \subseteq J_1 \subseteq \cdots \subseteq J_n \subseteq \cdots$. Thus, R satisfies the condition B_n for each $n \in \mathbb{N}$. Accordingly, $T_n(R)$ is strongly clean, by Corollary 3.1.17. □

Theorem 3.1.19. *Let R be a local ring. If $Bl\big(J(R)\big) \cup Bl\big(1 + J(R)\big) = R$ and $Bl^2\big(J(R)\big) \cup Bl^2\big(1 + J(R)\big) = R$, then $T_n(R)$ is strongly clean for all n.*

Proof. Clearly, $Bl\big(J_1\big) \cup Bl\big(1 - J_1\big) = R$ and $J_0 \subseteq J_1$. Suppose now that $Bl\big(J_k\big) \cup Bl\big(1 - J_k\big) = R$ and $J_0 \subseteq J_1 \subseteq \cdots \subseteq J_{k-1} \subseteq J_k$ $(k \geq 1)$. As in the proof of Corollary 3.1.18, $J_k \subseteq J_{k+1}$. Hence, $J_0 \subseteq J_1 \subseteq \cdots \subseteq J_k \subseteq J_{k+1}$. As $J_{k+1} \subseteq Bl\big(1 - J_k\big)$, it follows by Lemma 3.1.14 that $Bl^2\big(1 - J_k\big) \subseteq Bl\big(J_{k+1}\big)$. Likewise, $Bl^2\big(J_k\big) \subseteq Bl\big(1 - J_{k+1}\big)$. Hence,

$$\begin{aligned} R &= Bl^2\big(J(R)\big) \cup Bl^2\big(1 + J(R)\big) \\ &= Bl^2\big(J_1\big) \cup Bl^2\big(1 - J_1\big) \\ &= Bl^2\big(J_k\big) \cup Bl^2\big(1 - J_k\big) \\ &\subseteq Bl\big(J_{k+1}\big) \cup Bl\big(1 - J_{k+1}\big) \\ &\subseteq R. \end{aligned}$$

Thus, $Bl\big(J_{k+1}\big) \cup Bl\big(1 - J_{k+1}\big) = R$. By induction, $Bl\big(J_n\big) \cup Bl\big(1 - J_n\big) = R$ for any $n \in \mathbb{N}$, and we therefore complete the proof by Corollary 3.1.18.

□

3.2 Necessary Conditions

The aim of this section is to investigate necessary conditions on strong cleanness of triangular matrices over a local ring. We shall discuss when

the B_n conditions are not only sufficient but also necessary for $T_n(R)$ to be strongly clean.

Lemma 3.2.1. *Let R be a local ring. If $R = Bl\big(J(R)\big) \cup Bl\big(1 + J(R)\big)$, then $T_3(R)$ is strongly clean.*

Proof. Clearly, $J_0 \subseteq J_1$. As $Bl\big(J(R)\big) \cup Bl\big(1+J(R)\big) = R$, we get $Bl(J_1) \cup Bl(1 - J_1) = R$; hence, R satisfies the condition B_3. Accordingly, $T_3(R)$ is strongly clean, by Corollary 3.1.17. □

We say that a local ring R is a weak h-ring in case for any $a \in 1 + J(R), b \in J(R)$, whenever $l_a - r_b$ is surjective, $l_a - r_b$ is injective. For instance, every commutative local ring is a weak h-ring. Evidently, a local ring R is a weak h-ring if and only if for any $a \in J(R), b \in 1 + J(R)$, whenever $l_a - r_b$ is surjective, $l_a - r_b$ is injective. We next consider strongly clean rings of 3×3 triangular matrices over weak h-rings.

Theorem 3.2.2. *Let R be a weak h-ring. Then $T_3(R)$ is strongly clean if and only if $R = Bl\big(J(R)\big) \cup Bl\big(1 + J(R)\big)$.*

Proof. $\Longrightarrow$ Let $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then $T_2(R) \cong ET_3(R)E$. Hence $T_2(R)$ is strongly clean. According to Corollary 1.2.17, R is weakly bleached. Let $b \in R$. If $b \notin Bl\big(J(R)\big)$, then we can find some $a \in J(R)$ such that $l_a - r_b$ or $l_b - r_a$ is not surjective. Suppose that $l_a - r_b$ is not surjective. Then we can find some $a_{12} \in R$ such that $ax - xb = a_{12}$ is not solvable. Let $c \in 1 + J(R)$, let $v \in R$, and let $A = \begin{pmatrix} a & a_{12} & 0 \\ 0 & b & v \\ 0 & 0 & c \end{pmatrix} \in T_3(R)$. Then we can find an idempotent $E = (e_{ij}) \in T_3(R)$ such that $A - E \in U\big(T_3(R)\big)$ and $EA = AE$. This implies that $e_{11}, e_{22}, e_{33} \in R$ are all idempotents. As $a \in J(R), c \in 1 + J(R)$, we claim that $e_{11} = 1$ and $e_{33} = 0$; otherwise, $A - E$ is not invertible. In addition, $e_{22} = 0$ or $e_{22} = 1$. Thus, $E = \begin{pmatrix} 1 & e_{12} & e_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & e_{13} \\ 0 & 1 & e_{23} \\ 0 & 0 & 0 \end{pmatrix}$. If $E = \begin{pmatrix} 1 & e_{12} & e_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then

$$\begin{pmatrix} a & a_{12} + e_{12}b & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = EA = AE = \begin{pmatrix} a & ae_{12} & ae_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and so $ae_{12} - e_{21}b = a_{12}$. This gives a contradiction. Thus, we have an idempotent $E = \begin{pmatrix} 1 & 0 & e_{13} \\ 0 & 1 & e_{23} \\ 0 & 0 & 0 \end{pmatrix}$ such that $A - E \in T_2(R)$ is invertible. In addition,

$$\begin{pmatrix} a & a_{12} & * \\ 0 & b & v + e_{23}c \\ 0 & 0 & 0 \end{pmatrix} = EA = AE = \begin{pmatrix} a & a_{12} & * \\ 0 & b & be_{23} \\ 0 & 0 & 0 \end{pmatrix},$$

and so $be_{23} - e_{23}c = v$. Therefore $l_b - r_c$ is surjective.

Choose $B = \begin{pmatrix} c & 0 & -v \\ 0 & a & a_{12} \\ 0 & 0 & b \end{pmatrix} \in T_3(R)$. Then we can find an idempotent $F = (f_{ij}) \in T_3(R)$ such that $B - F \in U\big(T_3(R)\big)$ and $FB = BF$. Further, $f_{11} = 0$ and $f_{22} = 1$; otherwise, $B - F$ is not invertible. In addition, $f_{33} = 0$ or $f_{33} = 1$. Thus, $F = \begin{pmatrix} 0 & f_{12} & f_{12}f_{23} \\ 0 & 1 & f_{23} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & f_{12} & f_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. If $F = \begin{pmatrix} 0 & f_{12} & f_{12}f_{23} \\ 0 & 1 & f_{23} \\ 0 & 0 & 0 \end{pmatrix}$, then

$$\begin{pmatrix} 0 & f_{12}a & * \\ 0 & a & a_{12} + f_{23}b \\ 0 & 0 & 0 \end{pmatrix} = FB = BF = \begin{pmatrix} 0 & cf_{12} & * \\ 0 & a & af_{23} \\ 0 & 0 & 0 \end{pmatrix},$$

and so $af_{23} - f_{23}b = a_{12}$. This gives a contradiction. Thus, we can find an idempotent $F = \begin{pmatrix} 0 & f_{12} & f_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ such that

$$\begin{pmatrix} 0 & f_{12}a & f_{12}a_{12} + f_{13}b \\ 0 & a & a_{12} \\ 0 & 0 & b \end{pmatrix} = FB = BF = \begin{pmatrix} 0 & cf_{12} & cf_{13} - v \\ 0 & a & a_{12} \\ 0 & 0 & b \end{pmatrix}.$$

Hence, $cf_{12} = f_{12}a$, and so $(1-c)f_{12} = f_{12}(1-a)$, where $1-c \in J(R), 1-a \in 1 + J(R)$. Clearly, $l_{(1-c)} - r_{(1-a)}$ is surjective. As R is a weak h-ring, we get $f_{12} = 0$. Hence $cf_{13} - f_{13}b = v$. Consequently, $l_c - r_b$ is surjective. This implies that $b \in Bl\big(1 + J(R)\big)$. If $l_b - r_q$ is not surjective, we show that $b \in Bl\big(1 + J(R)\big)$ by a similar route. Therefore $R = Bl\big(J(R)\big) \cup Bl\big(1 + J(R)\big)$.

$\Longleftarrow$ This is clear from Lemma 3.2.1. $\square$

A local ring R is said to be an h-ring provided that for any $a, b \in R$, whenever $l_a - r_b$ is surjective, $l_a - r_b$ is injective (see [233]). Clearly, every h-ring is a weak h-ring, but the converse is not true as the following shows.

Example 3.2.3. There exists a weak h-ring R which is not an h-ring, while $T_3(R)$ is strongly clean.

Proof. Let Λ be an arbitrary division ring with *charp* > 0. As in the [379], one can construct an extension division ring R with the property that there exists an element $a \in R$ whose associated inner derivation D_a is an onto map: $D_a(R) = R$. This implies that $l_a - r_a$ is surjective. As $(l_a - r_a)(1_R) = 0$, we see that $l_a - r_a$ is not injective. Thus, R is not an h-ring. It is easy to verify that every division ring is a weak h-ring, and so is R. Also we see that every division ring is a bleached local ring. In light of Theorem 3.2.2, $T_3(R)$ is strongly clean. □

Proposition 3.2.4. *Let R be a weak h-ring. Then the following are equivalent:*

(1) *$T_3(R)$ is strongly clean.*
(2) *$T_3\big(R[[x]]\big)$ is strongly clean.*

Proof. (1) $\Rightarrow$ (2) Clearly, $R/J(R) \cong R[[x]]/J\big(R[[x]]\big)$. Thus, $R[[x]]$ is local. Let $a = \sum_{i=0}^{\infty} a_i x^i \in J\big(R[[x]]\big)$ and $c = \sum_{i=0}^{\infty} c_i x^i \in 1 + J\big(R[[x]]\big)$. Assume that $l_a - r_c$ is surjective and $d = \sum_{i=0}^{\infty} d_i x^i \in R[[x]]$ such that $ad = dc$. Then $l_{a_0} - r_{c_0}$ is surjective and $a_0 d_0 = d_0 c_0$. Clearly, $a_0 \in J(R)$ and $c_0 \in 1 + J(R)$. As R is a weak h-ring, we deduce that $d_0 = 0$. Further, $a_0 d_1 + a_1 d_0 = d_0 c_1 + d_1 c_0$, and so $a_0 d_1 = d_1 c_0$. Thus, we see that $d_1 = 0$. Moreover, $a_0 d_2 + a_1 d_1 + a_2 d_0 = d_0 c_2 + d_1 c_1 + d_2 c_0$, and then $a_0 d_2 = d_2 c_0$; hence, $d_2 = 0$. By iteration of this process, we deduce that $d_n = 0 (n = 3, 4, \ldots)$, and so $d = 0$. Thus, $R[[x]]$ is a weak h-ring.

Let $b = \sum_{i=0}^{\infty} b_i x^i \in R[[x]] \notin Bl\big(J(R[[x]])\big)$. Then we have some $a = \sum_{i=0}^{\infty} a_i x^i \in J\big(R[[x]]\big)$ such that $l_a - r_b$ or $l_b - r_a$ is not surjective. Suppose that $l_a - r_b : R[[x]] \to R[[x]]$ is not surjective. Let $e = \sum_{i=0}^{\infty} e_i x^i \in R[[x]]$. If $l_{a_0} - r_{b_0} : R \to R$ is surjective, then we can find some $h_0 \in R$ such that $a_0 h_0 - h_0 b_0 = e_0$. Further, we have some $h_1, h_2, \ldots, h_n, \ldots$ such that the

following hold:

$$\begin{aligned}
a_0h_0 - h_0b_0 &= e_0;\\
a_0h_1 - h_1b_0 &= e_1 - a_1h_0 + h_0b_1;\\
a_0h_2 - h_2b_0 &= e_2 - a_1h_1 - a_2h_0 + h_0b_2 + h_1b_1;\\
&\vdots\\
a_0h_n - h_nb_0 &= e_n - \Big(\sum_{i=1}^{n} a_ih_{n-i}\Big) + \sum_{i=0}^{n-1} h_ib_{n-i};\\
&\vdots
\end{aligned}$$

Thus, there exists $h = \sum\limits_{i=0}^{\infty} h_ix^i$ such that $ah - hb = e$. This implies that $l_a - r_b : R[[x]] \to R[[x]]$ is surjective, a contradiction. So $l_{a_0} - r_{b_0} : R \to R$ is not surjective. This shows that $b_0 \notin Bl\big(J(R)\big)$. In view of Theorem 3.2.2, $R = Bl\big(J(R)\big) \cup Bl\big(1 + J(R)\big)$, and so $b_0 \in Bl\big(1 + J(R)\big)$. Let $c = \sum\limits_{i=0}^{\infty} c_ix^i \in 1 + J\big(R[[x]]\big)$. Then $l_{b_0} - r_{c_0}, l_{c_0} - r_{b_0}$ are surjective. Hence, there exists $k_0 \in R$ such that $b_0k_0 - k_0c_0 = e_0$. As the preceding discussion, we can find some $k_1, k_2, \ldots \in R$ such that $bk - kc = e$, where $k = \sum\limits_{i=0}^{\infty} k_ix^i$. This means that $l_b - r_c$ is surjective. Likewise, $l_c - r_b$ is surjective. It follows that $b \in Bl\big(1 + J(R[[x]])\big)$. If $l_b - r_a : R[[x]] \to R[[x]]$ is not surjective, analogously, we deduce that $b \in Bl\big(1 + J(R[[x]])\big)$. Therefore $R[[x]] = Bl\big(J(R[[x]])\big) \cup Bl\big(1 + J(R[[x]])\big)$. In light of Theorem 3.2.2, $T_3\big(R[[x]]\big)$ is strongly clean.

(2) $\Rightarrow$ (1) Let $\varphi : R[[x]] \to R$ given by $\varphi\big(f(x)\big) = f(0)$ for any $f(x) \in R[[x]]$. Then φ is an epimorphism. This induces an epimorphism $\varphi^* : T_3\big(R[[x]]\big) \to T_3(R)$. Hence, $T_3(R) \cong T_3\big(R[[x]]\big)/Ker\varphi^*$. Since $T_3\big(R[[x]]\big)$ is strongly clean, then so is $T_3(R)$, as asserted. □

Corollary 3.2.5. *Let R be a weak h-ring. Then the following are equivalent:*

(1) *$T_3(R)$ is strongly clean.*
(2) *$T_3\big(R[x]/(x^n)\big)$ is strongly clean.*

Proof. (1) $\Rightarrow$ (2) According to Proposition 3.2.4, $T_3\big(R[[x]]\big)$ is strongly clean, and so is the homomorphic image $T_3\big(R[[x]]/(x^n)\big)$. Clearly, $R[[x]]/(x^n) \cong R[x]/(x^n)$. Therefore $T_3\big(R[x]/(x^n)\big)$ is strongly clean.

(2) $\Rightarrow$ (1) Let $\varphi : R[x]/(x^n) \to R$ given by $\varphi\big(\overline{f(x)}\big) = f(0)$ for any $f(x) \in R[x]$. Then we get an epimorphism $\varphi^* : T_3\big(R[x]/(x^n)\big) \to T_3\big(R\big)$

given by $\varphi^*\big((\overline{f_{ij}})\big) = (\varphi(f_{ij}))$. Hence, $T_3(R)$ is a homomorphic image of $T_3\big(R[x]/(x^n)\big)$, and thus yielding the result. □

An n-tuple $\{c_1, \ldots, c_n\} \subseteq R$ is bleached provided that $c_i \in Bl(c_j)$ for $i \neq j$ if and only if $|i-j| \geq 2$. A local ring R is said to be a DD_n-ring (i.e., Disel-Dorsey ring for n) provided that for any bleached n-tuple $(c_1, \ldots, c_n)$, there exists some $A \in T_n(R)$, where $\{A_{11}, \ldots, A_{nn}\}$ is a permutation of $\{c_{11}, \ldots, c_{nn}\}$, such that $AE = EA$ and $E = E^2 \in T_n(R) \Rightarrow E = 0$ or $E = I_n$. Clearly, every local ring is a DD_1-ring. Furthermore, we can derive

Lemma 3.2.6. *Every local ring is a DD_2-ring.*

Proof. Let (c_1, c_2) be bleached. Then $c_1 \notin Bl(c_2)$. Hence, $l_{c_2} - r_{c_1}$ or $l_{c_1} - r_{c_2}$ is not surjective. If $l_{c_2} - r_{c_1}$ is not surjective, we chose $y \in R \backslash Im\big(l_{c_2} - r_{c_1}\big)$. Choose $C = \begin{pmatrix} c_1 & y \\ 0 & c_2 \end{pmatrix}$. Let $B = (e_{ij}) \in T_2(R)$ be an idempotent matrix such that $CB = BC$. If $B \neq 0, I_2$, then $e_{11} = 1, e_{22} = 0$ or $e_{11} = 0, e_{22} = 1$. This implies that $c_1 e_{12} - e_{12} c_2 = \pm y$, a contradiction. Hence, $B = 0$ or I_2.

If $l_{c_1} - r_{c_2}$ is not surjective, applying the previous discussion to (c_2, c_1), we have a contradiction, too. Therefore we conclude that R is a DD_2 ring. □

Theorem 3.2.7. *Every h-ring is a DD_n-ring for all $n \in \mathbb{N}$.*

Proof. It will suffice to assume $n \geq 3$.

Case I. $n = 3$. Let (c_1, c_2, c_3) be bleached. Then $c_1 \notin Bl(c_2)$ and $c_2 \notin Bl(c_3)$. (i) If $l_{c_1} - r_{c_2}$ is not surjective, we consider (c_1, c_2, c_3). In this case, if $l_{c_2} - r_{c_3}$ is not surjective, we consider (c_1, c_2, c_3); if $l_{c_3} - r_{c_2}$ is not surjective, we consider (c_3, c_1, c_2). (ii) If $l_{c_2} - r_{c_1}$ is not surjective, we consider (c_2, c_1, c_3). In this case, if $l_{c_2} - r_{c_3}$ is not surjective, we consider (c_2, c_1, c_3); if $l_{c_3} - r_{c_2}$ is not surjective, we consider (c_3, c_2, c_1).

Let $E = E^2 := (e_{ij}) \in T_3(R) \neq 0, I_3$. Then E is one of the set

$$\left\{ \begin{pmatrix} 0 & * & * \\ & 0 & * \\ & & 1 \end{pmatrix}, \begin{pmatrix} 0 & * & * \\ & 1 & * \\ & & 0 \end{pmatrix}, \begin{pmatrix} 1 & * & * \\ & 0 & * \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & * & * \\ & 0 & * \\ & & 0 \end{pmatrix} \right\}.$$

(a) $l_{c_1} - r_{c_2}, l_{c_2} - r_{c_3}$ are not surjective. Choose $y \in R \backslash Im\big(l_{c_1} - r_{c_2}\big), z \in R \backslash Im\big(l_{c_2} - r_{c_3}\big)$, and $C = \begin{pmatrix} c_1 & y & \\ & c_2 & z \\ & & c_3 \end{pmatrix}$ with $CE = EC$. Then $c_1 e_{12} - e_{12} c_2 = \pm y$ or $c_2 e_{23} - e_{23} c_3 = \pm z$. This gives a contradiction.

(b) $l_{c_1}-r_{c_2}, l_{c_3}-r_{c_2}$ are not surjective. Choose $y \in R\backslash Im\big(l_{c_1}-r_{c_2}\big), z \in R\backslash Im\big(l_{c_3}-r_{c_2}\big)$, and $C = \begin{pmatrix} c_3 & & z \\ & c_1 & y \\ & & c_2 \end{pmatrix}$ with $CE = EC$. Then (i) $c_1e_{23} - e_{23}c_2 = \pm y$, a contradiction. (ii) $c_3e_{12} - e_{12}c_1 = 0$ and $c_3e_{13} - e_{13}c_2 = e_{12}y \pm z$. By hypothesis, $l_{c_3} - r_{c_1}$ is surjective; hence, $e_{12} = 0$ as R is an h-ring. Therefore, $c_3e_{13} - e_{13}c_2 = \pm z$, a contradiction.

(c) $l_{c_2}-r_{c_1}, l_{c_2}-r_{c_3}$ are not surjective. Choose $y \in R\backslash Im\big(l_{c_2}-r_{c_1}\big), z \in R\backslash Im\big(l_{c_2}-r_{c_3}\big)$, and $C = \begin{pmatrix} c_2 & y & z \\ & c_1 & \\ & & c_3 \end{pmatrix}$ with $CE = EC$. Then (i) $c_2e_{12} - e_{12}c_1 = \pm y$, a contradiction. (ii) $c_1e_{23} - e_{23}c_3 = 0$ and $c_2e_{13} - e_{13}c_3 = -ye_{23} \pm z$. By hypothesis, $l_{c_1} - r_{c_3}$ is surjective; hence, $e_{23} = 0$ as R is an h-ring. Therefore, $c_2e_{13} - e_{13}c_3 = \pm z$, a contradiction.

(d) $l_{c_2}-r_{c_1}, l_{c_3}-r_{c_2}$ are not surjective. Choose $y \in R\backslash Im\big(l_{c_2}-r_{c_1}\big), z \in R\backslash Im\big(l_{c_3}-r_{c_2}\big)$, and $C = \begin{pmatrix} c_3 & z & \\ & c_2 & y \\ & & c_1 \end{pmatrix}$ with $CE = EC$. Then (i) $c_2e_{23} - e_{23}c_1 = \pm y$, a contradiction. (ii) $c_3e_{12} - e_{12}c_2 = \pm z$, a contradiction.

Therefore we conclude that R is a DD_3-ring.

Case II. We now consider $(c_1, \ldots, c_n)$. Order the elements c_i by the following process. For each $i \in \{1, 2, \ldots, n-1\}$, if $l_{c_i}-r_{c_{i+1}}$ is not surjective, not change the position of c_{i+1}; if $l_{c_{i+1}} - r_{c_i}$ is not surjective, place c_{i+1} at the far left of the list. We note that the elements between c_i and c_{i+1} are exactly some permutation of $\{c_1, \ldots, c_{i-1}\}$.

Construct $C \in T_n(R)$ as follows: For each $i \in \{1, 2, \ldots, n-1\}$, if $l_{c_i} - r_{c_{i+1}}$ is not surjective, choose $y_i \in R\backslash Im\big(l_{c_i} - r_{c_{i+1}}\big)$, and place y_i in the same row as c_i and the same column as c_{i+1}; if $l_{c_{i+1}} - r_{c_i}$ is not surjective, choose $z_i \in R\backslash Im\big(l_{c_{i+1}} - r_{c_i}\big)$, and place z_i in the same row as c_{i+1} and the same column as c_i. All other entries of C are zero.

Let $E = E^2 \in T_n(R)$ and $E \neq 0, I_n$. For each k, define C_k to be the smallest square block in C that contains c_k and c_{k+1}, and define E_k to be the corresponding submatrix of E. Let e_i be the diagonal entry of E whose position corresponds to that of c_i in C. Let m be the smallest index such that $e_m \neq e_{m+1}$. Without loss of generality, we may assume that $e_m = 1$ and $e_{m+1} = 0$.

Case I. c_m is adjacent to c_{m+1}. If c_m is in the left of c_{m+1}, then

$$\begin{pmatrix} c_m & y_m \\ 0 & c_{m+1} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_m & y_m \\ 0 & c_{m+1} \end{pmatrix}.$$

This implies that $c_m x - x c_{m+1} = y_m$, a contradiction. If c_m is in the right of c_{m+1}, then we also deduce a contradiction.

Case II. c_m is not adjacent to c_{m+1}. It must be that c_{m+1} is at the $(1,1)$-entry and c_m is at (m,m)-entry in some block matrix C_m of C. Let j be the smallest index greater 1 such that $(1,j)$-entry of E_m is not zero. If $j \neq m$, then it follows from $C_m E_m = E_m C_m$ that $c_{m+1} z - z c_i = 0$ for some $1 \leq i \leq m$. As R is an h-ring, by hypothesis, $z = 0$, a contradiction. If $j = m$, then $c_{m+1} z - z c_{m+1} = z_m$, a contradiction. □

Following Dorsey, we say that a local ring R is a *gh*-ring if for all $a, b \in R$, if $l_a - r_b : R \to R$ is surjective, then $Ker(l_a - r_b)_{C(R)} \subseteq_s R_{C(R)}$. It is immediate from [241, Lemma 4.3.10] that every *gh*-ring is DD_n-ring.

Lemma 3.2.8. *Suppose that* $J_0 \subseteq J_1 \subseteq \cdots \subseteq J_k$ *and* $Bl(J_{k-1}) \cup Bl(1 - J_{k-1}) = R (k \geq 1)$. *If* $a_k \in J_k \backslash J_{k-1}$ *and* $b_k \in (1-J_k)\backslash(1-J_{k-1})$, *then there exist* $a_1, \ldots, a_{k-1}; b_1, \ldots, b_{k-1}$ *with* $a_i \in J_i \backslash J_{i-1}$ *and* $b_i \in (1-J_i)\backslash(1-J_{i-1})$ *such that* $a_i \notin Bl(a_{i+1})$ *and* $b_i \notin Bl(b_{i+1})$ *for* $i = 1, \ldots, k-1$.

Proof. As $a_k \in J_k \backslash J_{k-1}$, $a_k \notin Bl(J_{k-1})$. Thus, we can find some $a_{k-1} \in J_{k-1}$ such that $a_k \notin Bl(a_{k-1})$. In view of Lemma 3.1.14, $R = Bl(1-J_{k-2}) \cup Bl(J_{k-2})$. Hence, $R \backslash J_{k-1} \subseteq Bl(J_{k-2})$ by Lemma 3.1.15. Accordingly, $a_k \in Bl(J_{k-2})$, and then $a_{k-1} \notin J_{k-2}$. Therefore we get $a_{k-1} \in J_{k-1} \backslash J_{k-2}$. By iteration, we can find some $a_i \in J_i \backslash J_{i-1}$ such that $a_i \notin Bl(a_{i-1})$ for $i = k-2, \ldots, 1$. Likewise, for $i = 1, \ldots, k-1$, we have $b_i \in (1-J_i)\backslash(1-J_{i-1})$ such that $b_i \notin Bl(b_{i+1})$, as desired. □

Lemma 3.2.9. *Suppose that* $J_0 \subseteq J_1 \subseteq \cdots \subseteq J_k$ *and* $Bl(J_{k-1}) \cup Bl(1 - J_{k-1}) = R$. *Let* $K_i = J_i \backslash J_{i-1} (1 \leq i \leq k)$. *Then the following hold:*

(1) $K_i \subseteq Bl(K_j)$, *whenever* $|i - j| \geq 2$.
(2) $K_i \subseteq Bl(1 - K_j)$ *for all* $1 \leq i, j \leq k-1$.
(3) *If* $J_k \subseteq J_{k+1}$, *then* $K_i \subseteq Bl(1 - K_j)$ *for all* $1 \leq i, j \leq k$.

Proof. (1) Without the loss of generality, we assume that $i - j \geq 2$. In view of Lemma 3.1.14, $R = Bl(J_j) \cup Bl(1 - J_j)$. It follows by Lemma 3.1.15 that $R \backslash J_{j+1} \subseteq Bl(J_j)$. Since $K_j \subseteq J_j$, by using Lemma 3.1.14 again, $Bl(J_j) \subseteq Bl(K_j)$. Hence, $K_i \subseteq R \backslash J_{i-1} \subseteq R \backslash J_{j+1}$, as $i - 1 \geq j + 1$. This implies that $K_i \subseteq Bl(J_j) \subseteq Bl(K_j)$. Analogously, $1 - K_i \subseteq Bl(1 - K_j)$ for all $1 \leq i, j \leq n$.

(2) Clearly, $J_{k-1} \subseteq J_k \subseteq Bl(1-J_{k-1})$. As $K_i \subseteq J_{k-1} \subseteq Bl(1-J_{k-1})$ and $1-K_j \subseteq 1-J_{k-1}$, it follows by Lemma 3.1.14 that $K_i \subseteq Bl(1-K_j)$ for all $1 \leq i, j \leq k-1$.

(3) is proved as in the discussion in (2). □

Theorem 3.2.10. *Let R be a DD_n-ring. If $T_n(R)$ is strongly clean, then R satisfies the condition B_n.*

Proof. If $n = 1$, then $R = U(R) \cup J(R)$, as R is local. That is, $R = Bl\big(J_0\big) \cup Bl\big(1-J_0\big)$. Hence, R satisfies the condition B_1. If $n = 2$, it follows by Corollary 1.2.17 that R is weakly bleached, i.e., $J_1 \subseteq Bl\big(1-J_1\big)$. Thus, R satisfies the condition B_2. If the theorem is false, let m be the least integer for which $T_m(R)$ is strongly clean but R doesn't satisfy the condition B_m.

Case I. m is odd. Write $m = 2k+1 (k \geq 1)$. By hypothesis, $T_{2k+1}(R)$ is strongly clean, and then so is $T_{2k}(R)$. We infer that R satisfies the condition B_{2k}. That is,

$$\begin{aligned} &J_0 \subseteq J_1 \subseteq \cdots \subseteq J_k; \\ &J_k \subseteq Bl\big(1-J_k\big); \\ &R = Bl\big(J_{k-1}\big) \cup Bl\big(1-J_{k-1}\big). \end{aligned}$$

As R doesn't satisfy the condition B_{2k+1}, we get $R \neq Bl\big(J_k\big) \cup Bl\big(1-J_k\big)$. Choose $x \notin Bl\big(J_k\big) \cup Bl\big(1-J_k\big)$. Then we can find $a_k \in J_k$ and $b_k \in 1-J_k$ such that $x \notin Bl(a_k)$ and $x \notin Bl(b_k)$.

Clearly,

$$\begin{aligned} R &= J_k \cup \big(1-J_k\big) \cup \big(R \backslash (J_k \cup \big(1-J_k\big)\big) \\ &= J_k \cup \big(1-J_k\big) \cup \big(Bl(J_{k-1}) \cap Bl(1-J_{k-1})\big), \end{aligned}$$

by Lemma 3.1.15. As $J_k \subseteq Bl\big(1-J_k\big)$, we get $1-J_k \subseteq Bl\big(J_k\big)$; hence, $x \notin J_k \cup \big(1-J_k\big)$. This implies that $x \in Bl\big(J_{k-1}\big) \cap Bl\big(1-J_{k-1}\big)$. It follows that $a_k \notin J_{k-1}$ and $b_k \notin 1-J_{k-1}$. Therefore $a_k \in J_k \backslash J_{k-1}$ and $b_k \in \big(1-J_k\big) \backslash \big(1-J_{k-1}\big)$.

By virtue of Lemma 3.2.8, there exist $a_{k-1}, \ldots, a_1$ and $b_{k-1}, \ldots, b_1$ such that $a_i \in J_i \backslash J_{i-1}$ and $b_i \in \big(1-J_i\big) \backslash \big(1-J_{i-1}\big)$ such that, for any $1 \leq i \leq k-1$, $a_i \notin Bl(a_{i+1})$ and $b_i \notin Bl(b_{i+1})$. We claim that

$$a_i, b_i \in Bl(x) \text{ if and only if } i \neq k.$$

As $a_k, b_k \notin Bl(x)$, we have $a_i, b_i \in Bl(x) \Rightarrow i \neq k$. If $i \neq k$, $a_i \in J_i \subseteq J_{k-1}$, and then $a_i \in Bl(x)$. Likewise, $b_i \in Bl(x)$. Hence, $i \neq k \Rightarrow a_i, b_i \in Bl(x)$.

Let $\alpha = (a_1, \ldots, a_k, x, b_k, \ldots, b_1)$. In light of Lemma 3.2.9 and the previous discussion, we conclude that α is bleached.

Case II. m is even. Write $m = 2k(k \geq 2)$. Then $T_{2k}(R)$ is strongly clean, and then so is $T_{2k-1}(R)$. This implies that R satisfies the condition $B_{2(k-1)+1}$. That is,

$$\begin{aligned} &J_0 \subseteq J_1 \subseteq \cdots \subseteq J_{k-1}; \\ &R = Bl\big(J_{k-1}\big) \cup Bl\big(1 - J_{k-1}\big). \end{aligned}$$

In view of Lemma 3.1.14, $J_{k-1} \subseteq Bl(1 - J_{k-1})$. As $J_{k-1} \cap Bl\big(J_{k-1}\big) = \emptyset$, we get $J_{k-1} \subseteq Bl(1 - J_{k-1}) \backslash Bl\big(J_{k-1}\big) = J_k$. As R doesn't satisfy the condition B_{2k}, we have $J_k \nsubseteq Bl(1 - J_k)$. Choose $a_k \in J_k$ and $b_k \in 1 - J_k$ such that $a_k \notin Bl(b_k)$.

As $a_k \in J_k$, $a_k \in Bl(1 - J_{k-1})$, and so $b_k \notin 1 - J_{k-1}$. Likewise, $a_k \notin J_{k-1}$. Hence, $a_k \in J_k \backslash J_{k-1}$ and $b_k \in \big(1 - J_k\big) \backslash \big(1 - J_{k-1}\big)$. In view of Lemma 3.2.8, there exist $a_{k-1}, \ldots, a_1$ and $b_{k-1}, \ldots, b_1$ such that $a_i \in J_i \backslash J_{i-1}$ and $b_i \in \big(1 - J_i\big) \backslash \big(1 - J_{i-1}\big)$ such that, for any $1 \leq i \leq k-1$, $a_i \notin Bl(a_{i+1})$ and $b_i \notin Bl(b_{i+1})$.

Let $\alpha = (a_1, \ldots, a_k, b_k, \ldots, b_1)$. In view of Lemma 3.2.9, $a_i \in Bl(a_j)$ and $b_i \in Bl(b_j)$ if $|i - j| \geq 2$. Since $a_k \in J_k \subseteq Bl(1 - J_{k-1})$, $a_k \in Bl(b_i)$ as $b_i \in 1 - J_i \subseteq 1 - J_{k-1}$ for all $1 \leq i \leq k-1$. Likewise, $b_k \in Bl(a_i)$ for all $1 \leq i \leq k-1$. Furthermore, it follows by Lemma 3.2.9 that $a_i \in Bl(b_j)$ and $b_i \in Bl(a_j)$ for all $1 \leq i, j \leq k-1$. Therefore α is bleached.

In each of these two cases, we can find a bleached $(a_1, \ldots, b_1)$. As R is a DD_n-ring, there exists some $A \in T_n(R)$ with each $\big(A_{11}, \ldots, A_{nn}\big) = (a_1, \ldots, b_1)$ such that, $AE = EA$ and $E = E^2 \in T_n(R)$ implies that $E = 0$ or $E = I_n$. Since $T_n(R)$ is strongly clean, we can find an idempotent $F \in T_n(R)$ such that $AF = FA$ and $A - F \in U\big(T_n(R)\big)$. This implies that $F = 0$ or $F = I_n$. Hence, A or $A - I_n$ is invertible. As $a_1 \in J(R)$, we see that $A \notin U\big(T_n(R)\big)$. As $b_1 \in 1 - J(R)$, we get $I_n - A \notin U\big(T_n(R)\big)$. This gives a contradiction, and we therefore complete the proof. □

Corollary 3.2.11. *Let R be an h-ring, and let $n \in \mathbb{N}$. Then the following are equivalent:*

(1) *$T_n(R)$ is strongly clean.*
(2) *R satisfies the condition B_n.*

Proof. This is obvious, by Theorem 3.2.7, Theorem 3.2.10 and Corollary 3.1.17. □

3.3 Triangular Matrices over Commutative Rings

The goal of this section is to explore the strongly clean triangular matrix rings over commutative rings. We start by

Lemma 3.3.1. *Let R be a commutative clean ring, let $n \in \mathbb{N}$, and let $a_1, \ldots, a_n \in R$. Then there exist idempotents $e_1, \ldots, e_n \in R$ such that*

(1) $a_i - e_i \in U(R)$ *for every* i;
(2) $(a_i - a_j) - \left(1 - (e_i - e_j)^2\right) \in U(R)$ *for every* $i \neq j$.

Proof. This could be deduced by [237, Lemma 2.4]. □

Theorem 3.3.2. *Let R be a commutative clean ring. Then $T_n(R)$ is strongly clean for all $n \in \mathbb{N}$.*

Proof. Let $A = (a_{ij}) \in T_n(R)(n \in \mathbb{N})$. Applying Lemma 3.3.1 to $a_{11}, \ldots, a_{nn}$, we can find idempotents $e_1, \ldots, e_n \in R$ such that the conditions in Lemma 3.3.1 hold. Then $a_{ii} - e_i \in U(R)$ for each i. We shall construct an idempotent $E = (e_{ij}) \in T_n(R)$ which witnesses the strong cleanness of A. Choose $e_{ii} = e_i$. Then this ensures that $A - E \in GL_n(R)$. We next use induction to construct the other entries of E such that E is an idempotent and commutes with A.

Suppose that for some $k(0 \leq k \leq n-1)$, we have defined e_{ij} for all $i \leq j \leq i+k$ such that for all $i = 1, \ldots, n-k$,

$$\begin{pmatrix} a_{ii} & \cdots & a_{i,i+k} \\ & \ddots & \vdots \\ & & a_{i+k,i+k} \end{pmatrix}$$

commutes with

$$\begin{pmatrix} e_{ii} & \cdots & e_{i,i+k} \\ & \ddots & \vdots \\ & & e_{i+k,i+k} \end{pmatrix}$$

and the later matrix is idempotent.

Let $i \in 1, \ldots, n-(k+1)$. Consider the matrices

$$A' = \begin{pmatrix} a_{ii} & \cdots & a_{i,i+(k+1)} \\ & \ddots & \vdots \\ & & a_{i+(k+1),i+(k+1)} \end{pmatrix}$$

commutes with

$$E' = \begin{pmatrix} e_{ii} & \cdots & e_{i,i+(k+1)} \\ & \ddots & \vdots \\ & & e_{i+(k+1),i+(k+1)} \end{pmatrix}.$$

By induction hypothesis, we shall suffice to find $e_{i,i+(k+1)}$ such that $E' \in M_{k+1}(R)$ is an idempotent and $A'E' = E'A'$.

Set $a = a_{ii}, b = a_{i+(k+1),i+(k+1)}, e = e_{ii}, e' = 1 - e, f = e_{i+(k+1),i+(k+1)}, f' = 1 - f, c = a_{i,i+(k+1)}$ and $z = e_{i,i+(k+1)}$. We use $\alpha, \beta, \gamma, \delta$ to denote the entries of A' and E' as in Lemma 3.1.1. By virtue of Lemma 3.3.1, we have $u := (a-b) - \big(1-(e-f)^2\big) \in U(R)$. Hence, $(a-b)(e-f)^2u^{-1} = (e-f)^2$ as $(e-f)^3 = e-f$. Thus, we can find some $y \in R$ such that

$$(a-b)(e-f)^2(y) = (e-f)(c) + (e-f)^2(\gamma\beta - \alpha\delta).$$

Let $z = -e\gamma\delta f + e'\gamma\delta f' + (e-f)^2y$. Then z satisfies the equations in Lemma 3.1.1. Therefore $E' \in M_{k+1}(R)$ is an idempotent and $E'A' = A'E'$, in terms of Lemma 3.1.1. This completes the proof, by induction. □

We say that a ring R is π-regular provided that for any $a \in R$, there exists some $n \in \mathbb{N}$ such that $a^n \in a^nRa^n$. In view of Lemma 1.1.1, every strongly π-regular ring is π-regular. But the converse is not true, e.g., $R = \prod_{n=1}^{\infty} M_n(F)$ where F is a field.

Corollary 3.3.3. *Let R be a commutative π-regular ring. Then $T_n(R)$ is strongly clean for all $n \in \mathbb{N}$.*

Proof. Clearly, R is strongly π-regular. In view of Theorem 1.1.2, R is strongly clean, and so the proof is complete by Theorem 3.3.2. □

In [78], Borooah et al. posed a question: If $M_n(R)$ is strongly clean, should $T_n(R)$ be strongly clean? Next, we are concerned with this relatively strong cleanness: If $A \in T_n(R) \subseteq M_n(R)$ is strongly clean as an element of $M_n(R)$, must it be strongly clean as an element of $T_n(R)$?

Theorem 3.3.4. *Let R be an integral domain, and let $A \in T_n(R) \subseteq M_n(R)$ be strongly clean in $M_n(R)$. Then $A \in T_n(R)$ is strongly clean in $T_n(R)$, if all its principal diagonal entries are distinct.*

Proof. By hypothesis, we can find an idempotent $E \in M_n(R)$ such that $A - E \in GL_n(R)$ and $AE = EA$. Write $A = (a_{ij})$ and $E = (e_{ij})$. Then

$$\begin{pmatrix} e_{11} & \cdots & e_{1n} \\ e_{21} & \cdots & e_{2n} \\ \vdots & \ddots & \vdots \\ e_{n1} & \cdots & e_{nn} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} e_{11} & \cdots & e_{1n} \\ e_{21} & \cdots & e_{2n} \\ \vdots & \ddots & \vdots \\ e_{n1} & \cdots & e_{nn} \end{pmatrix}.$$

This yields that

$$\begin{gathered} e_{n1}a_{11} = a_{nn}e_{n1}, \\ e_{n1}a_{12} + e_{n2}a_{22} = a_{nn}e_{n2}, \\ e_{n1}a_{13} + e_{n2}a_{23} + e_{n3}a_{33} = a_{nn}e_{n3}, \\ \vdots \\ e_{n1}a_{1(n-1)} + e_{n2}a_{2(n-1)} + \cdots + e_{nn}a_{n(n-1)} = a_{nn}e_{n(n-1)}. \end{gathered}$$

By assumption, $a_{nn} - a_{11} \neq 0$. If $a_{nn}x - xa_{11} = 0$, then $(a_{nn} - a_{11})x = 0$, and so $x = 0$ as R is an integral domain. That is, $l_{a_{nn}} - r_{a_{11}}$ is injective, and so we get $e_{n1} = 0$. Likewise, $l_{a_{nn}} - r_{a_{22}}$ is injective. It follows from $e_{n2}a_{22} = a_{nn}e_{n2}$ that $e_{n2} = 0$. Similarly, $e_{n3} = \cdots = e_{n(n-1)} = 0$. Further, we have

$$\begin{aligned} &\begin{pmatrix} e_{11} & \cdots & e_{1n-1} \\ e_{21} & \cdots & e_{2(n-1)} \\ \vdots & \ddots & \vdots \\ e_{(n-1)1} & \cdots & e_{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1(n-1)} \\ 0 & a_{22} & \cdots & a_{2(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{(n-1)(n-1)} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1(n-1)} \\ 0 & a_{22} & \cdots & a_{2(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} e_{11} & \cdots & e_{1n-1} \\ e_{21} & \cdots & e_{2(n-1)} \\ \vdots & \ddots & \vdots \\ e_{(n-1)1} & \cdots & e_{(n-1)(n-1)} \end{pmatrix}. \end{aligned}$$

As in the previous discussion, we get $e_{(n-1)1} = e_{(n-1)2} = \cdots = e_{(n-1)(n-2)} = 0$. By iteration of this process, we have $e_{ij} = 0$ whenever $j < i$. That is, $E \in T_n(R)$. Accordingly, $A - E \in U\big(T_n(R)\big)$, as required.

□

Corollary 3.3.5. *Let R be an integral domain. If $A \in T_2(R) \subseteq M_2(R)$ is strongly clean in $M_2(R)$, then $A \in T_2(R)$ is strongly clean in $T_2(R)$.*

Proof. Write $A = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \in T_2(R)$. If $a \neq b$, then $A \in T_2(R)$ is strongly clean in $T_2(R)$, by Theorem 3.3.4. Now assume that $a = b$. Then we have

an idempotent $E \in M_2(R)$ such that $A - E \in GL_2(R)$ and $AE = EA$. If $E = 0$ or $E = I_2$, then the result holds. Otherwise, we may assume that $E = \begin{pmatrix} x & y \\ z & 1-x \end{pmatrix}$ with $yz = x - x^2$ by Lemma 1.3.1. It follows from

$$\begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \begin{pmatrix} x & y \\ z & 1-x \end{pmatrix} = \begin{pmatrix} x & y \\ z & 1-x \end{pmatrix} \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}$$

that $cz = 0$ and $(a-b)z = 0$, and so either $z = 0$ or $c = 0$. If $z = 0$, then $A - E \in U\big(T_2(R)\big)$ and $AE = EA$ with $E \in T_2(R)$. If $z \neq 0$, then $c = 0$ and $a = b$; hence, $\begin{pmatrix} a & \\ & a \end{pmatrix} - \begin{pmatrix} x & y \\ z & 1-x \end{pmatrix} \in GL_2(R)$. This implies that $a^2 - a \in U(R)$ as $yz = x - x^2$. Hence, $a = 0$ or $a = 1$. Thus, $A = 0$ or $A = I_2$ is strongly clean in $T_2(R)$, as asserted. □

Corollary 3.3.6. *Let R be an integral domain. If $M_2(R)$ is strongly clean, then so is $T_2(R)$.*

Proof. It is obvious by Corollary 3.3.5. □

Example 3.3.7. Let $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in T_2(\mathbb{Z})$. Then the following are equivalent:

(1) A is strongly clean in $T_2(\mathbb{Z})$.
(2) A is in one of the following cases:
   (a) $a = \pm 1; d = \pm 1$.
   (b) $a = 0, 2; d = 0, 2$.
   (c) $a = 1, d = 0, 2; a = -1, d = 0; a = -1, d = 2, 3|b$.
   (d) $a = 0, 2, d = 1; a = 0, d = -1; a = 2, d = -1, 3|b$.

Proof. In view of Corollary 3.3.5, A is strongly clean in $T_2(\mathbb{Z})$ if and only if A is strongly clean in $M_2(\mathbb{Z})$. Therefore we are done by Lemma 1.3.7. □

Lemma 3.3.8. *Let $E = E^2 \in M_3(\mathbb{Z})$. Then it is one of the following forms:*

$$0, I_2, \begin{pmatrix} 1 & a & b \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & a & ab \\ & 1 & b \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & a \\ & 0 & b \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & a \\ & 1 & b \\ & & 0 \end{pmatrix}, \begin{pmatrix} 1 & a & -ab \\ & 0 & b \\ & & 1 \end{pmatrix}, \begin{pmatrix} 0 & a & b \\ & 1 & 0 \\ & & 1 \end{pmatrix},$$

where $a, b \in \mathbb{Z}$.

Proof. Straightforward. □

Lemma 3.3.9. *Let $A \in T_3(\mathbb{Z})$. Then $A \in U(T_3(\mathbb{Z}))$ if and only if $det(A) = \pm 1$.*

Proof. This is obvious, as $U(\mathbb{Z}) = \{-1, 1\}$. □

Theorem 3.3.10. *Let $A = \begin{pmatrix} a & b & c \\ & d & e \\ & & f \end{pmatrix} \in T_3(\mathbb{Z})$. Then the following are equivalent:*

(1) *There exists $E^2 = E$ and $det(E) = 1$ such that $A - E \in U(T_3(\mathbb{Z}_3))$ and $AE = EA$.*
(2) $(a-1)(d-1)(f-1) = \pm 1$.

Proof. Clearly, $A - I_3 = \begin{pmatrix} a-1 & b & c \\ & d-1 & e \\ & & f-1 \end{pmatrix} \in U(T_3(\mathbb{Z}))$ if and only if $(a-1)(d-1)(f-1) = \pm 1$, desired. □

Example 3.3.11. Let $A = \begin{pmatrix} 2 & 3 & 2 \\ & 2 & -2 \\ & & 0 \end{pmatrix} \in T_3(\mathbb{Z})$. Then there exists $E^2 = E$ and $det(E) = 1$ such that $A - E \in U(T_3(\mathbb{Z}))$ and $AE = EA$. In fact, we have

$$A = I_2 + \begin{pmatrix} 1 & 3 & 2 \\ & 1 & -2 \\ & & -1 \end{pmatrix}.$$

Theorem 3.3.12. *Let $A = \begin{pmatrix} a & b & c \\ & d & e \\ & & f \end{pmatrix} \in T_3(\mathbb{Z})$. Then there exists $E^2 = E$ and $det(E) = 0$ such that $A - E \in U(T_3(\mathbb{Z}))$ and $AE = EA$ if and only if*

(1) *$adf = \pm 1$; or*
(2) *$(1-a)df = \pm 1$ and the equations*

$$\begin{aligned} (a-d)x - b &= 0, \\ -ex + (a-f)y - c &= 0 \end{aligned}$$

is solvable; or
(3) *$(1-a)df = \pm 1$ and the equations*

$$\begin{aligned} (d-a)x - b &= 0, \\ (d-f)y - e &= 0, \\ ex - by + (f-a)xy &= 0, \end{aligned}$$

is solvable; or

(4) $ad(1-f) = \pm 1$ *and the equations*

$$\begin{aligned}(f-a)x - by - c &= 0,\\ (f-d)y - e &= 0,\end{aligned}$$

is solvable; or

(5) $(1+ad-a-d)f = \pm 1$ *and the equations*

$$\begin{aligned}(a-f)x + by - c &= 0,\\ (d-f)y - e &= 0,\end{aligned}$$

is solvable; or

(6) $(1+af-a-f)d = \pm 1$ *and the equations*

$$\begin{aligned}(a-d)x - b &= 0,\\ ex - (f-a)xz - bz &= 0,\\ (f-d)z - e &= 0\end{aligned}$$

is solvable; or

(7) $a(1+df-d-f) = \pm 1$ *and the equations*

$$\begin{aligned}(d-a)x - b &= 0,\\ ex - (f-a)y - c &= 0,\end{aligned}$$

is solvable.

Proof. This is easily obtained by using Lemma 3.3.8 and Lemma 3.3.9. □

Example 3.3.13. Let $A = \begin{pmatrix} 1 & 2 & -5 \\ & 2 & 9 \\ & & -1 \end{pmatrix} \in T_3(\mathbb{Z})$. Then there exists $E^2 = E$ and $det(E) = 0$ such that $A - E \in U(T_3(\mathbb{Z}))$ and $AE = EA$. In fact, we have

$$A = \begin{pmatrix} 0 & 2 & 6 \\ & 1 & 3 \\ & & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & -11 \\ & 1 & 6 \\ & & -1 \end{pmatrix}.$$

3.4 Generalized Matrix Rings

In this section, we are concerned with the strong cleanness for generalized matrix rings. Let R be a ring, and let $s \in C(R)$. Let $M_{(s)}(R) =$

$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a,b,c,d \in R \right\}$, where the operations are defined as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a+a' & b+b' \\ c+c' & d+d' \end{pmatrix},$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} aa'+sbc' & ab'+bd' \\ ca'+dc' & scb'+dd' \end{pmatrix}.$$

Then $M_{(s)}(R)$ is a ring with the identity $I_2 = \begin{pmatrix} 1_R & 0 \\ 0 & 1_R \end{pmatrix}$.

Lemma 3.4.1. *Let R be a ring, and let $s \in C(R)$. Then $J\big(M_{(s)}(R)\big) = \left\{ \begin{pmatrix} c & x \\ y & d \end{pmatrix} \in M_{(s)}(R) \;\middle|\; c,d,sx,sy \in J(R) \right\}$.*

Proof. Let $A = \begin{pmatrix} c & x \\ y & d \end{pmatrix} \in M_{(s)}(R)$ with $c,d,sx,sy \in J(R)$. For any $X = \begin{pmatrix} p & q \\ k & l \end{pmatrix} \in M_{(s)}(R)$, we check that $I_2 - \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} X = \begin{pmatrix} 1-cp & -cq \\ 0 & 1 \end{pmatrix} \in U(M_{(s)}(R))$. Hence, $\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} \in J(M_{(s)}(R))$. Further, we see that $I_2 - \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} X = \begin{pmatrix} 1-sxk & -xd \\ 0 & 1 \end{pmatrix} \in U(M_{(s)}(R))$, and so $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in J(M_{(s)}(R))$. Likewise, $\begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} \in J(M_{(s)}(R))$. Therefore

$$A = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} \in J(M_{(s)}(R)).$$

Let $A = \begin{pmatrix} c & x \\ y & d \end{pmatrix} \in J\big(M_{(s)}(R)\big)$. For any $r \in R$, we see that $I_2 - A \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1-cr & 0 \\ -yr & 1 \end{pmatrix} \in U(M_{(s)}(R))$; hence, $1-cr \in U(R)$. This implies that $c \in J(R)$. Furthermore, $I_2 - A \begin{pmatrix} 0 & 0 \\ r & 0 \end{pmatrix} = \begin{pmatrix} 1-sxr & 0 \\ -dr & 1 \end{pmatrix} \in U(M_{(s)}(R))$, and then $1-sxr \in U(R)$. It follows that $sx \in J(R)$. Similarly, $sy, d \in J(R)$. This completes the proof. □

Theorem 3.4.2. *Let R be a local ring, and let $s \in C(R)$. Then $A \in M_{(s)}(R)$ is strongly clean if and only if*

(1) *A or $I_2 - A$ is invertible; or*
(2) *A is similar to a diagonal matrix.*

Proof. $\Longleftarrow$ If A or $I_2 - A$ is invertible, then $A \in M_{(s)}(R)$ is strongly clean. Suppose that A is similar to a diagonal matrix. Then there exists some $U \in U\big(M_{(s)}(R)\big)$ such that $U^{-1}AU = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. As R is local, it is strongly clean. Hence, we can find idempotents $e, f \in R$ such that $a - e, b - f \in U(R), ea = ae$ and $bf = fb$. Set $E = U \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} U^{-1}$. Then $E^2 = E \in M_{(s)}(R)$ and $EA = AE$. Further, $A - E \in U\big(M_{(s)}(R)\big)$, as desired.

$\Longrightarrow$ Suppose that A and $I_2 - A$ are not invertible. Write $A = E + U$ with $EA = AE, E^2 = E, U \in U\big(M_{(s)}(R)\big)$. Set $E = \begin{pmatrix} c & x \\ y & d \end{pmatrix}$. Then $E \notin J\big(M_{(s)}(R)\big)$.

Case I. $c \in U(R)$. Then we have $\begin{pmatrix} 1 & 0 \\ -yc^{-1} & 1 \end{pmatrix} E \begin{pmatrix} c^{-1} & -c^{-1}x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & d - syc^{-1}x \end{pmatrix}$. This implies that $\begin{pmatrix} 1 & 0 \\ 0 & d - syc^{-1}x \end{pmatrix} \in M_{(s)}(R)$ is regular, and then so is $d - syc^{-1} \in R$. As R is local, we easily check that $d - syc^{-1}$ is zero or invertible. Hence, we have $P, Q \in U\big(M_{(s)}(R)\big)$ such that PEQ is an idempotent diagonal matrix. In light of Lemma 2.4.10, E is similar to a diagonal matrix.

Case II. $d \in U(R)$. Similarly to the discussion in Case I, we easily verify that E is similar to a diagonal matrix.

Case III. $c, d \in J(R), sx \in U(R)$. We have $E \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} sx & c \\ d & sy \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \in U\big(M_{(s)}(R)\big)$. By using the argument in Case I, we prove that E is similar to a diagonal matrix.

Case IV. $c, d \in J(R), sy \in U(R)$. Analogously to the discussion in Case III, one easily checks that E is similar to a diagonal matrix.

Write $P^{-1}EP = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}$. We may assume that $e = 1$ and $f = 0$. Then $P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + P^{-1}UP$ and $P^{-1}AP \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} P^{-1}AP$. This forces that $P^{-1}AP$ is diagonal, as asserted. □

Lemma 3.4.3. *Let R be a local ring, and let $A \in M_{(s)}(R)$. Then $A \in U\big(M_{(s)}(R)\big)$, $I_2 - A \in U\big(M_{(s)}(R)\big)$, A or $I_2 - A$ is similar to a matrix $\begin{pmatrix} u & 1 \\ v & w \end{pmatrix}$, where $u \in 1 + J(R), v \in U(R), w \in J(R)$.*

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Suppose that $A, I_2 - A \notin U\big(M_{(s)}(R)\big)$. Then $a \in 1 + J(R), d \in J(R)$ or $a \in J(R), d \in 1 + J(R)$.

Case I. $a \in 1 + J(R), d \in J(R)$. Then A is similar to

$$\begin{aligned}
&\begin{pmatrix} 1 & a^{-1}(b-1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -a^{-1}(b-1) \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} a + sa^{-1}(b-1)c & -(a + sa^{-1}(b-1)c)a^{-1}(b-1) + (b + a^{-1}(b-1)d) \\ c & d - sca^{-1}(b-1) \end{pmatrix} \\
&= \begin{pmatrix} a_1 & b_1 \\ c & d_1 \end{pmatrix},
\end{aligned}$$

where $a_1, b_1 \in 1 + J(R), d_1 \in J(R)$. So we may assume that $b \in 1 + J(R)$. Hence, A is similar to

$$\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & b^{-1} \end{pmatrix} = \begin{pmatrix} a & 1 \\ bc & bdb^{-1} \end{pmatrix}.$$

If $c \in U(R)$, we are done. If $c \in J(R)$, A is similar to

$$\begin{aligned}
&\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & 1 \\ bc & bdb^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} a - s & 1 \\ a + bc - s - bdb^{-1} & s + bdb^{-1} \end{pmatrix},
\end{aligned}$$

and we are through.

Case II. $a \in J(R), d \in 1 + J(R)$. By Case I, we see that $I_2 - A$ is similar to a matrix $\begin{pmatrix} u' & 1 \\ v' & w' \end{pmatrix}$, where $u' \in 1 + J(R), v' \in U(R), w' \in J(R)$. This completes the proof. □

Lemma 3.4.4. *Let R be a local ring, let $s \in J(R) \cap C(R)$, and let $A = \begin{pmatrix} u & 1 \\ v & w \end{pmatrix}$ where $u \in 1 + J(R), v \in U(R), w \in J(R)$. Then the following are equivalent:*

(1) *$A \in M_{(s)}(R)$ is strongly clean.*
(2) *$t^2 - (vuv^{-1} + w)t + (vuv^{-1}w - sv)$ has a root in $1 + J(R)$ and $t^2 - (u + w)t + (wu - sv)$ has a root in $J(R)$.*

Proof. (1) $\Rightarrow$ (2) Since $A, I_2 - A \notin U(M_{(s)}(R))$, it follows by Theorem 3.4.2 that there exist $\begin{pmatrix} a & x \\ y & b \end{pmatrix} \in U(M_{(s)}(R))$ and $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \in M_{(s)}(R)$ such that

$$\begin{pmatrix} u & 1 \\ v & w \end{pmatrix} \begin{pmatrix} a & x \\ y & b \end{pmatrix} = \begin{pmatrix} a & x \\ y & b \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

Hence,

$$\begin{aligned} ua + sy &= a\alpha; \\ va + wy &= y\alpha; \\ ux + b &= x\beta; \\ svx + wb &= b\beta. \end{aligned}$$

Clearly, $a, b \in U(R)$; hence, $\beta \in J(R)$. As $I_2 - \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \notin U(M_{(s)}(R))$, we see that $\alpha \in 1 + J(R)$. Further, we check that $x, y \in U(R)$. Let $\lambda = y\alpha y^{-1} \in 1 + J(R)$ and $\mu = x\beta x^{-1} \in J(R)$. Then we verify that

$$\begin{aligned} &\lambda^2 - (vuv^{-1} + w)\lambda + vuv^{-1}w \\ &= \big((y\alpha)\alpha - (vuv^{-1} + w)y\alpha + vuv^{-1}wy\big)y^{-1} \\ &= \big((va + wy)\alpha - (vuv^{-1} + w)y\alpha + vuv^{-1}wy\big)y^{-1} \\ &= \big(va\alpha - vuv^{-1}(va + wy) + vuv^{-1}wy\big)y^{-1} \\ &= \big(va\alpha - vua\big)y^{-1} \\ &= \big(v(ua + sy) - vua\big)y^{-1} \\ &= sv, \end{aligned}$$

and

$$\begin{aligned} &\mu^2 - (u + w)\mu + wu \\ &= x\beta^2x^{-1} - (u + w)x\beta x^{-1} + wu \\ &= \big((ux + b)\beta - (u + w)x\beta + wux\big)x^{-1} \\ &= (b\beta - wx\beta + wux)x^{-1} \\ &= \big(svx + wb - w(ux + b) + wux\big)x^{-1} \\ &= sv. \end{aligned}$$

Thus proving (2).

(2) $\Rightarrow$ (1) Suppose that $\alpha \in 1 + J(R)$ is the root of $t^2 - (vuv^{-1} + w)t + (vuv^{-1}w - sv)$ and $\beta \in J(R)$ is the root of $t^2 - (u + w)t + (wu - sv)$. Let $P = \begin{pmatrix} v^{-1}(\alpha - w) & 1 \\ 1 & \beta - u \end{pmatrix}$. Then $P \in U(M_{(s)}(R))$. It is easy to verify that $AP = P\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$. Therefore we complete the proof by Theorem 3.4.2. $\square$

Theorem 3.4.5. *Let R be a local ring, and let $s \in J(R) \cap C(R)$. Then the following are equivalent:*

(1) $M_{(s)}(R)$ *is strongly clean.*
(2) *For any* $u \in 1+J(R), v \in U(R)$ *and* $w \in J(R)$, $t^2-(u+w)t+(uw-sv)$ *has a root in* $1+J(R)$ *and* $t^2-(u+w)t+(wu-sv)$ *has a root in* $J(R)$.
(3) *Every matrix* $\begin{pmatrix} u & 1 \\ v & w \end{pmatrix}$, *where* $u \in 1+J(R), v \in U(R)$ *and* $w \in J(R)$, *is strongly clean.*

Proof. (1) $\Rightarrow$ (2) Let $u \in 1+J(R), v \in U(R)$ and $w \in J(R)$. Set $A = \begin{pmatrix} u & 1 \\ v & w \end{pmatrix}$. Then $A, I_2 - A \notin U\big(M_{(s)}(R)\big)$. In view of Lemma 3.4.4, $t^2-(u+w)t+(wu-sv)$ has a root in $J(R)$.

Set $B = \begin{pmatrix} v^{-1}uv & 1 \\ v & w \end{pmatrix}$. Then $B, I_2 - B \notin U\big(M_{(s)}(R)\big)$. By virtue of Lemma 3.4.4, $t^2-(u+w)t+(uw-sv)$ has a root in $1+J(R)$.

(2) $\Rightarrow$ (1) Let $A \in M_{(s)}(R)$. Then A or $I_2 - A$ is invertible; hence, A is strongly clean; otherwise, A or $I_2 - A$ is similar to a matrix $\begin{pmatrix} u & 1 \\ v & w \end{pmatrix}$, where $u \in 1+J(R), v \in U(R), w \in J(R)$.

Case I. A is similar to a matrix $\begin{pmatrix} u & 1 \\ v & w \end{pmatrix}$. Since $vuv^{-1} \in 1+J(R)$, by hypothesis, $t^2-(vuv^{-1}+w)t+(vuv^{-1}w-sv)$ has a root in $1+J(R)$. In view of Lemma 3.4.4, $\begin{pmatrix} u & 1 \\ v & w \end{pmatrix}$ is strongly clean, and then so is A.

Case II. $I_2 - A$ is similar to a matrix $\begin{pmatrix} u & 1 \\ v & w \end{pmatrix}$. By the preceding discussion, $I_2 - A$ is strongly clean, and then so is A. Hence, $M_{(s)}(R)$ is strongly clean.

(1) $\Rightarrow$ (3) This is trivial.
(3) $\Rightarrow$ (1) We easily obtain the result, by Lemma 3.4.3. □

Corollary 3.4.6. *Let* R *be a commutative local ring, and let* $s \in J(R)$. *Then the following are equivalent:*

(1) $M_{(s)}(R)$ *is strongly clean.*
(2) *For any* $u \in 1+J(R), v \in U(R)$ *and* $w \in J(R)$, $t^2-(u+w)t+(uw-sv)$ *is solvable.*

Proof. (1) $\Rightarrow$ (2) This is obvious, by Theorem 3.4.5.

(2) $\Rightarrow$ (1) Suppose that $t^2-(u+w)t+(uw-sv)$ has a root α. Then $\beta = u+w-\alpha$ is the other root. One easily checks that $\alpha\beta = uw-sv \in J(R)$. This shows that one of α and β is in $J(R)$. Say $\alpha \in J(R)$. Then $\beta \in 1+J(R)$. This completes the proof by Theorem 3.4.5. □

Example 3.4.7. Let R be a weakly bleached local ring. Then $M_{(0)}(R)$ is strongly clean.

Proof. Let $A = \begin{pmatrix} u & 1 \\ v & w \end{pmatrix}$, where $u \in 1 + J(R), v \in U(R), w \in J(R)$. Since R is a weakly bleached local ring, there exist some $x, y \in R$ such that $ux - xw = -1$ and $wy - yu = -v$. Let $P = \begin{pmatrix} 1 & x \\ y & 1 \end{pmatrix}$. Then $P \in U(M_{(0)}(R))$. It is easy to verify that $AP = P \begin{pmatrix} u & 0 \\ 0 & w \end{pmatrix}$. Hence, A is similar to a diagonal matrix. Therefore $K_0(R)$ is strongly clean, by Theorem 3.4.2 and Theorem 3.4.5. □

Let R be a commutative ring, and let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{(s)}(R)$. Set $tr(A) = a + d$ and $det_s(A) = ad - sbc$. We now derive

Theorem 3.4.8. *Let R be a commutative local ring, let $s \in R$ and let $A \in M_{(s)}(R)$. Then $A \in M_{(s)}(R)$ is strongly clean if and only if*

(1) *A or $I_2 - A$ is invertible in $M_{(s)}(R)$; or*
(2) *$t^2 - tr(A)t + det_s(A) = 0$ is solvable in R.*

Proof. Suppose that $A, I_2 - A \notin U(M_{(s)}(R))$. In view of Lemma 3.4.3, A or $I_2 - A$ is similar to $\begin{pmatrix} u & 1 \\ v & w \end{pmatrix}$, where $u \in 1 + J(R), v \in U(R), w \in J(R)$.

Case I. A is similar to a matrix $\begin{pmatrix} u & 1 \\ v & w \end{pmatrix}$. Then $t^2 - tr(A)t + det_s(A) = 0$ is solvable if and only if $t^2 - (u + w)t + (uw - sv) = 0$ is solvable, if and only if $\begin{pmatrix} u & 1 \\ v & w \end{pmatrix}$ is strongly clean by Lemma 3.4.4, as desired.

Case II. $I_2 - A$ is similar to a matrix $\begin{pmatrix} u & 1 \\ v & w \end{pmatrix}$. Then $t^2 - tr(A)t + det_s(A) = 0$ is solvable if and only if $x^2 - tr(I_2 - A) + det_s(I_2 - A) = 0$ is solvable, if and only if $x^2 - (u + w)x + (uw - sv) = 0$ is solvable, if and only if $\begin{pmatrix} u & 1 \\ v & w \end{pmatrix}$ is strongly clean. As A is strongly clean if and only if so is $I_2 - A$, hence the result. □

Corollary 3.4.9. *Let R be a commutative local ring. Then the following are equivalent:*

(1) $M_2(R)$ *is strongly clean.*
(2) $M_{(s)}(R)$ *is strongly clean for all* $s \in J(R)$.

Proof. (1) $\Rightarrow$ (2) Let $s \in J(R)$, and let $u \in 1 + J(R), v \in U(R)$ and $w \in J(R)$. By Proposition 1.2.9, the equation $x^2 - x + \frac{uw-sv}{(u+w)^2} = 0$ is solvable in R. Let $t = (u + w)x$. Then $t^2 - (u + w)t + (uw - sv) = 0$ is solvable. In light of Corollary 3.4.6, $M_{(s)}(R)$ is strongly clean.

(2) $\Rightarrow$ (1) Let $w \in J(R)$. Then $M_{(w)}(R)$ is strongly clean. Choose $A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \in M_{(w)}(R)$. Then $tr(A) = 1$ and $det_w(A) = w$. By virtue of Theorem 3.4.8, the equation $t^2 - tr(A)t + det_w(A) = 0$ is solvable. That is, $t^2 - t + w = 0$ is solvable. According to Proposition 1.2.9, $M_2(R)$ is strongly clean, as asserted. □

Let R be a commutative local ring. We note that $M_{(s)}(R) \cong M_2(R)$ for all $s \in U(R)$ (see [376, Lemma 3]).

We conclude this chapter by investigating a type of generalized matrices which possess the similar form of both lower and upper triangular matrices. Let R be a ring, let $n = 2k - 1 (k \in \mathbb{N})$, and let

$$\Delta_n = \left\{ \left(\begin{array}{ccccccc} a_{11} & & & & & & \\ a_{12} & a_{22} & & & & & \\ \vdots & \vdots & \ddots & & & & \\ a_{1k} & a_{2k} & \cdots & a_{kk} = b_{11} & b_{12} & \cdots & b_{1k} \\ & & & & b_{22} & \cdots & b_{2k} \\ & & & & & \ddots & \vdots \\ & & & & & & b_{kk} \end{array} \right) \;\middle|\; \text{each } a_{ij}, b_{ij} \in R \right\}.$$

Then $\Delta_n(R)$ is a ring with the identity I_n under the usual addition and multiplication.

Lemma 3.4.10. *Let R be a ring. Then the following hold:*

(1) $\begin{pmatrix} e_{11} & & & & & \\ \vdots & \ddots & & & & \\ e_{1k} & \cdots & e_{kk} = f_{11} & \cdots & f_{1k} \\ & & & \ddots & \vdots \\ & & & & f_{kk} \end{pmatrix} \in \Delta_{2k-1}(R)$ *is an idempotent if and only*

if so are (e_{ij}) *and* (f_{ij}) *in triangular matrix rings, respectively.*

(2)

$$\begin{pmatrix} a_{11} & & & \\ \vdots & \ddots & & \\ a_{1k} & \cdots & a_{kk} = b_{11} & \cdots & b_{1k} \\ & & & \ddots & \vdots \\ & & & & b_{kk} \end{pmatrix} \begin{pmatrix} e_{11} & & & \\ \vdots & \ddots & & \\ e_{1k} & \cdots & e_{kk} = f_{11} & \cdots & f_{1k} \\ & & & \ddots & \vdots \\ & & & & f_{kk} \end{pmatrix}$$

$$= \begin{pmatrix} e_{11} & & & \\ \vdots & \ddots & & \\ e_{1k} & \cdots & e_{kk} = f_{11} & \cdots & f_{1k} \\ & & & \ddots & \vdots \\ & & & & f_{kk} \end{pmatrix} \begin{pmatrix} a_{11} & & & \\ \vdots & \ddots & & \\ a_{1k} & \cdots & a_{kk} = b_{11} & \cdots & b_{1k} \\ & & & \ddots & \vdots \\ & & & & b_{kk} \end{pmatrix}$$

if and only if $(a_{ij})(e_{ij}) = (e_{ij})(a_{ij})$ *and* $(b_{ij})(f_{ij}) = (f_{ij})(b_{ij})$.

Proof. Straightforward. □

Theorem 3.4.11. *Let* R *be a bleached local ring, and let* $n \geq 3$ *be an odd number. Then* $\Delta_n(R)$ *is strongly clean.*

Proof. Write $n = 2k + 1$. If $k = 0$, the result is true. Suppose the result holds for $k - 1 \in \mathbb{N}$. Let $A = (a_{ij}) \in T_{2k+1}(R)$. Write

$$A = \begin{pmatrix} a_{11} & & \\ \alpha & B & \beta \\ & & b_{kk} \end{pmatrix}, \text{where } a_{11}, b_{kk} \in R, \alpha, \beta \in M_{(2k-1)\times 1}(R), B \in \Delta_{2k-1}(R).$$

Choose $e_{11} = 0$ if $a_{11} \in U(R)$ and $e_{11} = 1$ if $a_{11} \in J(R)$. Choose $f_{kk} = 0$ if $b_{kk} \in U(R)$ and $f_{kk} = 1$ if $b_{kk} \in J(R)$. By hypothesis, $\Delta_{2k-1}(R)$ is strongly clean. Thus, we can find an idempotent $H \in \Delta_{2k-1}(R)$ such that $B - H \in U\big(\Delta_{2k-1}(R)\big)$ and $BH = HB$. Write

$$H = \begin{pmatrix} e_{22} & & & & & & & \\ e_{32} & e_{33} & & & & & & \\ \vdots & \vdots & \ddots & & & & & \\ e_{(k+1)2} & e_{(k+1)3} & \cdots & e_{(k+1)(k+1)} = f_{11} & f_{12} & \cdots & f_{1k} \\ & & & & f_{22} & \cdots & f_{2k} \\ & & & & & \ddots & \vdots \\ & & & & & & f_{kk} \end{pmatrix}.$$

Consider $\begin{pmatrix} a_{11} & \\ a_{21} & a_{22} \end{pmatrix} \in T_2(R)$. If $e_{11} = e_{22}$, we choose $e_{21} = 0$. If $e_{11} = 0$ and $e_{22} = 1$, then $l_{a_{22}} - r_{a_{11}}$ is surjective; hence, $a_{22}x - xa_{11} = a_{21}$ has a root e_{21}. If $e_{11} = 1$ and $e_{22} = 0$, similarly, $a_{22}x - xa_{11} = -a_{21}$ has a root e_{21}. Thus, in any case, we have some $e_{21} \in R$ and an idempotent $\begin{pmatrix} e_{11} & \\ e_{21} & e_{22} \end{pmatrix} \in T_2(R)$ such that $\begin{pmatrix} a_{11} & \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} e_{11} & \\ e_{21} & e_{22} \end{pmatrix} = \begin{pmatrix} e_{11} & \\ e_{21} & e_{22} \end{pmatrix}\begin{pmatrix} a_{11} & \\ a_{21} & a_{22} \end{pmatrix}$, and that $\begin{pmatrix} a_{11} & \\ a_{21} & a_{22} \end{pmatrix} - \begin{pmatrix} e_{11} & \\ e_{21} & e_{22} \end{pmatrix} \in U\big(T_2(R)\big)$.

Consider $\begin{pmatrix} a_{11} & & \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. If $e_{11} = e_{33}$, applying Lemma 3.1.2 to the transpose of the matrix above, we can find $e_{31} \in R$ such that $\begin{pmatrix} e_{11} & & \\ e_{21} & e_{22} & \\ e_{31} & e_{32} & e_{33} \end{pmatrix}$ is an idempotent, $\begin{pmatrix} a_{11} & & \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} e_{11} & & \\ e_{21} & e_{22} & \\ e_{31} & e_{32} & e_{33} \end{pmatrix} = \begin{pmatrix} e_{11} & & \\ e_{21} & e_{22} & \\ e_{31} & e_{32} & e_{33} \end{pmatrix}\begin{pmatrix} a_{11} & & \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, where $e_{31} = -e_{21}e_{32}$ or $e_{31} = e_{21}e_{32}$. If $e_{11} \neq e_{33}$, as is the preceding discussion, $a_{11}x - xa_{33} = z$ is solvable. Thus, we let $e_{31} \in R$ be the root of the equation $a_{11}e_{31} - e_{31}a_{33} = e_{21}a_{32} - a_{21}e_{32} + a_{31}$ or $a_{11}e_{31} - e_{31}a_{33} = e_{21}a_{32} - a_{21}e_{32} - a_{31}$. By applying Lemma 3.1.2 to the transpose of the preceding matrix, there is an idempotent $\begin{pmatrix} e_{11} & & \\ e_{21} & e_{22} & \\ e_{31} & e_{32} & e_{33} \end{pmatrix}$ such that $\begin{pmatrix} a_{11} & & \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} e_{11} & & \\ e_{21} & e_{22} & \\ e_{31} & e_{32} & e_{33} \end{pmatrix} = \begin{pmatrix} e_{11} & & \\ e_{21} & e_{22} & \\ e_{31} & e_{32} & e_{33} \end{pmatrix}\begin{pmatrix} a_{11} & & \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. By iteration of this process, we find $e_{41}, \ldots,$ $e_{(k+1)1}$ and an idempotent $\begin{pmatrix} e_{11} & & & \\ e_{21} & e_{22} & & \\ \vdots & \vdots & \ddots & \\ e_{(k+1)1} & e_{(k+1)2} & \cdots & e_{(k+1)(k+1)} \end{pmatrix} \in M_{k+1}(R)$

such that

$$\begin{pmatrix} a_{11} & & & \\ a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{(k+1)1} & a_{(k+1)2} & \cdots & a_{(k+1)(k+1)} \end{pmatrix} \begin{pmatrix} e_{11} & & & \\ e_{21} & e_{22} & & \\ \vdots & \vdots & \ddots & \\ e_{(k+1)1} & e_{(k+1)2} & \cdots & e_{(k+1)(k+1)} \end{pmatrix}$$
$$= \begin{pmatrix} e_{11} & & & \\ e_{21} & e_{22} & & \\ \vdots & \vdots & \ddots & \\ e_{(k+1)1} & e_{(k+1)2} & \cdots & e_{(k+1)(k+1)} \end{pmatrix} \begin{pmatrix} a_{11} & & & \\ a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{(k+1)1} & a_{(k+1)2} & \cdots & a_{(k+1)(k+1)} \end{pmatrix}.$$

Likewise, we have $f_{(k+1)(k+1)}, \ldots, f_{1(k+1)} \in R$ and an idempotent $\begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1(k+1)} \\ & f_{22} & \cdots & f_{2(k+1)} \\ & & \ddots & \vdots \\ & & & f_{(k+1)(k+1)} \end{pmatrix} \in M_{k+1}(R)$ such that

$$\begin{pmatrix} b_{11} & \cdots & b_{1(k+1)} \\ & \ddots & \vdots \\ & & b_{(k+1)(k+1)} \end{pmatrix} \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1(k+1)} \\ & f_{22} & \cdots & f_{2(k+1)} \\ & & \ddots & \vdots \\ & & & f_{(k+1)(k+1)} \end{pmatrix}$$
$$= \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1(k+1)} \\ & f_{22} & \cdots & f_{2(k+1)} \\ & & \ddots & \vdots \\ & & & f_{(k+1)(k+1)} \end{pmatrix} \begin{pmatrix} b_{11} & \cdots & b_{1(k+1)} \\ & \ddots & \vdots \\ & & b_{(k+1)(k+1)} \end{pmatrix}.$$

Choose

$$E = \begin{pmatrix} e_{11} & & & & & & \\ e_{12} & e_{22} & & & & & \\ \vdots & \vdots & \ddots & & & & \\ e_{1(k+1)} & e_{2(k+1)} & \cdots & e_{(k+1)(k+1)} = f_{11} & f_{12} & \cdots & f_{1(k+1)} \\ & & & & f_{22} & \cdots & f_{2(k+1)} \\ & & & & & \ddots & \vdots \\ & & & & & & f_{(k+1)(k+1)} \end{pmatrix}$$
$$\in \Delta_{2k+1}(R).$$

By virtue of Lemma 3.4.10, $E^2 = E$ and $AE = EA$. Further, $A - E \in U\big(T_{2k+1}(R)\big)$. By induction, we complete the proof. □

Corollary 3.4.12. *Let R be a commutative local ring, and let $n \geq 3$ be an odd number. Then $\Delta_n(R)$ is strongly clean.*

Proof. It is an immediate consequence of Theorem 3.4.11, as every commutative local ring is weakly bleached. □

Corollary 3.4.13. *Let R be a local ring. Then the following are equivalent:*

(1) *$\Delta_3(R)$ is strongly clean.*
(2) *For any $a \in J(R), b \in R, c \in 1 + J(R)$, $l_b - r_a$ or $l_b - r_c$ is surjective.*

Proof. (1) $\Rightarrow$ (2) Suppose that $\Delta_3(R)$ is strongly clean. Let $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then $T_2(R) \cong E\Delta_3(R)E$. Thus, $T_2(R)$ is strongly clean. According to Corollary 1.2.17, R is weakly bleached. Let $a \in J(R), b \in R, c \in 1 + J(R)$. Then $b \in U(R) \cap \big(1 + U(R)\big)$; otherwise, $b \in J(R)$ or $b \in 1 + J(R)$, and so $l_b - r_a$ or $l_b - r_c$ is surjective. Suppose that $l_b - r_a$ is not surjective. Then we can find some $a_{21} \in R$ such that $bx - xa = -a_{21}$ is not solvable. Let $v \in R$. Choose $A = \begin{pmatrix} a & 0 & 0 \\ a_{21} & b & v \\ 0 & 0 & c \end{pmatrix} \in \Delta_3(R)$. Then we can find an idempotent $E = (e_{ij}) \in \Delta_3(R)$ such that $A - E \in U\big(\Delta_3(R)\big)$ and $EA = AE$. This implies that $e_{11}, e_{22}, e_{33} \in R$ are all idempotents, $e_{21} = e_{21}e_{11} + e_{22}e_{21}$ and $e_{23} = e_{22}e_{23} + e_{23}e_{33}$. Clearly, $e_{11} = 1$ and $e_{33} = 0$; otherwise, $A - E$ is not invertible. Thus, $E = \begin{pmatrix} 1 & 0 & 0 \\ e_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e_{23} \\ 0 & 0 & 0 \end{pmatrix}$. If $E = \begin{pmatrix} 1 & 0 & 0 \\ e_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then

$$\begin{pmatrix} a & 0 & 0 \\ e_{21}a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = EA = AE = \begin{pmatrix} a & 0 & 0 \\ a_{21} + be_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and so $be_{21} - e_{21}a = -a_{21}$. This gives a contradiction, and thus $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e_{23} \\ 0 & 0 & 0 \end{pmatrix}$. Hence,

$$\begin{pmatrix} a & 0 & 0 \\ a_{21} & b & v + e_{23}c \\ 0 & 0 & 0 \end{pmatrix} = EA = AE = \begin{pmatrix} a & 0 & 0 \\ a_{21} & b & be_{23} \\ 0 & 0 & 0 \end{pmatrix},$$

and so $be_{23} - e_{23}c = v$. This implies that $l_b - r_c$ is surjective.

(2) $\Rightarrow$ (1) Clearly, R is weakly bleached. Let $A = (a_{ij}) \in \Delta_3(R)$. Case 1. $a_{11}, a_{22}, a_{33} \in J(R)$. Then $A = I_2 + (A - I_2)$, and so $A - I_2 \in U(\Delta_3(R))$. Then $A \in \Delta_3(R)$ is strongly clean.

Case 2. $a_{11} \in U(R), a_{22}, a_{33} \in J(R)$. If $a_{11} - 1 \in J(R)$, by hypothesis, we can find some $e_{21} \in R$ such that $a_{22}e_{21} - e_{21}a_{11} = a_{21}$. Choose $E = \begin{pmatrix} 0 & 0 & 0 \\ e_{21} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \Delta_3(R)$. Then $E = E^2$, and that $A = E + (A - E)$, where $A - E \in U(\Delta_3(R))$. It is easy to verify that $EA = AE$. In addition,

$$EA = \begin{pmatrix} 0 & 0 & 0 \\ e_{21}a_{11} + a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ a_{22}e_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = AE.$$

Hence $A \in \Delta_3(R)$ is strongly clean. If $a_{11} - 1 \in U(R)$, we can choose $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \Delta_3(R)$. Then $E = E^2$, and that $A = E + (A - E)$, where $A - E \in U(\Delta_3(R))$. It is easy to verify that $EA = AE$. Hence $A \in \Delta_3(R)$ is strongly clean.

Case 3. $a_{11} \in J(R), a_{22} \in U(R), a_{33} \in J(R)$. If $a_{22} - 1 \in J(R)$, by hypothesis, we can find some $e_{21}, e_{23} \in R$ such that $a_{22}e_{21} - e_{21}a_{11} = -a_{21}$ and $a_{22}e_{23} - e_{23}a_{33} = -a_{23}$. Choose $E = \begin{pmatrix} 1 & 0 & 0 \\ e_{21} & 0 & e_{23} \\ 0 & 0 & 1 \end{pmatrix} \in \Delta_3(R)$. Then $E = E^2$, and that $A = E + (A - E)$, where $A - E \in U(\Delta_3(R))$. In addition,

$$EA = \begin{pmatrix} a_{11} & 0 & 0 \\ e_{21}a_{11} & 0 & e_{23}a_{33} \\ 0 & 0 & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} + a_{22}e_{21} & 0 & a_{22}e_{23} + a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = AE.$$

Hence $A \in \Delta_3(R)$ is strongly clean. If $a_{22} - 1 \in U(R)$, we choose $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \Delta_3(R)$. Then $E = E^2$, and that $A = E + (A - E)$, where $A - E \in U(\Delta_3(R))$. Hence $A \in \Delta_3(R)$ is strongly clean.

Case 4. $a_{11}, a_{22} \in J(R), a_{33} \in U(R)$. If $a_{33} - 1 \in J(R)$, by hypothesis, we can find some $e_{21} \in R$ such that $a_{22}e_{23} - e_{23}a_{33} = a_{23}$. Choose $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e_{23} \\ 0 & 0 & 0 \end{pmatrix} \in \Delta_3(R)$. Then $E = E^2$, and that $A = E + (A - E)$, where $A - E \in U(\Delta_3(R))$. In addition,

$$EA = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} + e_{23}a_{33} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{22} & a_{22} & a_{22}e_{23} \\ 0 & 0 & 0 \end{pmatrix} = AE.$$

Hence $A \in \Delta_3(R)$ is strongly clean. If $a_{33} - 1 \in U(R)$, we choose $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \Delta_3(R)$. Then $E = E^2$, and that $A = E + (A - E)$, where $A - E \in U(\Delta_3(R))$. Hence $A \in \Delta_3(R)$ is strongly clean.

Case 5. $a_{11} \in J(R), a_{22}, a_{33} \in U(R)$. If $a_{22} - 1 \in J(R)$, by hypothesis, we can find some $e_{21} \in R$ such that $a_{22}e_{21} - e_{21}a_{11} = -a_{21}$. Choose $E = \begin{pmatrix} 1 & 0 & 0 \\ e_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \Delta_3(R)$. Then $E = E^2$, and that $A = E + (A - E)$, where $A - E \in U(\Delta_3(R))$. In addition,

$$EA = \begin{pmatrix} a_{11} & 0 & 0 \\ e_{21}a_{11} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} + a_{22}e_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = AE.$$

Hence $A \in \Delta_3(R)$ is strongly clean. If $a_{22} - 1, a_{33} - 1 \in U(R)$, we choose $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \Delta_3(R)$. Then $E = E^2$, and that $A = E + (A - E)$, where $A - E \in U(\Delta_3(R))$. Obviously, $EA = AE$. Hence $A \in \Delta_3(R)$ is strongly clean. If $a_{22} - 1 \in U(R), a_{33} - 1 \in J(R)$, by hypothesis, $l_{a_{22}} - r_{a_{11}}$ or $l_{a_{22}} - r_{a_{33}}$ is surjective. Thus, we can find some $e_{21} \in R$ such that $a_{22}e_{21} - e_{21}a_{11} = -a_{21}$ or some $e_{23} \in R$ such that $a_{22}e_{23} - e_{23}a_{33} = a_{23}$. Assume that $a_{22}e_{21} - e_{21}a_{11} = -a_{21}$. Choose $E = \begin{pmatrix} 1 & 0 & 0 \\ e_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \Delta_3(R)$. Then $E = E^2$, and that $A = E + (A - E)$, where $A - E \in U(\Delta_3(R))$. In addition,

$$EA = \begin{pmatrix} a_{11} & 0 & 0 \\ e_{21}a_{11} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} + a_{22}e_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = AE.$$

Hence $A \in \Delta_3(R)$ is strongly clean. Assume that $a_{22}e_{23} - e_{23}a_{33} = a_{23}$. Choose $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e_{23} \\ 0 & 0 & 0 \end{pmatrix} \in \Delta_3(R)$. Then $E = E^2$, and that $A = E + (A - E)$, where $A - E \in U(\Delta_3(R))$. In addition,

$$EA = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} + e_{23}a_{33} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{22}e_{23} \\ 0 & 0 & 0 \end{pmatrix} = AE.$$

Hence $A \in \Delta_3(R)$ is strongly clean.

Case 6. $a_{11} \in U(R), a_{22} \in J(R), a_{33} \in U(R)$. If $a_{11} - 1, a_{33} - 1 \in J(R)$, then we can find some $e_{21}, e_{23} \in R$ such that $a_{22}e_{21} - e_{21}a_{11} = a_{21}$ and $a_{22}e_{23} - e_{23}a_{33} = a_{23}$. Choose $E = \begin{pmatrix} 0 & 0 & 0 \\ e_{21} & 1 & e_{23} \\ 0 & 0 & 0 \end{pmatrix} \in \Delta_3(R)$. Then $E = E^2$, and that $A = E + (A - E)$, where $A - E \in U(\Delta_3(R))$. In addition,

$$EA = \begin{pmatrix} 0 & 0 & 0 \\ e_{21}a_{11} + a_{21} & a_{22} & a_{23} + e_{23}a_{33} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ a_{22}e_{21} & a_{22} & a_{22}e_{23} \\ 0 & 0 & 0 \end{pmatrix} = AE.$$

If $a_{11} - 1 \in J(R), a_{33} - 1 \in U(R)$, then we can find some e_{21} such that $a_{22}e_{21} - e_{21}a_{11} = a_{21}$. Choose $E = \begin{pmatrix} 0 & 0 & 0 \\ e_{21} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \Delta_3(R)$. Then $E = E^2$, and that $A = E + (A - E)$, where $A - E \in U(\Delta_3(R))$. In addition,

$$EA = \begin{pmatrix} 0 & 0 & 0 \\ e_{21}a_{11} + a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ a_{22}e_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = AE.$$

If $a_{11} - 1 \in U(R), a_{33} - 1 \in J(R)$, then we can find some $e_{23} \in R$ such that $a_{22}e_{23} - e_{23}a_{33} = a_{23}$. Choose $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e_{23} \\ 0 & 0 & 0 \end{pmatrix} \in \Delta_3(R)$. Then $E = E^2$, and that $A = E + (A - E)$, where $A - E \in U(\Delta_3(R))$. In addition,

$$EA = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} + e_{23}a_{33} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{22}e_{23} \\ 0 & 0 & 0 \end{pmatrix} = AE.$$

If $a_{11} - 1, a_{33} - 1 \in U(R)$, we choose $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \Delta_3(R)$. Then $E = E^2$, and that $A = E + (A - E)$, where $A - E \in U(\Delta_3(R))$. Obviously, $EA = AE$. Hence $A \in \Delta_3(R)$ is strongly clean.

Case 7. $a_{11}, a_{22} \in U(R), a_{33} \in J(R)$. If $a_{22} - 1 \in J(R)$, by hypothesis, we can find some $e_{23} \in R$ such that $a_{22}e_{23} - e_{23}a_{33} = -a_{23}$. Choose $E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_{23} \\ 0 & 0 & 1 \end{pmatrix} \in \Delta_3(R)$. Then $E = E^2$, and that $A = E + (A - E)$, where $A - E \in U(\Delta_3(R))$. In addition,

$$EA = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_{23}a_{33} \\ 0 & 0 & a_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_{22}e_{23} + a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = AE.$$

Hence $A \in \Delta_3(R)$ is strongly clean. If $a_{22} - 1 \in U(R), a_{11} - 1 \in J(R)$, by hypothesis, $l_{a_{22}} - r_{a_{11}}$ or $l_{a_{22}} - r_{a_{33}}$ is surjective. Thus, we can find some $e_{21} \in R$ such that $a_{22}e_{21} - e_{21}a_{11} = a_{21}$ or some $e_{23} \in R$ such that $a_{22}e_{23} - e_{23}a_{33} = -a_{23}$. Assume that $a_{22}e_{21} - e_{21}a_{11} = a_{21}$. Choose $E = \begin{pmatrix} 0 & 0 & 0 \\ e_{21} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \Delta_3(R)$. Then $E = E^2$, and that $A = E + (A - E)$, where $A - E \in U(\Delta_3(R))$. In addition,

$$EA = \begin{pmatrix} 0 & 0 & 0 \\ e_{21}a_{11} + a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ a_{22}e_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = AE.$$

Hence $A \in \Delta_3(R)$ is strongly clean. Assume that $a_{22}e_{23} - e_{23}a_{33} = -a_{23}$. Choose $E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_{23} \\ 0 & 0 & 1 \end{pmatrix} \in \Delta_3(R)$. Then $E = E^2$, and that $A = E + (A - E)$, where $A - E \in U(\Delta_3(R))$. In addition,

$$EA = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_{23}a_{33} \\ 0 & 0 & a_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_{22}e_{23} + a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = AE.$$

Hence $A \in \Delta_3(R)$ is strongly clean.

Case 8. $a_{11}, a_{22}, a_{33} \in U(R)$. Then $A = 0 + A$, where $A \in U(\Delta_3(R))$. Hence $A \in \Delta_3(R)$ is strongly clean.

Therefore $\Delta_3(R)$ is strongly clean. □

Notes. Theorem 3.1.4 is a slight generalization of [77, Theorem 9]. Corollary 3.1.17 and Corollary 3.1.18 were proved by Dorsey [241, Theorem 4.2.10 and Corollary 4.2.13]. Theorem 3.1.19 is due to Borooah et al. [77, Corollary 28]. Theorem 3.2.8 is due to Diesl [233, Lemma 2.3.4]. Theorem 3.2.11

is a version for DD_n of [77, Theorem 33]. Theorem 3.3.2 was proved by Diesl et al. [237, Theorem 2.5]. Theorem 3.3.4 was observed by Rajeswari and Aziz [450, Theorem 1.1]. Theorem 3.3.12 is due to Nikkhah [441, Theorem 2.8]. Theorem 3.4.2, Theorem 3.4.5 and Theorem 3.4.8 are due to Tang and Zhou [504, Lemma 4, Theorem 8 and Theorem 15].

Chapter 4

Strong J-Cleanness

This chapter is mainly concerned with the conditions that are stronger than strong cleanness. Replacing $U(R)$ by $J(R)$, we will parallel to introduce the concept of strong J-cleanness. We say that $a \in R$ is strongly J-clean provided that there exists an idempotent $e \in R$ such that $a - e \in J(R)$ and $ae = ea$. A ring R is strongly J-clean in case every element in R is strongly J-clean. A ring R has stable range 1 provided that $aR + bR = R$ implies that there exists $y \in R$ such that $a + by \in U(R)$. A long standing question asks whether strongly clean rings have stable range 1. The motivation of introducing strong J-cleanness is to construct a natural subclass of strongly clean rings which have stable range one. In Section 4.1, we investigate elementary properties of strongly J-clean rings. In Section 4.2, we get criteria on the strong J-cleanness of 2×2 matrices over local rings in terms of the solvability of quadratic equations. In Section 4.3 we introduce strongly $J^{\#}$-clean matrices, and then derive the criteria for an $n \times n$ matrix over a projective-free ring is strongly $J^{\#}$-clean. In the last section we discuss the unique cleanness for a ring R. This forms a subclass of strongly J-clean rings.

4.1 Strongly J-Clean Rings

The aim of this section is to provide necessary and sufficient conditions under which a ring is strongly J-clean. The following elementary property will be used repeatedly.

Theorem 4.1.1. *A ring R is strongly J-clean if and only if*

(1) *R is strongly clean;*
(2) *$R/J(R)$ is Boolean.*

Proof. Suppose that R is strongly J-clean. For any $a \in R$, there exists an idempotent $e \in R$ and $w \in J(R)$ such that $a = e + w$ and $ae = ea$. Thus, $a = (1-e)+(2e-1+w)$. Clearly, $2e-1+w = (2e-1)\big(1+(2e+1)w\big) \in U(R)$; hence, $a \in R$ is strongly clean. Since $\overline{a} \in R/J(R)$, we see that $R/J(R)$ is Boolean.

Conversely, assume that (1) and (2) hold. Since $R/J(R)$ is Boolean, $(\overline{2})^2 = \overline{2}$; hence, $2 \in J(R)$. For any $a \in R$, there exists an idempotent $e \in R$ such that $u := a - e \in U(R)$ and $ae = ea$. But $\overline{u} = \overline{u}^2$, and so $u \in 1 + J(R)$. Therefore $a = (1-e) + \big(2e + (a-e-1)\big)$, the result follows. □

Let $R = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, (m,n) = 1, 3 \nmid n\}$. Then R is a local ring with the Jacobson radical $J(R) = 3R$. Clearly, R is strongly clean. Since $\big(\frac{2}{1}\big)^2 - \frac{2}{1} = \frac{2}{1} \notin J(R)$, $\overline{\big(\frac{2}{1}\big)} \in R/J(R)$ is not an idempotent. In light of Theorem 4.1.1, R is not strongly J-clean. Let R be a strongly J-clean. By the preceding discussion, we note that $2 \in J(R)$.

Corollary 4.1.2. *Let R be a ring. Then the following are equivalent:*

(1) *R is Boolean.*
(2) *For any $a, b \in R$, $\ell(a) = \ell(b)$ implies that $a = b$.*
(3) *R is strongly J-clean and $J(R) = 0$.*

Proof. (1) $\Rightarrow$ (2) Let $x, y \in R$. Then $x+y = (x+y)^2 = x^2+xy+yx+y^2 = x+xy+yx+y$, and so $xy+yx = 0$. As $2 = 2^2 = 4$, R has characteristic 2. Thus, $xy = yx$. That is, R is commutative. For any $a, b \in R$, $\ell(a) = \ell(b)$ implies that $R(1-a) = R(1-b)$. This shows that $(1-b)a = (1-a)b = 0$, and so $a = ba = ab = b$, as required.

(2) $\Rightarrow$ (1) Given $x^2 = 0$, then $(1+x)(1-x) = 1$, and so $1 + x \in U(R)$. This implies that $\ell(1+x) = \ell(1)$; hence, $1 + x = 1$, i.e., $x = 0$. Thus,

R is reduced. For any $a \in R$, we see that $\ell(a) \subseteq \ell(a^2)$. If $ca^2 = 0$, then $\big((ac)a\big)^2 = aca^2ca = 0$. By the preceding discussion, we get $aca = 0$, and so $(ac)^2 = acac = 0$. Thus, $ac = 0$. This implies that $(ca)^2 = c(ac)a = 0$, and so $ca = 0$. Therefore $\ell(a) = \ell(a^2)$. Accordingly, $a = a^2$, as desired.

(1) $\Leftrightarrow$ (3) is immediate from Theorem 4.1.1. □

Corollary 4.1.3. *Let R be a ring, and let $e \in R$ be an idempotent. If R is strongly J-clean, then so is eRe.*

Proof. Suppose that R is strongly J-clean. In view of Theorem 4.1.1, R is strongly clean and $R/J(R)$ is Boolean. According to Theorem 1.1.12, eRe is strongly clean. As every Boolean ring is commutative, $eRe/J\big(eRe\big) \cong \overline{e}\big(R/J(R)\big)\overline{e}$ is Boolean, thus completing the proof. □

We recall a simple elementary result which is due to [280].

Lemma 4.1.4. *Let R be a ring. Then the following are equivalent:*

(1) *$R/J(R)$ is commutative.*
(2) *For all $a, b, c \in R$, $c + ab \in U(R)$ implies that $c + ba \in U(R)$.*
(3) *The transpose of any invertible matrix over R is invertible.*

Proof. (1) $\Rightarrow$ (3) Let $A \in GL_n(R)$. Then $\overline{A} \in GL_n\big(R/J(R)\big)$. Hence, $\overline{A^T} \in GL_n\big(R/J(R)\big)$ as $R/J(R)$ is commutative. Thus, we can find some $\overline{B} \in M_n\big(R/J(R)\big)$ such that $\overline{A^TB} = \overline{BA^T} = \overline{I_n}$. So $I_n - A^TB \in M_n\big(J(R)\big) = J\big(M_n(R)\big)$. It follows that $A^TB = I_n - \big(I_n - A^TB\big) \in GL_n(R)$, and then $A^T \in M_n(R)$ is right invertible. Likewise, $A^T \in M_n(R)$ is left invertible. Thus, $A^T \in GL_n(R)$.

(3) $\Rightarrow$ (2) Given $c + ab \in U(R)$ with $a, b, c \in R$, then

$$\begin{pmatrix} 1 & b \\ -a & c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -a & c+ab \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in GL_2(R).$$

By hypothesis, we get

$$\begin{pmatrix} 1 & b \\ -a & c \end{pmatrix}^T = \begin{pmatrix} 1 & -a \\ b & c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b & c+ba \end{pmatrix} \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} \in GL_2(R).$$

Thus, $c + ba \in U(R)$, as desired.

(2) $\Rightarrow$ (1) Let $S = R/J(R)$. Then S is semiprimitive, i.e., $J(S) = 0$. As is well known, a semiprimitive ring is commutative if and only if any additive commutator lies in the central (see [377, Theorem 12.11]). Thus, it suffices to show that $[\overline{a}, \overline{b}] \in C(S)$ for any $a, b \in R$. Let $w = 1 + [a, b]$ and

$r \in R$. Then $w = (1+ab) - ba \in U(R)$, by hypothesis. For any $d \in R$, we check that $(1+wrd) - d(rw) = w\big(w^{-1} + rd - (w^{-1}d)(rw)\big) \in U(R)$, and so $1+(wr-rw)d = (1+wrd)-(rw)d \in U(R)$. It follows that $wr - rw \in J(R)$, so $\overline{wr} = \overline{rw}$. Therefore $\overline{w} \in C(S)$, and we are done. □

Proposition 4.1.5. *The transpose of any invertible matrix over strongly J-clean rings is invertible.*

Proof. Let R be strongly J-clean. In view of Theorem 4.1.1, $R/J(R)$ is Boolean. As in the proof of Corollary 4.1.2, every Boolean ring is commutative. Thus $R/J(R)$ is commutative. According to Lemma 4.1.4, the proof is true. □

Lemma 4.1.6. *A ring R is Boolean if and only if R is isomorphic to a subdirect product of copies of the field $\mathbb{Z}_2$.*

Proof. Suppose that R is Boolean. Then R is commutative. Let M be a maximal ideal of R. Then R/M is simple, and so a field. This implies that $R/M \cong \mathbb{Z}_2$. Construct a surjective ring morphism $\varphi_M : R \to R/M$, then $Ker\big(\varphi_M\big) = \bigcap\limits_{M \in Max(R)} M = J(R) = 0$. Therefore R is isomorphic to a subdirect product of copies of the field $\mathbb{Z}_2$. Conversely, we have surjective ring morphisms $\varphi_i : R \to \mathbb{Z}_2$ such that $\bigcap\limits_i Ker(\varphi_i) = 0$. For any $a \in R$, we see that $a - a^2 \in \bigcap\limits_i Ker(\varphi_i)$, and so $a = a^2$. Therefore R is Boolean, as asserted. □

Let $J\text{-}spec(R) = \{P \in Spec(R) \mid J(R) \subseteq P\}$. As is well known, the Jacobson radical $J(R)$ is semiprime, and so $J(R)$ is the intersection of some prime ideals . Hence, $J(R) = \bigcap\limits_{P \in J\text{-}spec(R)} P$.

Theorem 4.1.7. *A ring R is strongly J-clean if and only if*

(1) *R is strongly clean;*
(2) *$R/P \cong \mathbb{Z}_2$ for all $P \in J\text{-}spec(R)$.*

Proof. Suppose that R is strongly J-clean. Then R is strongly clean and $R/J(R)$ is Boolean by Theorem 4.1.1. Further, $R/J(R)$ is Boolean. Let $P \in J\text{-}spec(R)$. Then $R/P \cong R/J(R)/P/J(R)$, and so R/P is Boolean. For any $\overline{a} \in R/P$, $\overline{a}\big(R/P\big)\overline{(1-a)} = \overline{0}$ as every Boolean ring is commutative. Since R/P is prime, $\overline{a} = \overline{0}$ or $\overline{a} = \overline{1}$. Consequently, $R/P \cong \mathbb{Z}_2$.

Conversely, assume that (1) and (2) hold. Let $P \in J\text{-}spec(R)$. Construct a surjective ring morphism $\varphi_P : R/J(R) \to R/P, x + J(R) \mapsto x + P$. For any $x + J(R) \in \bigcap_{P \in J\text{-}spec(R)} Ker\varphi_P$, $x \in \bigcap_{P \in J\text{-}spec(R)} P = J(R)$, and so $\bigcap_{P \in J\text{-}spec(R)} Ker\varphi_P = \overline{0}$. We infer that $R/J(R)$ is isomorphic to a subdirect product of copies of the field $\mathbb{Z}_2$. An appeal to Lemma 4.1.6 completes the proof. □

Corollary 4.1.8. *Let R be strongly J-clean. Then every prime ideal containing $J(R)$ is maximal.*

Proof. Let $P \in J$ -spec (R). In view of Theorem 4.1.7, $R/P \cong \mathbb{Z}_2$; hence, R/P is a field. Therefore P is maximal, as asserted. □

Theorem 4.1.9. *A ring R is strongly J-clean if and only if*

(1) R is strongly clean;
(2) $J(R) = \{x \in R \mid 1 - x \in U(R)\}$.

Proof. Suppose that R is strongly J-clean. Then R is strongly clean by Theorem 4.1.1. Obviously, $J(R) \subseteq \{x \in R \mid 1 - x \in U(R)\}$. If $1 - x \in U(R)$, we have an idempotent $e \in R$ and an element $w \in J(R)$ such that $x = e + w$ and $xe = ex$. Thus, $1 - e = (1 - x) + w \in U(R)$, and so $1 - e = 1$. This implies that $e = 0$; whence, $x = w \in J(R)$. Therefore $J(R) = \{x \in R \mid 1 - x \in U(R)\}$.

Conversely, assume that (1) and (2) hold. For any $a \in R$, there exists an idempotent $e \in R$ such that $-a = e + u$ and $eu = ue$ for $u \in U(R)$. This implies that $a = (1 - e) - (1 + u)$. By hypothesis, $1 + u \in J(R)$. Therefore $a \in R$ is strongly J-clean, as desired. □

Corollary 4.1.10. *Let R be a commutative ring. Then the following are equivalent:*

(1) $T_n(R)$ is strongly J-clean.
(2) R is strongly J-clean.

Proof. (1) $\Rightarrow$ (2) Let $E = diag(1, 0, \ldots, 0) \in M_n(R)$. Then $R \cong EM_n(R)E$. In view of Corollary 4.1.3, R is strongly J-clean.

(2) $\Rightarrow$ (1) Since R is strongly J-clean, it follows by Theorem 4.1.9, R is strongly clean and $J(R) = \{x \in R \mid 1 - x \in U(R)\}$. In light of

Theorem 3.3.2, $T_n(R)$ is strongly clean. One easily checks that $J\big(T_n(R)\big) = \{X \in M_n(R) \mid I_n - X \in U\big(T_n(R)\big)\}$. By using Theorem 4.1.9 again, $T_n(R)$ is strongly J-clean. □

Proposition 4.1.11. *Let A and B be local rings and ${}_AV_B$ is a bimodule, let $R = \begin{pmatrix} A & V \\ 0 & B \end{pmatrix}$. Then R is strongly J-clean if and only if*

(1) $A/J(A) \cong B/J(B) \cong \mathbb{Z}_2$;
(2) *If $a \in 1_A + J(A), b \in J(B)$ and $v \in V$, there exists $x \in V$ such that $v = ax - xb$.*

Proof. Suppose that R is strongly J-clean. Clearly, A and B are both corners of R. In view of Corollary 4.1.3, A and B are strongly J-clean. Thus, $A/J(A) \cong B/J(B) \cong \mathbb{Z}_2$ from Theorem 4.1.1. Since R is strongly J-clean, it is strongly clean. If $a \in 1_A + J(A), b \in J(B)$ and $v \in V$, it follows by Corollary 1.2.16 that there exists $x \in V$ such that $v = ax - xb$.

Suppose that (1) and (2) are satisfied. In view of Corollary 1.2.16, R is strongly clean. Obviously, $R/J(R) \cong A/J(A) \times B/J(B)$, and so $R/J(R)$ is Boolean. Therefore R is strongly J-clean by Theorem 4.1.1. □

Let R be a local ring. As an immediate consequence, we deduce that $T_2(R)$ is strongly J-clean if and only if $R/J(R) \cong \mathbb{Z}_2$ and R is weakly bleached.

Corollary 4.1.12. *Let A and B be local rings, let ${}_AV_B$ be a bimodule, and let $R = \begin{pmatrix} A & V \\ 0 & B \end{pmatrix}$. Then R is strongly J-clean if and only if*

(1) *R is strongly clean;*
(2) $A/J(A) \cong B/J(B) \cong \mathbb{Z}_2$.

Proof. According to Corollary 1.2.16, R is strongly clean if and only if whenever $a - 1 \in J(A), b \in J(B)$ and $v \in V$, there exists $x \in V$ such that $v = ax - xb$. Therefore we complete the proof by Proposition 4.1.11. □

Corollary 4.1.13. *Let R be a commutative local ring and ${}_RV_R$ be a bimodule, let $T = \begin{pmatrix} R & V \\ 0 & R \end{pmatrix}$. Then the following are equivalent:*

(1) *T is strongly J-clean.*
(2) $R/J(R) \cong \mathbb{Z}_2$.

Proof. Let $a-1 \in J(R), b \in J(R)$ and $v \in V$. Then $a-b \in 1+J(R)$; hence, $v = ax - xb$ has a solution $x = (a-b)^{-1}v$. In light of Proposition 4.1.11, the result follows. □

Lemma 4.1.14. *Let R be a local ring. Then the following are equivalent:*

(1) *R is strongly J-clean.*
(2) *$R/J(R) \cong \mathbb{Z}_2$.*

Proof. (1) $\Rightarrow$ (2) By virtue of Theorem 4.1.1, $R/J(R)$ is Boolean, and so $R/J(R)$ is a field. Accordingly, $R/J(R) \cong \mathbb{Z}_2$.

(2) $\Rightarrow$ (1) For any $a \in R$, $a \in J(R)$ or $a-1 \in J(R)$. Therefore $a \in R$ is strongly J-clean, the result follows. □

Theorem 4.1.15. *Let R be a local ring, and let $n \geq 2$. Then the following are equivalent:*

(1) *$T_n(R)$ is strongly J-clean.*
(2) *R is bleached and $R/J(R) \cong \mathbb{Z}_2$.*

Proof. (1) $\Rightarrow$ (2) In view of Corollary 4.1.3, R is strongly J-clean. It follows from Lemma 4.1.14 that $R/J(R) \cong \mathbb{Z}_2$. By Corollary 4.1.3 again, $T_2(R)$ is strongly J-clean. Let $a \in U(R)$ and $b \in J(R)$. We claim that $l_a - r_b : R \to R$ is surjective. For any $v \in R$, it suffices to find some $x \in R$ such that $ax - xb = v$. Let $r = \begin{pmatrix} a & v \\ 0 & b \end{pmatrix}$. As $R/J(R) \cong \mathbb{Z}_2$, we see that $a \in 1+J(R)$. Since R is local, all idempotents in R are 0 and 1. Thus, any idempotent in $T_2(R)$ is one of the following:

$$\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

By hypothesis, there exists an idempotent $e \in T_2(R)$ such that $r - e \in J\big(T_2(R)\big)$ and $er = re$. This implies that $e = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}$; otherwise, $r - e \notin J\big(T_2(R)\big)$. As $er = re$, we deduce that $ax - xb = v$. Thus, $l_a - r_b : R \to R$ is surjective. Analogously, $l_b - r_a : R \to R$ is surjective. Therefore R is bleached.

(2) $\Rightarrow$ (1) Since R is bleached, $T_n(R)$ is strongly clean by Corollary 3.1.5. Clearly, $T_n(R)/J\big(T_n(R)\big) \cong \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, and so $T_n(R)/J\big(T_n(R)$ is Boolean. According to Theorem 4.1.1, we complete the proof. □

Corollary 4.1.16. *Let R be a commutative local ring, and let $n \in \mathbb{N}$. Then the following are equivalent:*

(1) $T_n(R)$ *is strongly J-clean.*
(2) $R/J(R) \cong \mathbb{Z}_2$.

Proof. As R is a commutative local ring, it is bleached. Therefore the result follows from Theorem 4.1.15. □

4.2 Solvability of Quadratic Equations

The main purpose of this section is to characterize a single strongly J-clean matrix be means of the solvability of quadratic equations. We begin with an elementary result on strongly J-clean endomorphisms of a module.

Theorem 4.2.1. *Let $E = End({}_RM)$, and let $\alpha \in E$. Then the following are equivalent:*

(1) *α is strongly J-clean in E.*
(2) *There exists $\pi^2 = \pi \in E$ such that $\alpha\pi = \pi\alpha$, $\alpha\pi \in J\big(\pi E\pi\big)$ and $(1-\alpha)(1-\pi) \in J\big((1-\pi)E(1-\pi)\big)$.*
(3) *$M = P \oplus Q$ where P and Q are α-invariant, and $\alpha|_P \in J\big(End(P)\big)$ and $(1_M - \alpha)|_Q \in J\big(End(Q)\big)$.*
(4) *$M = P_1 \oplus \cdots \oplus P_n$ for some $n \geq 1$ where P_i is α-invariant and $\alpha|_{P_i}$ is strongly J-clean in $End(P_i)$ for each i.*

Proof. (1) $\Rightarrow$ (2) Since α is strongly J-clean in E, there exists an idempotent $\pi \in E$ and a $\sigma \in J(E)$ such that $\alpha = (1-\pi) + \sigma$ and $\pi\sigma = \sigma\pi$. Thus, $\pi\alpha = \pi\sigma \in J\big(\pi E\pi\big)$. Further, $1 - \alpha = \pi + (-\sigma)$, and so $(1-\pi)(1-\alpha) = (1-\pi)(-\sigma) \in J\big((1-\pi)E(1-\pi)\big)$, as required.

(2) $\Rightarrow$ (3) Set $P = M\pi$ and $Q = M(1-\pi)$. Then $M = P \oplus Q$. As $\alpha\pi = \pi\alpha$, we see that P and Q are α-invariant. Let $\gamma \in End(P)$. Define $\overline{\gamma} : M \to M$ given by $(p+q)\overline{\gamma} = (p)\gamma$ for any $p \in P, q \in Q$. Then

$$\begin{aligned} 1_P - \alpha|_P\gamma : P &\to P \\ p &\mapsto p - (p)\alpha\gamma. \end{aligned}$$

If $(p)\big(1_P - \alpha|_P\gamma\big) = 0$, then $(p\pi)\big(\pi - (\pi\alpha)\pi\overline{\gamma}\pi\big) = 0$. As $\alpha\pi \in J\big(\pi E\pi\big)$, we get $p\pi = 0$. This implies that $1_P - \alpha|_P \in End(P)$ is an R-monomorphism. For any $p \in P$. Choose $q = (p\pi)\big(\pi - (\pi\alpha)\pi\overline{\gamma}\pi\big)^{-1} \in P$. It is easy to verify that $(q)\big(1_P - \alpha|_P\gamma\big) = p$; hence, $1_P - \alpha|_P\gamma \in End(P)$ is an R-epimorphism. Thus, $1_P - \alpha|_P\gamma \in Aut(P)$, and so $\alpha|_P \in J\big(End(P)\big)$. Likewise, $(1_M - \alpha)|_Q \in J\big(End(Q)\big)$, as required.

(3) $\Rightarrow$ (4) Obviously, $\alpha|_P \in End(P)$ is strongly J-clean. As $(1_M-\alpha)|_Q \in J\big(End(Q)\big)$, we see that $\alpha\ |_Q = 1_Q - (1_M - \alpha)|_Q \in End(Q)$ is strongly J-clean, as required.

(4) $\Rightarrow$ (1) Let $\lambda_i \in End(P_i)$. We construct an R-morphism $\overline{\lambda_i} \in End(M)$ given by $\left(\sum\limits_{j=1}^{n} p_j\right)\overline{\lambda_i} = (p_i)\lambda_i$ for any $p_j \in P_j$. By hypothesis, there exist $\pi_j^2 = \pi_j \in End(P_j), \sigma_j \in J\big(End(P_j)\big)$ such that $\alpha|_{P_j} = \pi_j + \sigma_j$ and $\pi_j\sigma_j = \sigma_j\pi_j$. Thus, $\alpha = \sum\limits_{j=1}^{n} \overline{\alpha|_{P_j}} = \sum\limits_{j=1}^{n} \overline{\pi_j} + \sum\limits_{j=1}^{n} \overline{\sigma_j}$. Obviously, $\overline{\pi_j\pi_k} = \overline{\pi_k\pi_j} = 0 (j \neq k)$, we see that $\sum\limits_{j=1}^{n} \overline{\pi_j} \in End(M)$ is an idempotent. In addition, $\overline{\pi_j\sigma_k} = \overline{\sigma_k\pi_j} = 0 (j \neq k)$. Thus,

$$\left(\sum_{j=1}^{n} \overline{\pi_j}\right)\left(\sum_{j=1}^{n} \overline{\sigma_j}\right) = \sum_{j=1}^{n} \overline{\pi_j\sigma_j} = \sum_{j=1}^{n} \overline{\sigma_j\pi_j} = \left(\sum_{j=1}^{n} \overline{\sigma_j}\right)\left(\sum_{j=1}^{n} \overline{\pi_j}\right).$$

Let $\gamma \in End(M)$. Then

$$\begin{aligned} 1_M - \overline{\sigma_j}\gamma : M &\to M \\ \sum_{i=1}^{n} p_i &\mapsto \sum_{i=1}^{n} p_i - (p_j)\sigma_j\gamma. \end{aligned}$$

If $\left(\sum\limits_{i=1}^{n} p_i\right)\big(1_M - \overline{\sigma_j}\gamma\big) = 0$, then $\sum\limits_{i=1}^{n} p_i - (p_j)\sigma_j\gamma = 0$. Let $\pi_j : M \to P_j$ be the projection. Then $p_j - (p_j)\sigma_j\gamma\pi_j = 0$, i.e., $(p_j)\big(1_{P_j} - \sigma_j\gamma|_{P_j}\pi_j\big) = 0$, and so $p_j = 0$. Hence, $1_M - \overline{\sigma_j}\gamma \in End(M)$ is an R-monomorphism. Let $\sum\limits_{i=1}^{n} p_i \in M$. Then $(p_j)\big(1_{P_j} - \sigma_j\gamma|_{P_j}\pi_j\big)^{-1}\big(1_M - \overline{\sigma_j}\gamma\big)\pi_j = (p_j)\pi_j$, and so

$$q := p_j - (p_j)\big(1_{P_j} - \sigma_j\gamma|_{P_j}\pi_j\big)^{-1}\big(1_M - \overline{\sigma_j}\gamma\big) \in Ker\pi_j.$$

Therefore

$$\sum_{i=1}^{n} p_i = \left(q + \sum_{i\neq j} p_i + (p_j)\big(1_{P_j} - \sigma_j\gamma|_{P_j}\pi_j\big)^{-1}\right)\big(1_M - \overline{\sigma_j}\gamma\big).$$

This implies that $1_M - \overline{\sigma_j}\gamma \in Aut(M)$. Hence, $\overline{\sigma_j} \in J\big(End(M)\big)$. Consequently, $\sum\limits_{i=1}^{n} \overline{\sigma_i} \in J\big(End(M)\big)$. Therefore $\alpha \in End(M)$ is strongly J-clean, as asserted. □

Corollary 4.2.2. *Let R be a ring, and let $\alpha \in R$. Then the following are equivalent:*

(1) *α is strongly J-clean in R.*

(2) *There exists* $\pi^2 = \pi \in R$ *such that* $\alpha\pi = \pi\alpha$, $\alpha\pi \in J\big(\pi R\pi\big)$ *and* $(1-\alpha)(1-\pi) \in J\big((1-\pi)R(1-\pi)\big)$.

(3) *There exists a decomposition* $1 = e_1 + \cdots + e_n$ *where* $n \geq 1$, *the* e_i *are orthogonal idempotents, and* $e_i\alpha = \alpha e_i$ *is strongly J-clean in* e_iRe_i *for each* i.

Proof. (1) $\Leftrightarrow$ (2) is obvious from Theorem 4.2.1.

(1) $\Rightarrow$ (3) In view of Theorem 4.2.1, $R = P_1 \oplus \cdots \oplus P_n$ for some $n \geq 1$ where P_i is α-invariant and $\alpha|_{P_i}$ is strongly J-clean in $End(P_i)$ for each i. Write $1_R = e_1 + \cdots + e_n$ where each $e_i \in P_i$. Then $e_i - e_i^2 = \sum_{j\neq i} e_je_i \in P_i \cap \left(\sum_{j\neq i} P_j\right)$. Hence, $e_i^2 = e_i$ and $e_je_i = 0 (j \neq i)$. Further, each $P_i = Re_i$. As Re_i is α-invariant, we get $e_i\alpha = e_i\alpha e_i$. Further, $R(1-e_i)$ is α-invariant, we get $\alpha e_i = e_i\alpha e_i$. Therefore $e_i\alpha = \alpha e_i$. Since $End(Re_i) \cong e_iRe_i$, we see that $e_i\alpha = \alpha e_i$ is strongly J-clean in e_iRe_i for each i.

(3) $\Rightarrow$ (1) By hypothesis, there exists a decomposition $1 = e_1 + \cdots + e_n$ where $n \geq 1$, the e_i are orthogonal idempotents, and $e_i\alpha = \alpha e_i$ is strongly J-clean in e_iRe_i for each i. Choose $P_i = Re_i$. Then $R = P_1 \oplus \cdots \oplus P_n$ for some $n \geq 1$ where P_i is α-invariant and $\alpha|_{P_i}$ is strongly J-clean in $End(P_i)$ for each i. According to Theorem 4.2.1, $\alpha \in R$ is strongly J-clean. □

Lemma 4.2.3. *Let* R *be a local ring, and let* $A \in M_2(R)$ *be strongly J-clean. Then* $A \in J\big(M_2(R)\big)$ *or* $I_2 - A \in J\big(M_2(R)\big)$ *or* A *is similar to a matrix* $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, *where* $\lambda \in J(R), \mu \in 1 + J(R)$.

Proof. If $A \notin J\big(M_2(R)\big)$ and $I_2 - A \notin J\big(M_2(R)\big)$, then there exists an idempotent $E \in M_2(R)$ such that $AE = EA$ and $A - E \in J\big(M_2(R)\big)$. Clearly, E is a nontrivial idempotent matrix. In view of Lemma 1.2.12, E is similar to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Thus, A is similar to $\begin{pmatrix} 1+a & b \\ c & d \end{pmatrix}$, where $a, b, c, d \in J(R)$. This implies that A is similar to $\begin{pmatrix} 1+a' & b' \\ 1+c' & d' \end{pmatrix}$ where $a', b', c', d' \in J(R)$. Therefore A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in J(R), \mu \in 1 + J(R)$. □

Theorem 4.2.4. *Let* R *be a local ring. Then* $A \in M_2(R)$ *is strongly J-clean if and only if*

(1) $A \in J\big(M_2(R)\big)$, *or*
(2) $I_2 - A \in J\big(M_2(R)\big)$, *or*

(3) *A is similar to* $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$ *where* $\lambda \in J(R), \mu \in 1+J(R)$, *and the equation* $x^2 - x\mu - \lambda = 0$ *has a root in* $J(R)$ *and a root in* $1 + J(R)$.

Proof. Suppose that $A \in M_2(R)$ is strongly J-clean. Assume that $A \notin J\big(M_2(R)\big)$ and $I_2 - A \notin J\big(M_2(R)\big)$. In view of Lemma 4.2.3, A is similar to the matrix $B = \begin{pmatrix} 0 & \lambda' \\ 1 & \mu' \end{pmatrix} \in M_2(R)$, where $\lambda' \in J(R), \mu' \in 1 + J(R)$. As A is strongly J-clean, so is B. Write $B = E + W, E = E^2$ and $EW = WE$ where $W \in J\big(M_2(R)\big)$. Then $B = \big(I_2 - E\big) + \big(W + 2E - I_2\big)$. It is easy to verify that $\big(2E - I_2\big)^2 = I_2$. Hence, $W + 2E - I_2 \in GL_2(R)$. Clearly, $\big(I_2 - E\big)\big(W + 2E - I_2\big) = \big(W + 2E - I_2\big)\big(I_2 - E\big)$, thus $B \in M_2(R)$ is strongly clean. Obviously, $B, I_2 - B \notin GL_2(R)$. In view of Theorem 1.2.4, B is similar to the matrix $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$ where $\lambda, \mu \in 1 + J(R)$, and the equation $x^2 - x\mu - \lambda = 0$ has a root in $J(R)$ and a root in $1 + J(R)$. One easily checks that A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$ where $\lambda \in J(R), \mu \in 1 + J(R)$, and the equation $x^2 - x\mu - \lambda = 0$ has a root in $J(R)$ and a root in $1 + J(R)$, as desired.

We now prove the converse. If (1) or (2) holds, then $A \in M_2(R)$ is strongly J-clean. Assume that (3) holds. It will suffice to show that $B = \begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$ is strongly J-clean. By hypothesis, the equation $x^2 - x\mu - \lambda = 0$ has solutions $c \in J(R)$ and $d \in 1 + J(R)$. Then $c^2 - c\mu - \lambda = 0$ and $d^2 - d\mu - \lambda = 0$. Choose $C = R(1, c)$ and $D = R(1, d)$. Clearly,

$$(1, c)\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix} = c(1, c) \in C,$$

thus C is B-invariant. Likewise, D is B-invariant. Write $r(1, c) = s(1, d) \in C \cap D$, then $r = s$ and $rc = sd$, and so $r(c - d) = 0$. This implies that $r = 0$, and so $C \cap D = 0$. Let $(a, b) \in 2R$. Choose $s = (b - ac)(d - c)^{-1}$ and $r = a - s$. Then it is easy to verify that $(a, b) = r(1, c) + s(1, d) \in C \oplus D$. As a result, we deduce that $2R = C \oplus D$. Let $\gamma \in End(C)$. Then

$$\begin{aligned} 1_C - B|_C\gamma : C &\to C; \\ r(1, c) &\mapsto r(1, c) - rc(1, c)\gamma. \end{aligned}$$

Write $(1, c)\gamma = b(1, c)$ where $b \in R$. If $\big(r(1, c)\big)\big(1_C - B|_C\gamma\big) = 0$, then $r(1, c) - rcb(1, c) = 0$, and so $r(1 - cb)(1, c) = 0$. As $c \in J(R)$, we get $r = 0$. Hence, $r(1, c) = 0$. This implies that $1_C - B|_C\gamma$ is an R-monomorphism.

For any $r(1,c) \in C$, one easily checks that $\big(r(1-cb)^{-1}(1,c)\big)\big(r(1,c)\big)\big(1_C - B|_C\gamma\big) = r(1,c)$; hence, $1_C - B|_C\gamma$ is an R-epimorphism. Therefore $1_C - B|_C\gamma$ is an R-isomorphism. This infers that

$$\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}\Big|_C \in J\big(End(C)\big).$$

Likewise,

$$\left(I_2 - \begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}\right)\Big|_D \in J\big(End(D)\big).$$

In light of Theorem 4.2.1, $B \in M_2(R)$ is strongly J-clean, as required. □

Corollary 4.2.5. *Let R be a commutative local ring, and let $A \in M_2(R)$. Then the following are equivalent:*

(1) *$A \in M_2(R)$ is strongly J-clean.*
(2) *$A \in J\big(M_2(R)\big)$ or $I_2 - A \in J\big(M_2(R)\big)$, or the equation $x^2 - x = \frac{det(A)}{tr^2(A)-4det(A)}$ has a root in $J(R)$.*
(3) *$A \in J\big(M_2(R)\big)$ or $I_2 - A \in J\big(M_2(R)\big)$, or the equation $x^2 - tr(A)x + det(A) = 0$ has a root in $J(R)$ and a root in $1 + J(R)$.*

Proof. (1) $\Rightarrow$ (2) Suppose that $A, I_2 - A \notin J(M_2(R))$. In view of Theorem 4.2.4, A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$ where $\lambda \in J(R), \mu \in 1+J(R)$, and the equation $x^2 - x\mu - \lambda = 0$ has a root in $J(R)$ and a root in $1+J(R)$. Hence, $tr(A) = \mu$ and $det(A) = -\lambda$. Thus, the equation $x^2 - tr(A)x + det(A) = 0$ has a root α in $J(R)$ and a root β in $1 + J(R)$. That is,

$$\alpha^2 - tr(A)\alpha + det(A) = 0 \text{ and } \beta^2 - tr(A)\beta + det(A) = 0;$$

hence, $tr(A) = \alpha + \beta$ and $det(A) = \alpha\beta$. Thus, $tr^2(A) - 4det(A) = (\alpha - \beta)^2 \in U(R)$. Accordingly, we directly verify that the equation $x^2 - x = \frac{det(A)}{tr^2(A)-4det(A)}$ has a root $\alpha(\alpha - \beta)^{-1}$ in $J(R)$.

(2) $\Rightarrow$ (3) Suppose that $A, I_2 - A \notin J\big(M_2(R)\big)$. Then the equation $x^2 - x = t_A^{-1}det(A)$ has a root α in $J(R)$, where $t_A = tr^2(A) - 4det(A)$. Clearly, $1 - \alpha \in 1 + J(R)$ is a root. Let $C = \begin{pmatrix} 0 & t_A^{-1}det(A) \\ 1 & 1 \end{pmatrix} \in M_2(R)$. Then the equation $x^2 - tr(C)x + det(C) = 0$ has a root in $1+J(R)$ and a root in $J(R)$. Clearly, $det(A) \in J(R)$. In view of Corollary 1.2.6, $C = (c_{ij})$ is strongly clean. As in the proof of Lemma 1.3.1, there exists a nontrivial idempotent $F = \begin{pmatrix} v & s \\ t & 1-v \end{pmatrix} \in M_2(R)$ such that $st = v - v^2, C - F \in GL_2(R)$ and

$FC = CF$. As in the proof of Lemma 1.3.2, we get $(s_C^2 + 4c_{12}c_{21})(v - v^2) = c_{12}c_{21}$, and so $\big(1 + 4t_A^{-1}det(A)\big)(v - v^2) = t_A^{-1}det(A)$. Hence, $v - v^2 = \big(1 + 4t_A^{-1}det(A)\big)^{-1} t_A^{-1}det(A) = \big(t_A + 4det(A)\big)^{-1} det(A) = tr^{-2}(A)det(A)$. This implies that $(1-v)-(1-v)^2 = tr^{-2}(A)det(A)$. Since $v(1-v) \in J(R)$, we have $v \in J(R)$ or $1 - v \in J(R)$. As a result, $y - y^2 = tr^{-2}(A)det(A)$ has a root in $1 + J(R)$ and a root in $J(R)$. By hypothesis, $tr(A) \in 1 + J(R)$. Choose $x = tr(A)y$. Then $x^2 - tr(A)x + det(A) = 0$ has a root in $1 + J(R)$ and a root in $J(R)$.

(3) $\Rightarrow$ (1) Suppose that $A, I_2 - A \notin J(M_2(R))$. Then $x^2 - tr(A)x + det(A) = 0$ has a root in $1 + J(R)$ and a root in $J(R)$. This implies that $det(A) \in J(R)$ and $tr(A) \in 1 + J(R)$. Hence, $det(I_2 - A) = 1 - tr(A) + det(A) \in J(R)$, and so $A, I_2 - A \notin GL_2(R)$. In light of Lemma 1.2.1, A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$. Hence, $\lambda = -det(A) \in J(R)$ and $\mu = tr(A) \in 1 + J(R)$. Thus, the equation $x^2 - \mu x - \lambda = 0$ has a root in $J(R)$ and a root in $1 + J(R)$. Therefore the result follows by Theorem 4.2.4. □

Let $R = \{\frac{m}{n} \in \mathbb{Q} \mid m, n \in \mathbb{Z}, (m, n) = 1, 2 \nmid n\}$. Then R is a commutative local ring. Choose $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \in M_2(R)$. Clearly, the equation $x^2 - x = \frac{det(A)}{tr^2(A) - 4det(A)}$ has a root $\frac{4}{5}$ in $J(R)$ and a root $\frac{1}{5}$ in $1 + J(R)$. According to Corollary 4.2.5, $A \in M_2(R)$ is strongly J-clean. In fact, we have the strongly J-clean decomposition $A = \begin{pmatrix} \frac{3}{5} & -\frac{2}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{pmatrix} + \begin{pmatrix} \frac{2}{5} & \frac{12}{5} \\ \frac{18}{5} & \frac{8}{5} \end{pmatrix}$. In this case, $M_2(R)$ is not strongly clean.

Corollary 4.2.6. *Let R be a local ring, and let $A \in M_2(R)$. Then the following are equivalent:*

(1) *$A \in M_2(R)$ is strongly clean.*
(2) *$A \in GL_2(R)$ or $I_2 - A \in GL_2(R)$, or $A \in M_2(R)$ is strongly J-clean.*

Proof. (1) $\Rightarrow$ (2) If $A \in M_2(R)$ is strongly clean, it follows from Theorem 1.2.4 that $A \in GL_2(R)$ or $I_2 - A \in GL_2(R)$, or A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$ where $\lambda \in J(R), \mu \in 1 + J(R)$, and the equation $x^2 - x\mu - \lambda = 0$ has a root in $J(R)$ and a root in $1 + J(R)$. In the last case, it follows from Theorem 4.2.4 that $A \in M_2(R)$ is strongly J-clean.

(2) $\Rightarrow$ (1) is obvious. □

Let $\mathbb{Z}_{(2)} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, (m,n) = 1, 2 \nmid n\}$, and let $A = \begin{pmatrix} 1 & 1 \\ \frac{2}{9} & 0 \end{pmatrix} \in M_2(\mathbb{Z}_{(2)})$. Then $A, I_2 - A \notin GL_2(R)$, but $A \in M_2(R)$ is strongly clean. In light of Corollary 4.2.6, A is strongly J-clean.

Proposition 4.2.7. *Let R be a local ring. Then $A \in M_2(R)$ is strongly J-clean if and only if $A \in J\big(M_2(R)\big)$ or $I_2 - A \in J\big(M_2(R)\big)$ or A is similar to a matrix $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $\alpha \in J(R), \beta \in 1 + J(R)$.*

Proof. One direction is obvious. Conversely, assume that $A \in M_2(R)$ is strongly J-clean. Then there exists an idempotent $E \in M_2(R)$ and $W \in J\big(M_2(R)\big)$ such that $A = E + W$ with $EW = WE$. Suppose that A and $I_2 - A$ are not in $J\big(M_2(R)\big)$. In view of Lemma 1.2.12, there exists $U \in GL_2(R)$ such that $UEU^{-1} = diag(1,0)$. Hence, $UAU^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + UWU^{-1}$. Set $V = (v_{ij}) := UWU^{-1}$. Then $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} V = V \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$; whence, $v_{12} = v_{21} = 0$ and $v_{11}, v_{22} \in J(R)$. Thus, A is similar to $\begin{pmatrix} 1 + v_{11} & 0 \\ 0 & v_{22} \end{pmatrix}$, which completes the proof. □

Example 4.2.8. For an arbitrary ring R, $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in M_2(R)$ is not strongly J-clean. Otherwise, $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = E + W$ and $EW = WE$, where $W \in J\big(M_2(R)\big)$. Hence, $E \in GL_2(R)$, and then $E = I_2$. This implies that $W = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$, an absurd.

Lemma 4.2.9. *Let R be a ring, and let $f(x) \in R[[x]]$. If there exists an idempotent $e \in R$ such that $f(0) - e \in J(R), f(0)e = ef(0)$ and that for any $b \in R$ there exists $c \in R$ such that $[a,c] = [e,b]$, then $f(x) \in R[[x]]$ is strongly J-clean.*

Proof. Write $f(x) = \sum_{i=0}^{\infty} r_i x^i$. Then we have an idempotent $e_0 \in R$ such that $r_0 = e_0 + (r_0 - e_0), r_0 - e_0 \in J(R)$ and that for any $b \in R$ there exists $c \in R$ such that $[r_0, c] = [e_0, b]$. As in the proof of Lemma 1.4.1, there exists some $e_1 \in (1 - e_0)Re_0 + e_0R(1 - e_0)$ such that $[r_0, e_1] + [e_0, r_1] = 0$, and so

$e_1 = e_0e_1 + e_1e_0$. Similarly to Theorem 1.4.4, we have $e_2, \ldots, e_k, \ldots \in R$ such that

$$e_k = e_0e_k + e_1e_{k-1} + \cdots + e_ke_0 \text{ and } [r_0, e_k] + [r_1, e_{k-1}] + \cdots + [r_k, e_0] = 0.$$

Consequently, $\sum\limits_{i=0}^{\infty} e_ix^i = \left(\sum\limits_{i=0}^{\infty} e_ix^i\right)^2 \in R[[x]]$ and $f(x)\left(\sum\limits_{i=0}^{\infty} e_ix^i\right) = \left(\sum\limits_{i=0}^{\infty} e_ix^i\right)f(x)$. Furthermore, we have a $g(x) \in R[[x]]$ such that $f(x) - \sum\limits_{i=0}^{\infty} e_ix^i = \big(f(0) - e_0\big) + xg(x)$, and so $f(x) - \sum\limits_{i=0}^{\infty} e_ix^i \in J\big(R[[x]]\big)$. Accordingly, $f(x) \in R[[x]]$ is strongly J-clean. □

Theorem 4.2.10. *Let R be a weakly bleached local ring. Then the following are equivalent:*

(1) $A(x) \in M_2\big(R[[x]]\big)$ *is strongly J-clean.*
(2) $A(0) \in M_2(R)$ *is strongly J-clean.*

Proof. (1) $\Rightarrow$ (2) This is obvious.

(2) $\Rightarrow$ (1) We may assume that $A(x) \in M_2(R)[[x]]$. Then $A(0) \in M_2(R)$ is J-strongly clean. By virtue of Proposition 4.2.7, $A(0) \in J\big(M_2(R)\big)$ or $I_2 - A(0) \in J\big(M_2(R)\big)$, or $A(0)$ is similar to a matrix $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $\alpha \in J(R), \beta \in 1 + J(R)$. If $A(0) \in J\big(M_2(R)\big)$ or $I_2 - A(0) \in J\big(M_2(R)\big)$, we can find an idempotent $E \in M_2(R)$ such that $A(0) - E \in J\big(M_2(R)\big), A(0)E = EA(0)$ and that for any $B \in M_2(R)$ there exists $C \in M_2(R)$ such that $[A(0), C] = [E, B]$. We now assume that $UA(0)U^{-1} = diag(\alpha, \beta)$ for some $U \in GL_2(R)$, where $\alpha \in J(R)$ and $\beta \in 1 + J(R)$. For any $B = (b_{ij}) \in M_2(R)$, we have $c_1, c_2 \in R$ such that $\alpha c_1 - c_1\beta = -b_{12}$ and $\beta c_2 - c_2\alpha = b_{21}$. Choose $C = \begin{pmatrix} 0 & c_1 \\ c_2 & 0 \end{pmatrix}$. Then

$$\big[diag(\alpha, \beta), C\big] = \big[diag(0, 1), B\big].$$

It is easy to check that $diag(\alpha, \beta) - diag(0, 1) \in J\big(M_2(R)\big)$, thus completing the proof. □

Corollary 4.2.11. *Let R be a commutative local ring. Then the following are equivalent:*

(1) $A(x) \in M_2\big(R[[x]]\big)$ *is strongly J-clean.*
(2) $A(0) \in M_2(R)$ *is strongly J-clean.*

Proof. Since R is a commutative local ring, it is weakly bleached, hence the result by Theorem 4.2.10. □

Corollary 4.2.12. *Let R be a local ring, and let $\overline{A(x)} \in M_2\big(R[[x]]/(x^m)\big)$ $(m \geq 1)$. If R is weakly bleached, then the following are equivalent:*

(1) $\overline{A(x)} \in M_2\big(R[[x]]/(x^m)\big)$ *is strongly J-clean.*
(2) $A(0) \in M_2(R)$ *is strongly J-clean.*

Proof. It is obvious from Theorem 4.2.10. □

Example 4.2.13. Let $R = \mathbb{Z}_4[x]/(x^2)$, and let $\overline{A(x)} = \begin{pmatrix} \overline{2} & \overline{2}+\overline{2}x \\ \overline{2}+x & \overline{3}+\overline{3}x \end{pmatrix} \in M_2(R)$. Obviously, $\mathbb{Z}_4$ is a commutative local ring, and that $R = \mathbb{Z}_4[[x]]/(x^2)$. Since we have the strongly J-clean decomposition $A(0) = \begin{pmatrix} \overline{0} & \overline{2} \\ \overline{2} & \overline{1} \end{pmatrix} + \begin{pmatrix} \overline{2} & \overline{0} \\ \overline{0} & \overline{2} \end{pmatrix}$ in $M_2(\mathbb{Z}_4)$, it follows by Corollary 4.2.12 that $\overline{A(x)} \in M_2(R)$ is strongly J-clean.

A Morita context $(A, B, M, N, \psi, \varphi)$ consists of two rings A and B, two bimodules ${}_AN_B$ and ${}_BM_A$, and a pair of bimodule homomorphisms $\psi : N \bigotimes_B M \to A$ and $\varphi : M \bigotimes_A N \to B$ which satisfy the following associativity: $\psi\big(n \otimes m\big)n' = n\varphi\big(m \otimes n'\big)$ and $\varphi\big(m \otimes n\big)m' = m\psi\big(n \otimes m'\big)$ for any $m, m' \in M, n, n' \in N$. These conditions ensure that the set T of generalized matrices $\begin{pmatrix} a & n \\ m & b \end{pmatrix}; a \in A, b \in B, m \in M, n \in N$ will form a ring, called the ring of the Morita context. We also use $\begin{pmatrix} A & N \\ M & B \end{pmatrix}$ to denote the Morita context. The class of rings of the Morita contexts includes all 2×2 matrix rings and all triangular matrix rings.

For further use, we record the following elementary result (see [463]).

Lemma 4.2.14. *Let T be the ring of a Morita context $\begin{pmatrix} A & N \\ M & B \end{pmatrix}$. Then $J(T) = \begin{pmatrix} J(A) & N_0 \\ M_0 & J(B) \end{pmatrix}$, where $M_0 = \{m \mid mN \subseteq J(B)$ and $N_0 = \{n \mid nM \subseteq J(A)\}$.*

Proof. Let $u \in U(A), v \in U(B), m \in M_0$ and $n \in N_0$. One easily checks that

$$\begin{pmatrix} u & n \\ m & v \end{pmatrix}^{-1} = \begin{pmatrix} (u-nv^{-1}m)^{-1} & -u^{-1}n(-mu^{-1}n+v)^{-1} \\ -v^{-1}m(u-nv^{-1}m)^{-1} & (-mu^{-1}n+v)^{-1} \end{pmatrix}^{-1}.$$

Let $a \in J(A), b \in J(B), m \in M_0$ and $n \in N_0$. For any $\begin{pmatrix} c & q \\ p & d \end{pmatrix} \in T$, we verify that

$$1_T - \begin{pmatrix} a & n \\ m & b \end{pmatrix}\begin{pmatrix} c & q \\ p & d \end{pmatrix} \in U(T),$$

where $mc + bp \in M_0$ and $aq + nd \in N_0$. Hence, $\begin{pmatrix} a & n \\ m & b \end{pmatrix} \in J(T)$. Thus, $\begin{pmatrix} J(A) & N_0 \\ M_0 & J(B) \end{pmatrix} \subseteq J(T)$.

Let $\begin{pmatrix} a & n \\ m & b \end{pmatrix} \in J(T)$. For any $r \in A$, we have

$$1_T - \begin{pmatrix} a & n \\ m & b \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} \in U(T).$$

It follows that $1 - ar \in U(A)$, and then $a \in J(A)$. For any $q \in M$ and $s \in B$, we have

$$1_T - \begin{pmatrix} a & n \\ m & b \end{pmatrix}\begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & s \end{pmatrix} \in U(T).$$

This shows that $1 - (mq)s \in U(B)$, and so $mq \in J(B)$. Hence, $mN \subseteq J(B)$, and then $m \in M_0$. Likewise, $b \in J(B)$ and $n \in N_0$. We infer that $J(T) \subseteq \begin{pmatrix} J(A) & N_0 \\ M_0 & J(B) \end{pmatrix}$. So the lemma is true. □

Let $T = (A,\ B,\ M,\ N,\ \psi, \varphi)$. If $Im(\psi) \subseteq J(A)$ and $Im(\varphi) \subseteq J(B)$, it follows by Lemma 4.2.14 that $U(T) = \begin{pmatrix} U(A) & N \\ M & U(B) \end{pmatrix}$ and $J(T) = \begin{pmatrix} J(A) & N \\ M & J(B) \end{pmatrix}$.

Lemma 4.2.15. *Let $T = (A,\ B,\ M,\ N,\ \psi, \varphi)$, let A and B be local rings, and let $e \in T$ be an idempotent. If $Im(\psi) \subseteq J(A)$ and $Im(\varphi) \subseteq J(B)$, then $e = 0$ or $e = 1$ or there exist some $\beta, \gamma \in U(T)$ such that $\beta e \gamma = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$.*

Proof. Write $e = \begin{pmatrix} a & n \\ m & b \end{pmatrix}$. As A and B are local, $a \in J(A)$ or $U(A)$, and $b \in J(B)$ or $b \in U(B)$.

Case I. $a \in U(A), b \in U(B)$. As in the proof of Lemma 4.2.14, $e \in U(T)$; whence, $e = 1$.

Case II. $a \in J(A), b \in J(B)$. In view of Lemma 4.2.14, $e \in J(T)$, and so $e = 0$.

Case III. $a \in U(A), b \in J(B)$. Choose $\beta = \begin{pmatrix} 1_A & 0 \\ -ma^{-1} & 1_B \end{pmatrix}$ and $\gamma = \begin{pmatrix} 1 & -a^{-1}n \\ 0 & 1_B \end{pmatrix}$. Then $\beta e \gamma = \begin{pmatrix} a & 0 \\ 0 & b - \varphi(ma^{-1} \otimes n) \end{pmatrix}$.

Case IV. $a \in J(A), b \in U(B)$. Choose $\beta = \begin{pmatrix} 1_A & -nb^{-1} \\ 0 & 1_B \end{pmatrix}$ and $\gamma = \begin{pmatrix} 1_A & 0 \\ -b^{-1}m & 1_B \end{pmatrix}$. Then $\beta e \gamma = \begin{pmatrix} a - \psi(nb^{-1} \otimes m) & 0 \\ 0 & b \end{pmatrix}$.

Therefore we complete the proof. □

Theorem 4.2.16. *Let $T = (A, B, M, N, \psi, \varphi)$, and let A and B be local rings. If $Im(\psi) \subseteq J(A)$ and $Im(\varphi) \subseteq J(B)$, then $\alpha \in T$ is strongly J-clean if and only if $\alpha \in J(T)$, or $1 - \alpha \in J(T)$, or there exists some $\beta \in U(T)$ such that $\beta\alpha\beta^{-1} = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$, where $c \in 1_A + J(A), d \in J(B)$, or $c \in J(A), d \in 1_B + J(B)$.*

Proof. Suppose $\alpha \in T$ is strongly J-clean. Then there exists an idempotent $e \in T$ such that $\alpha - e \in J(T)$ and $\alpha e = e\alpha$. In view of Lemma 4.2.15, $e = 0$ or $e = 1$ or there exist some $\beta, \gamma \in U(T)$ such that $\beta e \gamma = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$. If $e = 0$, then $\alpha \in J(T)$. If $e = 1$, then $1 - \alpha \in J(T)$. Otherwise, there exist some $\beta, \gamma \in U(T)$ such that $\beta e \gamma = \sigma$, where $\sigma = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$. Let $\zeta = (\zeta_{ij}) = \gamma^{-1}\beta^{-1}$. Then $\beta e \beta^{-1} = \sigma\zeta$. We easily check that $(\sigma\zeta)^2 = \sigma\zeta$ and $\sigma\zeta\sigma = \sigma$. Hence,

$$c\zeta_{11}c = c, d\zeta_{22}d = d \text{ and } c\zeta_{12}d = d\zeta_{21}c = 0.$$

Let $f = \begin{pmatrix} c\zeta_{11} & \\ & d\zeta_{22} \end{pmatrix}$. Then $f = f^2$ and $f\sigma = \sigma$. Let $\mu = \begin{pmatrix} 1_A & \\ & 1_B \end{pmatrix} - f + \sigma\zeta$. Clearly, $\sigma\zeta f = \sigma\zeta\sigma \begin{pmatrix} \zeta_{11} & \\ & \zeta_{22} \end{pmatrix} = \sigma \begin{pmatrix} \zeta_{11} & \\ & \zeta_{22} \end{pmatrix} = f$. We verify that

$\mu^{-1} = \begin{pmatrix} 1_A & \\ & 1_B \end{pmatrix} + f - \sigma\zeta$, and so $\mu \in U(T)$. Further, $\mu(\sigma\zeta)\mu^{-1} = f$. Let $\rho = \mu\beta$. Then $\rho e\rho^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. If $\rho e\rho^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then $w := \rho\alpha\rho^{-1} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in J(T)$. Write $w = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}$. Then

$$w \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} w;$$

hence, $w_{21} = 0$ in M, $w_{12} = 0$ in N, $w_{11} \in J(A)$ and $w_{22} \in J(B)$. Therefore

$$\rho\alpha\rho^{-1} = \begin{pmatrix} 1 + w_{11} & 0 \\ 0 & w_{22} \end{pmatrix}.$$

If $\rho e\rho^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, as in the previous discussion,

$$\rho\alpha\rho^{-1} = \begin{pmatrix} w_{11} & 0 \\ 0 & 1 + w_{22} \end{pmatrix},$$

where $w_{11} \in J(A)$ and $w_{22} \in J(B)$, as required.

The converse is obvious. □

Corollary 4.2.17. *Let $T = (A, B, M, N, \psi, \varphi)$, and let A and B be local rings. If $Im(\psi) \subseteq J(A)$ and $Im(\varphi) \subseteq J(B)$, then the following are equivalent:*

(1) *$\alpha \in T$ is strongly clean.*
(2) *$\alpha \in U(T)$, or $1 - \alpha \in U(T)$, or there exists some $\beta \in U(T)$ such that $\beta\alpha\beta^{-1} = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$, where $c \in 1_A + J(A), d \in J(B)$, or $c \in J(A), d \in 1_B + J(B)$.*
(3) *$\alpha \in U(T)$, or $1 - \alpha \in U(T)$, or $\alpha \in T$ is strongly J-clean.*

Proof. (1) ⇒ (2) is analogous to the proof of Theorem 4.2.16.

(2) ⇒ (3) If $\alpha, 1 - \alpha \notin U(T)$, then there exists some $\beta \in U(T)$ such that $\beta\alpha\beta^{-1} = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$, where $c \in 1_A + J(A), d \in J(B)$, or $c \in J(A), d \in 1_B + J(B)$. By using Theorem 4.2.16, $\alpha \in T$ is strongly J-clean.

(3) ⇒ (1) This is clear. □

Corollary 4.2.18. *Let R be a local ring, and let $s \in J(R) \cap C(R)$. Then $A \in M_{(s)}(R)$ is strongly J-clean if and only if $A \in J(M_{(s)}(R))$ or*

$I_2 - A \in J\big(M_{(s)}(R)\big)$ *or* A *is similar to a matrix* $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, *where* α *and* β *are in* $J(R)$ *or* $1 + J(R)$.

Proof. Let $\psi : R \otimes R \to R, n \otimes m \mapsto snm$ and $\varphi : R \otimes R \to R, m \otimes n \mapsto smn$. Then $M_s(R) = (R, R, R, R, \psi, \varphi)$. As $s \in J(R)$, we see that $Im(\varphi), Im(\psi) \subseteq J(R)$, and therefore we complete the proof by Theorem 4.2.16. □

Corollary 4.2.19. *Let* R *be a local ring, and let* $s \in J(R) \cap C(R)$. *Then* $A \in M_{(s)}(R)$ *is strongly clean if and only if* $A \in U\big(M_{(s)}(R)\big)$ *or* $I_2 - A \in U\big(M_{(s)}(R)\big)$ *or* A *is similar to a matrix* $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, *where* α *and* β *are in* $J(R)$ *or* $1 + J(R)$.

Proof. As in the proof of Corollary 4.2.18, we complete the proof by Corollary 4.2.17. □

Let A be a commutative ring, and let M be an R-module. We use $[A, M, \varphi]$ to denote $(A, A, M, M, \varphi, \varphi)$. Let $T = [A, M, \varphi]$, and let $\alpha = \begin{pmatrix} a & m \\ n & b \end{pmatrix} \in T$. We use $det(\alpha)$ to stand for $ab - \varphi(m \otimes n)$, and that $tr(\alpha) = a + b$.

Lemma 4.2.20. *Let* $T = [A, M, \varphi]$, *and let* $\alpha, \beta \in T$. *Then* $det(\alpha\beta) = det(\alpha)det(\beta)$.

Proof. Straightforward. □

Theorem 4.2.21. *Let* $T = [A, M, \varphi]$, *and let* A *be a commutative local ring. If* $Im(\varphi) \subseteq J(A)$, *then* $\alpha \in T$ *is strongly J-clean if and only if* $\alpha \in J(T)$, *or* $1 - \alpha \in J(T)$, *or the equation* $x^2 - tr(\alpha)x + det(\alpha) = 0$ *has a root in* $J(A)$ *and a root in* $1 + J(A)$.

Proof. Suppose $\alpha \in T$ is strongly J-clean. In view of Theorem 4.2.16, $\alpha \in J(T)$, or $1 - \alpha \in J(T)$, or there exists some $\beta \in U(T)$ such that $\beta\alpha\beta^{-1} = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$, where $c \in 1_A + J(A), d \in J(B)$, or $c \in J(A), d \in 1_B + J(B)$. Hence, we may assume that $\beta(xI_2 - \alpha)\beta^{-1} = \begin{pmatrix} x - c & 0 \\ 0 & x - d \end{pmatrix}$. In light of Lemma 4.2.20, $x^2 - tr(\alpha)x + det(\alpha) = det(xI_2 - \alpha) = (x - c)(x - d)$. Therefore $x^2 - tr(\alpha)x + det(\alpha) = 0$ has a root c and a root d, as required.

Conversely, if $\alpha \in J(T)$, or $1-\alpha \in J(T)$, then α is strongly J-clean. We now assume that the equation $x^2 - tr(\alpha)x + det(\alpha) = 0$ has a root c in $J(A)$ and a root d in $1+J(A)$. Then $tr(\alpha) = c+d \in 1+J(R)$ and $det(\alpha) = cd \in J(R)$. Write $\alpha = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Then $a_{11}+a_{22} \in 1+J(R)$ and $a_{11}a_{22} = det(\alpha) + sa_{12}a_{21} \in J(R)$. Since R is local, either a_{11} or a_{22} is invertible. Without loss of generality, we may assume that $a_{11} \in U(R)$, and then $a_{22} \in J(R)$. This shows that $a_{11} \in 1+J(R)$

Let $\beta = \begin{pmatrix} 1 & a_{12}(c-a_{11})^{-1} \\ a_{21}(d-a_{22})^{-1} & 1 \end{pmatrix}$. Clearly, $det(\beta) = 1 - sa_{12}a_{21}(c-a_{11})^{-1}(d-a_{22})^{-1} = 1+\big(det(\alpha)-a_{11}a_{22}\big)(c-a_{11})^{-1}(d-a_{22})^{-1} \in 1+J(R)$. It is easy to verify that

$$\beta^{-1} = \frac{1}{det(\beta)} \begin{pmatrix} 1 & -a_{12}(c-a_{11})^{-1} \\ -a_{21}(d-a_{22})^{-1} & 1 \end{pmatrix}.$$

Furthermore,

$$\begin{aligned} &\beta^{-1}\alpha\beta \\ &= \begin{pmatrix} p & \frac{a_{12}}{(c-a_{11})^2 det(\beta)}\big(c^2 - tr(\alpha)c + det(\alpha)\big) \\ \frac{a_{21}}{(d-a_{22})^2 det(\beta)}\big(d^2 - tr(\alpha)d + det(\alpha)\big) & q \end{pmatrix} \\ &= \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}, \end{aligned}$$

where $p = \frac{a_{11} - sa_{12}a_{21}\left((c-a_{11})^{-1} - (d-a_{22})^{-1}(1-a_{22}(c-a_{11})^{-1}\right)}{det(\beta)} \in 1+J(R)$, and $q = \frac{a_{22} - sa_{12}a_{21}\left((d-a_{22})^{-1} - (c-a_{11})^{-1}(1-a_{11}(d-a_{22})^{-1}\right)}{det(\beta)} \in J(R)$. According to Theorem 4.2.16, $\alpha \in T$ is strongly J-clean, as desired. □

Let $A = (a_{ij}) \in M_{(s)}(R)$. We use $det_{(s)}(A)$ to stand for $a_{11}a_{22} - sa_{12}a_{21}$, and that $tr_{(s)}(A) = a_{11} + a_{22}$.

Corollary 4.2.22. *Let R be a commutative local ring, let $s \in J(R)$, and let $A \in M_{(s)}(R)$. Then the following are equivalent:*

(1) *$A \in M_{(s)}(R)$ is strongly J-clean.*
(2) *$A \in J\big(M_{(s)}(R)\big)$ or $I_2 - A \in J\big(M_{(s)}(R)\big)$ or the equation $x^2 - tr_{(s)}(A)x + det_{(s)}(A) = 0$ has a root in $J(R)$ and a root in $1+J(R)$.*

Proof. It is an immediate consequence of Theorem 4.2.21. □

Let R be a commutative local ring, let $s \in J(R)$, and let $A \in M_{(s)}(R)$. Analogously, $A \in M_{(s)}(R)$ is strongly clean if and only if $A \in U\big(M_{(s)}(R)\big)$

or $I_2 - A \in U\big(M_{(s)}(R)\big)$ or the equation $x^2 - tr_{(s)}(A)x + det_{(s)}(A) = 0$ has a root in $J(R)$ and a root in $1 + J(R)$.

Theorem 4.2.23. *Let R be a commutative local ring, and let $A(x) \in M_{(x^n)}\big(R[[x]]\big)(n \geq 1)$. Then the following are equivalent:*

(1) $A(x) \in M_{(x^n)}\big(R[[x]]\big)$ *is strongly J-clean.*
(2) $A(0) \in M_{(0)}(R)$ *is strongly J-clean.*

Proof. (1) $\Rightarrow$ (2) By hypothesis, there exists an idempotent $E(X) \in M_{(x^n)}\big(R[[x]]\big)$ and $W(x) \in J\big(M_{(x^n)}(R[[x]])\big)$ such that $A(x) = E(x)+W(x)$ and $E(x)W(x) = W(x)E(x)$. Thus, $A(0) = E(0) + W(0)$ and $E(0)W(0) = W(0)E(0)$, as required.

(2) $\Rightarrow$ (1) Obviously, $R[[x]]$ is a commutative local ring. As $A(0) \in M_{(0)}(R)$ is strongly J-clean, it follows from Corollary 4.2.19 that $A(0) \in J\big(M_{(0)}(R)\big)$ or $I_2 - A(0) \in J\big(M_{(0)}(R)\big)$ or the equation $t^2 - tr_{(0)}A(0)t + det_{(0)}A(0) = 0$ has a root in $1 + J(R)$ and a root in $J(R)$. If $A(0) \in J\big(M_{(0)}(R)\big)$, then $A(x) \in J\big(M_{(x^n)}(R[[x]])\big)$. If $I_2 - A(0) \in J\big(M_{(0)}(R)\big)$, then $I_2 - A(x) \in J\big(M_{(x^n)}(R[[x]])\big)$. Thus, we assume that the equation $t^2 - tr_{(0)}A(0)t + det_{(0)}A(0) = 0$ has a root $\alpha \in 1 + J(R)$ and a root $\beta \in J(R)$. Write $y = \sum\limits_{i=0}^{\infty} b_i x^i$. Then $y^2 = \sum\limits_{i=0}^{\infty} c_i x^i$ where $c_i = \sum\limits_{k=0}^{i} b_k b_{i-k}$. Let $tr_{(x^n)}\big(A(x)\big) = \sum\limits_{i=0}^{\infty} \mu_i x^i, det_{(x^n)}\big(A(x)\big) = \sum\limits_{i=0}^{\infty} \lambda_i x^i \in R[[x]]$ where $\mu_0 = tr_{(0)}A(0)$ and $\lambda_0 = det_{(0)}A(0)$. Then, $y^2 - tr_{(x^n)}\big(A(x)\big)y + det_{(x^n)}\big(A(x)\big) = 0$ holds in $R[[x]]$ if the following equations are satisfied:

$$\begin{gathered} b_0^2 - b_0\mu_0 + \lambda_0 = 0; \\ (b_0b_1 + b_1b_0) - (b_0\mu_1 + b_1\mu_0) + \lambda_1 = 0; \\ (b_0b_2 + b_1^2 + b_2b_0) - (b_0\mu_2 + b_1\mu_1 + b_2\mu_0) + \lambda_2 = 0; \\ \vdots \end{gathered}$$

Clearly, $\mu_0 = \alpha + \beta \in U(R)$. Let $b_0 = \alpha$. Since R is commutative, there exists some $b_1 \in R$ such that

$$b_0b_1 + b_1(b_0 - \mu_0) = -\lambda_1 + b_0\mu_1.$$

Further, there exists some $b_2 \in R$ such that

$$b_0b_2 + b_2(b_0 - \mu_0) = -\lambda_2 - b_1^2 + b_0\mu_2 + b_1\mu_1.$$

By iteration of this process, we get $b_3, b_4, \ldots$. Then $y^2 - tr_{(x^n)}\big(A(x)\big)y + det_{(x^n)}\big(A(x)\big) = 0$ has a root $y_0(x) \in 1 + J\big(R[[x]]\big)$. If $b_0 = \beta \in J(R)$,

analogously, we show that $y^2 - tr_{(x^n)}\big(A(x)\big)y + det_{(x^n)}\big(A(x)\big) = 0$ has a root $y_1(x) \in J\big(R[[x]]\big)$. Therefore the result follows by Corollary 4.2.22. □

Let R be a commutative local ring. Then $\begin{pmatrix} xf(x) & xg(x) \\ h(x) & 1+xk(x) \end{pmatrix} \in M_{(x^n)}\big(R[[x]]\big)(n \geq 1)$ is strongly J-clean, where $f(x), g(x), h(x), k(x) \in R[[x]]$. As $\begin{pmatrix} 0 & 0 \\ h(0) & 1 \end{pmatrix} \in M_{(0)}(R)$ is strongly J-clean, we are done by Theorem 4.2.23.

Corollary 4.2.24. *Let R be a commutative local ring, and let $A(x) \in M_{(x^n)}\big(R[[x]]\big)(n \geq 1)$. Then the following are equivalent:*

(1) $A(x) \in M_{(x^n)}\big(R[[x]]\big)$ *is strongly clean.*
(2) $A(0) \in M_{(0)}(R)$ *is strongly clean.*

Proof. (1) $\Rightarrow$ (2) This is obvious as in the proof of Theorem 4.2.23.

(2) $\Rightarrow$ (1) In virtue of Corollary 4.2.18, $A(0) \in U\big(M_{(0)}(R)\big)$, or $I_2 - A(0) \in U\big(M_{(0)}(R)\big)$, or $A(0) \in M_{(0)}(R)$ is strongly J-clean. Thus, $A(x) \in U\big(M_{(x^n)}(R[[x]])\big)$, or $I_2 - A(x) \in U\big(M_{(x^n)}(R[[x]])\big)$; otherwise, $A(0) = \begin{pmatrix} \alpha & * \\ * & \beta \end{pmatrix} \in M_{(0)}(R)$ where one of α and β is in $J(R)$ and the other is in $1 + J(R)$. Thus, $tr_{(0)}A(0) = \alpha + \beta$ and $detA_{(0)}(0) = \alpha\beta$. This shows that the equation $t^2 - tr_{(0)}A(0)t + det_{(0)}A(0) = 0$ has a root in $J(R)$ and a root in $1 + J(R)$. In view of Corollary 4.2.22, $A(0) \in M_{(0)}(R)$ is strongly J-clean. Thus, $A(x) \in M_{(x^n)}(R)$ is strongly J-clean by Theorem 4.2.23. This completes the proof. □

4.3 Strongly $J^{\#}$-Clean Matrices

The purpose of this section is to introduce a subclass of strongly clean matrices which behaves like strong J-cleanness but one can characterize such matrices by a kind of factorizations of the corresponding characteristic polynomials, and so get more explicit decompositions for many types of matrices over projective-free rings.

Let $J(R)$ be the Jacobson radical of R. Set

$$J^{\#}(R) = \{x \in R \mid \exists\, n \in \mathbb{N} \text{ such that } x^n \in J(R)\}.$$

That is, $J^{\#}(R) = \sqrt{J(R)}$. For instance, let $R = M_2(\mathbb{Z}_2)$. Then

$$J^{\#}(R) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\},$$

while $J(R) = 0$. Thus, $J^{\#}(R)$ and $J(R)$ are distinct in general. We say that an element $a \in R$ is strongly $J^{\#}$-clean provided that there exists an idempotent $e \in R$ such that $a - e \in J^{\#}(R)$ and $ae = ea$. The prime radical $P(R)$ of a ring R is defined as the intersection of the prime ideals of R. Let $a \in R$ be strongly nilpotent if for every sequence $a_0, a_1, \ldots, a_i, \ldots$ such that $a_0 = a$ and $a_{i+1} \in a_i R a_i$, there exists n with $a_n = 0$. We begin with

Lemma 4.3.1. *The prime radical of a ring is exactly the set of all its strongly nilpotents.*

Proof. Let R be a ring, and let $a \in P(R)$. Suppose there exists a sequence $a_0, a_1, \ldots, a_i, \ldots$ such that $a_0 = a$ and $a_{i+1} \in a_i R a_i$, while $a_n \neq 0$ for all n. We set $S = \{a_0, a_1, a_2, \ldots\}$. Let

$$\Omega = \{I \mid I \trianglelefteq R \text{ such that } I \cap S = \emptyset\}.$$

Then Ω is a nonempty inductive set. By Zorn's Lemma, we have an ideal Q which is maximal in Ω. If Q is not prime, then there are some $b, c \in R$ such that $bRc \subseteq Q$, while $b, c \notin Q$. Thus, we can find $m, n \in \mathbb{N}$ such that $a_m \in Q + RbR$ and $a_n \in Q + RcR$. This shows that $a_{m+n} \in Q + RbRcR \subseteq Q$. This gives a contradiction, and so $Q \in Spec(R)$. Thus, $a \in Q$, a contradiction. Hence, every element in the prime radical $P(R)$ is strongly nilpotent. Clearly, every strongly nilpotent in R is contained in $P(R)$. Therefore $P(R)$ is the set of all strongly nilpotents in R. □

Every $n \times n$ matrix with nilpotent entries in a commutative ring is nilpotent. But the converse is not true.

Theorem 4.3.2. *Let R be a commutative ring, and let $\varphi \in M_n(R)$. Then the following are equivalent:*

(1) $\varphi \in N\big(M_n(R)\big)$.
(2) $\chi(\varphi) \equiv t^n \big(mod\ N(R)\big)$, *i.e.,* $\chi(\varphi) - t^n \in N(R)[t]$.
(3) *There exists a monic polynomial* $h \in R[t]$ *such that* $h \equiv t^{degh} \big(mod\ N(R)\big)$ *for which* $h(\varphi) = 0$.

Proof. It follows immediately from the fact that the characteristic polynomial of a nilpotent $n \times n$ matrix over a filed has the form t^n and from the equality $N(R) = \bigcap\{P \mid P \in Spec(R)\ \}$, by Lemma 4.3.1. □

Lemma 4.3.3. *Let R be a commutative ring, and let $\varphi \in M_n(R)$. Then the following are equivalent:*

(1) $\varphi \in J^{\#}\big(M_n(R)\big)$.
(2) $\chi(\varphi) \equiv t^n \big(mod\ J(R)\big)$, *i.e.*, $\chi(\varphi) - t^n \in J(R)[t]$.
(3) *There exists a monic polynomial* $h \in R[t]$ *such that* $h \equiv t^{degh} \big(mod\ J(R)\big)$ *for which* $h(\varphi) = 0$.

Proof. (1) $\Rightarrow$ (2) Since $\varphi \in J^{\#}\big(M_n(R)\big)$, there exists some $m \in \mathbb{N}$ such that $\varphi^m \in J\big(M_n(R)\big)$. As $J\big(M_n(R)\big) = M_n\big(J(R)\big)$, we get $\overline{\varphi} \in N\big(M_n(R/J(R))\big)$. In view of Theorem 4.3.2, $\chi\big(\overline{\varphi}\big) \equiv t^n \big(mod\ N(R/J(R))\big)$. Write $\chi(\varphi) = t^n + a_1 t^{n-1} + \cdots + a_n$. Then $\chi\big(\overline{\varphi}\big) = t^n + \overline{a_1} t^{n-1} + \cdots + \overline{a_n}$. We infer that each $a_i^{m_i} + J(R) = 0 + J(R)$ where $m_i \in \mathbb{N}$. This implies that $a_i \in J^{\#}(R)$. That is, $\chi(\varphi) \equiv t^n \big(mod\ J^{\#}(R)\big)$. As R is commutative, $J^{\#}(R) = J(R)$, as required.

(2) $\Rightarrow$ (3) Choose $h = \chi(\varphi)$. Then $h \equiv t^{degh} \big(mod\ J(R)\big)$. In light of the Cayley-Hamilton Theorem, $h(\varphi) = 0$.

(3) $\Rightarrow$ (1) By hypothesis, there exists a monic polynomial $h \in R[t]$ such that $h \equiv t^{degh} \big(mod\ J(R)\big)$ for which $h(\varphi) = 0$. Write $h = t^n + a_1 t^{n-1} + \cdots + a_n$. Choose $\overline{h} = t^n + \overline{a_1} t^{n-1} + \cdots + \overline{a_n} \in \big(R/J(R)\big)[t]$. Then $\overline{h} \equiv t^n \in \big(R/J(R)\big)\big)[t]$; hence, $\overline{h} = t^n\ N\big(R/J(R)\big)$ for which $\overline{h}\big(\overline{\varphi}\big) = 0$. In light of Theorem 4.3.2, $\overline{\varphi} \in N(M_n\big(R/J(R)\big)$. Therefore $\varphi \in J^{\#}\big(M_n(R)\big)$. □

The next lemma is an analog of Theorem 4.2.1.

Lemma 4.3.4. *Let M be a left R-module, let $E = End(M)$, and let $\alpha \in E$. Then the following are equivalent:*

(1) $\alpha \in E$ *is strongly* $J^{\#}$*-clean.*
(2) *There exists a left R-module decomposition $M = P \oplus Q$ where P and Q are α-invariant, and $\alpha|_P \in J^{\#}\big(End(P)\big)$ and $(1_M - \alpha)|_Q \in J^{\#}\big(End(Q)\big)$.*

Proof. (1) $\Rightarrow$ (2) Since α is strongly $J^{\#}$-clean in E, there exists an idempotent $\pi \in E$ and $u \in J^{\#}(E)$ such that $\alpha = (1 - \pi) + u$ and $\pi u = u\pi$. Thus, $\pi\alpha = \pi u \in J^{\#}\big(\pi E \pi\big)$. Further, $1 - \alpha = \pi + (-u)$, and so $(1 - \pi)(1 - \alpha) \in J^{\#}\big((1 - \pi)E(1 - \pi)\big)$. Set $P = M\pi$ and $Q = M(1 - \pi)$. Then $M = P \oplus Q$ and P, Q are α-invariant. Since $\alpha\pi \in J^{\#}\big(\pi E \pi\big)$, we have some $t \in \mathbb{N}$ such that $(\alpha\pi)^t \in J\big(\pi E \pi\big)$. Let $\gamma \in End(P)$. For any $x \in M$, it is easy to see that $(x)\pi\big(1_P - \gamma\big(\alpha|_P\big)^t\big) = (x)\pi\big(\pi - (\pi\overline{\gamma}\pi)(\pi\alpha\pi)^t\big)$

where $\overline{\gamma} : M \to M$ given by $(m)\overline{\gamma} = (m)\pi\gamma$ for any $m \in M$. Hence, $1_P - \gamma\big(\alpha|_P\big)^t \in Aut(P)$, and so $\big(\alpha|_P\big)^t \in J\big(End(P)\big)$. This implies that $\alpha|_P \in J^{\#}\big(End(P)\big)$. Likewise, $(1-\alpha)|_Q \in J^{\#}\big(End(Q)\big)$, as desired.

(2) $\Rightarrow$ (1) For any $\lambda \in End(Q)$, we construct an R-morphism $\overline{\lambda} \in End(M)$ given by $\big(p+q\big)\overline{\lambda} = (q)\lambda$. By hypothesis, $\alpha|_P \in J^{\#}\big(End(P)\big)$ and $(1_M - \alpha)|_Q \in J^{\#}\big(End(Q)\big)$. Thus, $\alpha = \overline{1_Q} + \overline{\alpha|_P} - \overline{(1_M-\alpha)|_Q}$. As P and Q are α-invariant, we see that $\alpha\overline{1_Q} = \overline{1_Q}\alpha$. In addition, $\overline{1_Q} \in End(M)$ is an idempotent. As $\big(\overline{\alpha|_P}\big)\big(\overline{(1_M-\alpha)|_Q}\big) = 0 = \big(\overline{(1_M-\alpha)|_Q}\big)\big(\overline{\alpha|_P}\big)$, we have $\overline{\alpha|_P} - \overline{(1_M-\alpha)|_Q} \in J^{\#}\big(End(M)\big)$, as required. □

For any $r \in R$, we define $\mathbb{J}_r = \{f \in R[t] \mid f \text{ monic, and } f \equiv (t-r)^{degf}\big(mod\ J(R)\big)\}$.

Lemma 4.3.5. *Let R be a projective-free ring, let $\varphi \in M_n(R)$, and let $h \in R[t]$ be a monic polynomial of degree n. If $h(\varphi) = 0$ and there exists a factorization $h = h_0h_1$ such that $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_1$, then φ is strongly $J^{\#}$-clean.*

Proof. Suppose that $h = h_0h_1$ where $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_1$. Write $h_0 = t^p + a_1t^{p-1} + \cdots + a_p$ and $h_1 = (t-1)^q + b_1t^{q-1} + \cdots + b_q$. Then each $a_i, b_j \in J(R)$. Thus, $\overline{h_0} = t^p$ and $\overline{h_1} = (t-\overline{1})^q$ in $\big(R/J(R)\big)[t]$. Hence, $\big(\overline{h_0}, \overline{h_1}\big) = \overline{1}$. As in the proof of Theorem 2.2.13, we see that $(h_0, h_1) = 1$. Write $u_0h_0 + u_1h_1 = 1$ for some $u_0, u_1 \in R[t]$. Then $u_0(\varphi)h_0(\varphi) + u_1(\varphi)h_1(\varphi) = 1_{nR}$. By hypothesis, $h(\varphi) = h_0(\varphi)h_1(\varphi) = h_1(\varphi)h_0(\varphi) = 0$. Clearly, $u_0(\varphi)h_1(\varphi) = h_1(\varphi)u_0(\varphi)$ and $h_0(\varphi)u_1(\varphi) = u_1(\varphi)h_0(\varphi)$. In light of Lemma 2.2.1, $nR = Ker\big(h_0(\varphi)\big) \oplus Ker\big(h_1(\varphi)\big)$. As $h_0t = th_0$ and $h_1t = th_1$, we see that $h_0(\varphi)\varphi = \varphi h_0(\varphi)$ and $h_1(\varphi)\varphi = \varphi h_1(\varphi)$, and so $Ker\big(h_0(\varphi)\big)$ and $Ker\big(h_1(\varphi)\big)$ are both φ-invariant. It is easy to verify that $h_0\big(\varphi\ |_{ker(h_0(\varphi))}\big) = 0$. Since $h_0 \in \mathbb{J}_0$, we see that $h_0 \equiv t^{degh_0}\big(mod\ J(R)\big)$; hence, $\varphi\ |_{ker(h_0(\varphi))} \in J^{\#}\big(End(kerh_0(\varphi))\big)$.

It is easy to verify that $h_1\big(\varphi\ |_{ker(h_1(\varphi))}\big) = 0$. Set $g(u) = (-1)^{degh_1}h_1(1-u)$. Then $g\big(\ (\ 1-\varphi\ )|_{Ker(h_1(\varphi))}\big) = 0$. As $h_1 \in \mathbb{J}_1$, we have $h_1 \equiv (t-1)^{degh_1}\big(mod\ J(R)\big)$. Hence, $g(u) \equiv (-1)^{degh_1}(-u)^{degg}\big(mod\ J(R)\big)$. This implies that $g \in \mathbb{J}_0$. By virtue of Lemma 4.3.3, $(1-\varphi)\ |_{Ker(h_1(\varphi))} \in J^{\#}\big(End(Ker(h_1(\varphi)))\big)$. According to Lemma 4.3.4, $\varphi \in M_n(R)$ is strongly $J^{\#}$-clean. □

Theorem 4.3.6. *Let R be a projective-free ring, and let $h \in R[t]$ be a monic polynomial of degree n. Then the following are equivalent:*

(1) *Every* $\varphi \in M_n(R)$ *with* $\chi(\varphi) = h$ *is strongly* $J^{\#}$*-clean.*
(2) *The companion matrix* C_h *of* h *is strongly* $J^{\#}$*-clean.*
(3) *There exists a factorization* $h = h_0h_1$ *such that* $h_0 \in \mathbb{J}_0$ *and* $h_1 \in \mathbb{J}_1$.

Proof. (1) $\Rightarrow$ (2) Write $h = t^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0 \in R[t]$. Then $\chi(C_h) = h$; hence, $C_h \in M_n(R)$ is strongly $J^{\#}$-clean.

(2) $\Rightarrow$ (3) Set $\varphi = C_h$. In view of Lemma 4.3.4, there exists a decomposition $nR = A \oplus B$ such that A and B are φ-invariant, $\varphi \mid_A \in J^{\#}\big(End_R(A)\big)$ and $(1-\varphi) \mid_B \in J^{\#}\big(End_R(B)\big)$. Since R is a projective-free ring, there exist $p, q \in \mathbb{N}$ such that $A \cong pR$ and $B \cong qR$. Regarding $End_R(A)$ as $M_p(R)$, we see that $\varphi \mid_A \in J^{\#}\big(M_p(R)\big)$. By virtue of Lemma 4.3.3, $\chi(\varphi \mid_A) \equiv t^p \big(mod\ J(R)\big)$. Thus $\chi(\varphi \mid_A) \in \mathbb{J}_0$. Analogously, $(1-\varphi) \mid_B \in J^{\#}\big(M_q(R)\big)$. It follows from Lemma 4.3.3 that $\chi\big((1-\varphi) \mid_B\big) \equiv t^q \pmod{J(R)}$. This implies that $det\big(\lambda I_q - (1-\varphi) \mid_B\big) \equiv \lambda^q \pmod{J(R)}$. Hence, $det\big((1-\lambda)I_q - \varphi \mid_B\big) \equiv (-\lambda)^q \pmod{J(R)}$. Set $t = 1 - \lambda$. Then $det\big(tI_q - \varphi \mid_B\big) \equiv (t-1)^q \pmod{J(R)}$. Therefore we get $\chi(\varphi \mid_B) \equiv (t-1)^q \pmod{J(R)}$. We infer that $\chi(\varphi \mid_B) \in \mathbb{J}_1$. Clearly, $\chi(\varphi) = \chi(\varphi \mid_A)\chi(\varphi \mid_B)$. Choose $h_0 = \chi(\varphi \mid_A)$ and $h_1 = \chi(\varphi \mid_B)$. Then there exists a factorization $h = h_0h_1$ such that $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_1$, as desired.

(3) $\Rightarrow$ (1) For every $\varphi \in M_n(R)$ with $\chi(\varphi) = h$, it follows by the Cayley-Hamilton Theorem that $h(\varphi) = 0$. Therefore φ is strongly $J^{\#}$-clean by Lemma 4.3.5. □

Corollary 4.3.7. *Let* R *be a projective-free ring, and let* $\varphi \in M_2(R)$. *Then* φ *is strongly* $J^{\#}$*-clean if and only if*

(1) $\chi(\varphi) \equiv t^2 \big(mod\ J(R)\big)$; *or*
(2) $\chi(\varphi) \equiv (t-1)^2 \big(mod\ J(R)\big)$; *or*
(3) $\chi(\varphi)$ *has a root in* $J(R)$ *and a root in* $1 + J(R)$.

Proof. $\Longrightarrow$ Write $\varphi = E + W$, where $E^2 = E, W^n \in J(M_2(R))$ and $EW = WE$. In light of Theorem 2.4.7, there exists $U \in GL_2(R)$ such that $UEU^{-1} = 0, I_2$ or $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $\varphi \in J^{\#}(R), I_2 - \varphi \in J^{\#}(R)$, or $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Thus we may assume that $UEU^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Hence, $U\varphi U^{-1} = UEU^{-1} + UWU^{-1}$. It follows from $(UEU^{-1})(UWU^{-1} = (UWU^{-1})(UEU^{-1})$ that $UWU^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ for some $\lambda, \mu \in J(R)$. Hence, $U\varphi U^{-1} = \begin{pmatrix} 1+\lambda & 0 \\ 0 & \mu \end{pmatrix}$. Therefore $\chi(\varphi)$ has a root μ in $J(R)$ and a root $1+\lambda$ in $1+J(R)$, as required.

$\Longleftarrow$ If $\chi(\varphi) \equiv t^2 (mod\ J(R))$ or $\chi(\varphi) \equiv (t-1)^2 (mod\ J(R))$, then $\varphi \in J^{\#}(M_2(R))$ or $I_2 - \varphi \in J^{\#}(M_2(R))$. This implies that φ is strongly $J^{\#}$-clean. Otherwise, $\chi(\varphi)$ has a root in $J(R)$ and a root in $1 + J(R)$. Thus, we can find some $\alpha \in J(R)$ and $\beta \in 1 + J(R)$ such that $\chi(\varphi) = h_0 h_1$, where $h_0 = t - \alpha \in \mathbb{J}_0$ and $h_1 = t - \beta \in \mathbb{J}_1$. In light of Theorem 4.3.6, φ is strongly $J^{\#}$-clean, hence the result. □

Corollary 4.3.8. *Let R be a projective-free ring, and let $f(t) = t^2 + at + b \in R[t]$ be a degree 2 polynomial with $1 + a \in J(R), b \notin J(R)$. Then the following are equivalent:*

(1) *Every $\varphi \in M_2(R)$ with $\chi(\varphi) = f(t)$ is strongly $J^{\#}$-clean.*
(2) *There exist $r_1 \in J(R)$ and $r_2 \in 1 + J(R)$ such that $f(r_1) = f(r_2) = 0$.*
(3) *There exists $r \in J(R)$ such that $f(r) = 0$.*

Proof. (1) $\Rightarrow$ (2) Since every $\varphi \in M_2(R)$ with $\chi(\varphi) = f(t)$ is strongly $J^{\#}$-clean, it follows from Corollary 4.3.7 that $f(t) = (t - r_1)(t - r_2)$ with $r_1 \in J(R), r_2 \in 1 + J(R)$.

(2) $\Rightarrow$ (3) This is trivial.

(3) $\Rightarrow$ (1) As $r^2 + ar + b = 0$, we see that $f(t) = (t - r)(t + a + r)$. Clearly, $t - r \in \mathbb{J}_0$. It follows from $1 + a + r \in J(R)$ that $t + a + r \in \mathbb{J}_1$. According to Theorem 4.3.6, we complete the proof. □

Corollary 4.3.9. *Let R be a commutative local ring. Then $A \in M_2(R)$ is strongly $J^{\#}$-clean if and only if $A^2 \in J(M_2(R))$, $I_2 - A^2 \in J(M_2(R))$, or A is strongly J-clean.*

Proof. $\Longleftarrow$ This is obvious.

$\Longrightarrow$ In light of Corollary 4.3.7, $A^2 \in J(M_2(R))$, $I_2 - A^2 \in J(M_2(R))$, or $\chi(\varphi)$ has a root in $J(R)$ and a root in $1 + J(R)$. Therefore we complete the proof, by Corollary 4.2.5. □

Theorem 4.3.10. *Let R be a projective-free ring, and let $A(x) \in M_2(R[[x]])$. Then the following are equivalent:*

(1) *$A(x) \in M_2(R[[x]])$ is strongly $J^{\#}$-clean.*
(2) *$A(0) \in M_2(R)$ is strongly $J^{\#}$-clean.*

Proof. (1) $\Rightarrow$ (2) Since $A(x)$ is strongly $J^{\#}$-clean in $M_2(R[[x]])$, there exists $E(x) = E^2(x) \in M_2(R[[x]])$ and $U(x) \in J^{\#}(M_2(R[[x]]))$ such that $A(x) = E(x) + U(x)$ and $E(x)U(x) = U(x)E(x)$. This implies that

$A(0) = E(0) + U(0)$ and $E(0)U(0) = U(0)E(0)$ where $E(0) = E^2(0) \in M_2(R)$ and $U(0) \in J^{\#}\big(M_2(R)\big)$. As a result, $A(0)$ is strongly $J^{\#}$-clean in $M_2(R)$.

(2) $\Rightarrow$ (1) In view of Corollary 2.4.3, $R[[x]]$ is projective-free. Since $A(0)$ is strongly $J^{\#}$-clean in $M_2(R)$, it follows from Corollary 4.3.7 that $A(0) \in J^{\#}\big(M_2(R)\big)$, or $I_2 - A(0) \in J^{\#}\big(M_2(R)\big)$, or the characteristic polynomial $\chi\big(A(0)\big) = y^2 + \mu y + \lambda$ has a root $\alpha \in 1 + J(R)$ and a root $\beta \in J(R)$. If $A(0) \in J^{\#}\big(M_2(R)\big)$, then $A(x) \in J^{\#}\big(M_2(R[[x]])\big)$. If $I_2 - A(0) \in J^{\#}\big(M_2(R)\big)$, then $I_2 - A(x) \in J^{\#}\big(M_2(R[[x]])\big)$. Otherwise, we write $y = \sum\limits_{i=0}^{\infty} b_i x^i$ and $\chi(A(x)) = y^2 - \mu(x)y - \lambda(x)$. Then $y^2 = \sum\limits_{i=0}^{\infty} c_i x^i$ where $c_i = \sum\limits_{k=0}^{i} b_k b_{i-k}$. Let $\mu(x) = \sum\limits_{i=0}^{\infty} \mu_i x^i, \lambda(x) = \sum\limits_{i=0}^{\infty} \lambda_i x^i \in R[[x]]$ where $\mu_0 = \mu$ and $\lambda_0 = \lambda$. Then, $y^2 - \mu(x)y - \lambda(x) = 0$ holds in $R[[x]]$ if the following equations are satisfied:

$$\begin{gathered} b_0^2 - b_0\mu_0 - \lambda_0 = 0; \\ (b_0b_1 + b_1b_0) - (b_0\mu_1 + b_1\mu_0) - \lambda_1 = 0; \\ (b_0b_2 + b_1^2 + b_2b_0) - (b_0\mu_2 + b_1\mu_1 + b_2\mu_0) - \lambda_2 = 0; \\ \vdots \end{gathered}$$

Obviously, $\mu_0 = \alpha + \beta \in U(R)$. Let $b_0 = \alpha$. Since R is commutative, there exists some $b_1 \in R$ such that

$$b_0b_1 + b_1(b_0 - \mu_0) = \lambda_1 + b_0\mu_1.$$

Further, there exists some $b_2 \in R$ such that

$$b_0b_2 + b_2(b_0 - \mu_0) = \lambda_2 - b_1^2 + b_0\mu_2 + b_1\mu_1.$$

By iteration of this process, we get $b_3, b_4, \ldots$. Then $y^2 - \mu(x)y - \lambda(x) = 0$ has a root $y_0(x) \in 1 + J\big(R[[x]]\big)$. If $b_0 = \beta \in J(R)$, analogously, we prove that $y^2 - \mu(x)y - \lambda(x) = 0$ has a root $y_1(x) \in J\big(R[[x]]\big)$. In light of Corollary 4.3.7, the result follows. □

Corollary 4.3.11. *Let R be a projective-free ring, and let $A(x) \in M_2\big(R[[x]]/(x^m)\big)$ $(m \in \mathbb{N})$. Then the following are equivalent:*

(1) *$A(x) \in M_2\big(R[[x]]/(x^m)\big)$ is strongly $J^{\#}$-clean.*
(2) *$A(0) \in M_2(R)$ is strongly $J^{\#}$-clean.*

Proof. (1) $\Rightarrow$ (2) This is obvious.

(2) $\Rightarrow$ (1) Let $\psi : R[[x]] \to R[[x]]/(x^m), \psi(f) = \overline{f}$. Then it reduces a surjective ring homomorphism $\psi^* : M_2\big(R[[x]]\big) \to M_2\big(R[[x]]/(x^m)\big)$. Hence, we have $B \in M_2\big(R[[x]]\big)$ such that $\psi^*\big(B(x)\big) = A(x)$. According to Theorem 4.3.10, we complete the proof. □

4.4 Uniquely Clean Rings

The aim of this section is to consider a subclass of strongly J-clean rings. An element $a \in R$ is uniquely clean if there exists a unique idempotent $e \in R$ such that $a - e \in U(R)$. A ring R is uniquely clean if every element in R is uniquely clean. This concept was introduced by Anderson and Camillo [20]. We start by

Lemma 4.4.1. *Every uniquely clean ring is abelian.*

Proof. Let R be uniquely clean. Let $e \in R$ be an idempotent and $r \in R$. Choose $a = 1 - \big(e + er(1-e)\big)$. One easily checks that $a = e + \big(1 - 2e - er(1-e)\big) = \big(e+er(1-e)\big) + \big(1-2(e+er(1-e))\big)$. Clearly, $e = e^2 \in R$, $\big(1 - 2e + er(1-e)\big)^{-1} = \big(1-er(1-e)\big)(1-2e)$, $\big(e+er(1-e)\big) = \big(e+er(1-e)\big)^2$ and $\big(1-2(e+er(1-e))\big)^2 = 1$. By the uniqueness, we get $e = e+er(1-e)$, and then $er = ere$. Likewise, $re = ere$. Thus, $er = re$, and so R is abelian. □

Lemma 4.4.2. *Let R be an exchange ring. If $I \nsubseteq J(R)$ is a right ideal of R, there exists $0 \neq e = e^2 \in I$.*

Proof. Suppose $I \nsubseteq J(R)$ is a right ideal containing no nonzero idempotent. Choose $a \in I$, but $a \notin J(R)$. For any $r \in R$, there exists an idempotent $e \in arR$ and $1 - e \in (1 - ar)R$. This implies that $e \in I$; whence, $e = 0$. We infer that $1 - ar \in R$ is right invertible. Thus, $a \in J(R)$, a contradiction. This completes the proof. □

Lemma 4.4.3. *Let R be a uniquely clean ring. Then $2 \in J(R)$.*

Proof. If $2 \notin J(R)$, then $2R := \{2r \mid r \in R\} \nsubseteq J(R)$. By virtue of Lemma 4.4.2, there exists $0 \neq e = e^2 \in 2R$. Write $e = 2s, s \in R$. We may assume that $es = s = se$. It is easy to check that $u := (1 - e) - 2e = \big((1 - e) - s\big)^{-1}$. In addition, $(1 - 2e)^2 = 1$. By the uniqueness, it follows from $(1 - e) + (1 - 2e) = 1 + u$ that $e = 0$, a contradiction. □

Lemma 4.4.4. *A ring R is Boolean if and only if*

(1) $J(R) = 0$;.
(2) *R is uniquely clean.*

Proof. $\Longrightarrow$ This is obvious.

$\Longleftarrow$ Let $a \in R$. Then there exists a unique idempotent $e \in R$ and a unit $u \in R$ such that $a = e + u$. If $u \neq 1$, then $(1-u)R \nsubseteq J(R)$. In view of Lemma 4.4.2, there exists $0 \neq f = f^2 \in (1-u)R$. Write $f = (1-u)r, r \in R$. According to Lemma 4.4.1, every idempotent in R is central. Hence, $f(1-u)f + 1 - f, 1 - f - fuf \in U(R)$. Clearly, $1 - f + \big(f(1-u)f + 1 - f\big) = 1 + \big(1 - f - fuf\big)$. By the uniqueness, we get $1 - f = 1$, and so $f = 0$. This gives a contradiction. Thus, $u = 1$; whence, $a = e + 1$. By virtue of Lemma 4.4.3, $2e = 0$; hence, $a = 1 + e = 1 - e$. Therefore $a \in R$ is an idempotent, as required. □

Lemma 4.4.5. *Let R be a ring. Then the following are equivalent:*

(1) *R is an exchange ring.*
(2) *For any right ideal I of R, every idempotent lifts modulo I.*
(3) *$R/J(R)$ is an exchange ring and every idempotent lifts modulo $J(R)$.*

Proof. (1) $\Rightarrow$ (2) Let I be a right ideal of R such that $e - e^2 \in I$. Then we have an idempotent $f \in eR$ such that $1 - f \in (1-e)R$. Write $f = es$ and $1 - f = (1-e)t$. Then $e - f = e(1-f) - (1-e)f = (e-e^2)t - (e-e^2)s \in I$, as required.

(2) $\Rightarrow$ (1) Let $a \in R$. Choose $I = (a - a^2)R$. If $I = R$, then $a \in R$ is right invertible, and so we have an idempotent $e = 1 \in R$ such that $e \in aR$ and $1 - e \in (1-a)R$. If $I \neq R$, then I is a right ideal of R and that $a - a^2 \in I$. Thus, we have some $e = e^2 \in R$ such that $a - e \in I$. Write $a - e = (a - a^2)x$. Then $e = a - (a - a^2)x \in aR$ and $1 - e = (1 - a)(1 + ax) \in (1 - a)R$, as desired.

(1) $\Rightarrow$ (3) This is obvious.

(3) $\Rightarrow$ (1) Let $a \in R$. Then there exists an idempotent $\overline{e} \in \overline{a}(R/J(R))$ such that $\overline{1-e} \in (\overline{1-a})(R/J(R))$. We may assume $e \in aR$. Choose $f^2 = f \in R$ such that $\overline{f} = \overline{e}$. Then $u := 1 - f + e = 1 + (e - f) \in U(R)$. Set $g = ufu^{-1}$. Then $g^2 = g \in eR \subseteq aR$.

Write $\overline{1-e} = \overline{(1-a)c}$. As $\overline{u} = \overline{1}$, we see that $\overline{g} = \overline{f} = \overline{e}$. Thus, $(1-g) - (1-a)c \in J(R)$, and so $g + (1-a)c \in U(R)$. Write $1 = gt + (l-a)s$. Define $h = g + gt(1-g)$. Then $h^2 = h \in gR \subseteq aR$. Further, $1 - h = (1 - gt)(1 - g) = (1 - a)(1 - g) \in (1 - a)R$. Therefore R is an exchange ring. □

Theorem 4.4.6. *A ring R is uniquely clean if and only if*

(1) *R is an abelian exchange ring;*
(2) *$R/J(R)$ is Boolean.*

Proof. $\Longrightarrow$ By Lemma 4.4.1, R is strongly clean, and so it is an abelian exchange ring. Clearly, $R/J(R)$ is clean. If $\overline{a} = \overline{e+u} = \overline{f+v}$ where $\overline{e}, \overline{f} \in R/J(R)$ are idempotents and $\overline{u}, \overline{v} \in U(R/J(R))$. Since R is an exchange ring, idempotents lift modulo $J(R)$ by Lemma 4.4.5. Thus, we may assume that $e, f \in R$ are idempotents. As every unit lifts modulo $J(R)$, we see that $u, v \in U(R)$. Further, $a = e + (u + r) = f + (v + s)$ for some $r, s \in J(R)$. Thus, $e = f$, and so $\overline{u} = \overline{v}$. Hence, $R/J(R)$ is uniquely clean. As $J\big(R/J(R)\big) = 0$, we see that $R/J(R)$ is Boolean from Lemma 4.4.4.

$\Longleftarrow$ Let $a \in R$. Then $a - a^2 \in J(R)$. In view of Lemma 4.4.5, we have an idempotent $e \in R$ such that $r := a - e \in J(R)$. Hence, $a = (1 - e) + (2e - 1 + r)$. Clearly, $2e - 1 + r \in U(R)$. If $a = f + v$ with $f^2 = f \in R$ and $v \in U(R)$, then $e + r = f + v$, and so $e - f = v - r \in U(R)$. As R is abelian, we see that $(e - f)^3 = e - f$, and so $(e - f)^2 = 1$. Thus, $f = (1 - e)(1 - 2e)^{-1} = 1 - e$. This completes the proof. □

Corollary 4.4.7. *A ring R is uniquely clean if and only if*

(1) *R is abelian;*
(2) *$R/J(R)$ is Boolean and idempotent lifts modulo $J(R)$.*

Proof. One direction is clear. Conversely, $R/J(R)$ is Boolean, and so $R/J(R)$ is an exchange ring. In light of Lemma 4.4.5, R is an exchange ring. So the corollary is true from Theorem 4.4.6. □

Corollary 4.4.8. *A ring R is uniquely clean if and only if*

(1) *R is abelian;*
(2) *For any $a \in R$, there exists a unique idempotent $e \in aR$ such that $1 - e \in (1 - a)R$.*

Proof. Suppose R is uniquely clean. Then R is an abelian exchange ring by Theorem 4.4.6. Thus, for any $a \in R$, there exists an idempotent $e \in aR$ such that $1 - e \in (1 - a)R$. Write $e = ax, 1 - e = (1 - a)y$ with $x \in eRe, y \in (1 - e)(1 - a)(1 - e)$. One easily checks that $(xa)^2 = x(ax)a = xea = xa$ and $xa = (ex)a = (ax)(xa) = a(xa)x = e$. Likewise, $y(1 - a) = 1 - e$. Thus, $a - (1 - e) \in U(R)$. We now assume that $f \in aR$ and $1 - f \in (1 - a)R$ for an idempotent $f \in R$. Similarly, $a - (1 - f) \in U(R)$. Therefore $1 - e = 1 - f$; hence, $e = f$, as desired.

Conversely, assume that (1) and (2) hold. By Theorem 1.1.8, R is strongly clean. Write $a - e, a - f \in U(R)$ where $e, f \in R$ are idempotents.

Analogously to Theorem 1.1.8, we show that $1-e, 1-f \in aR$ and $e, f \in (1-a)R$. By the uniqueness, we get $e = f$, as required. □

Theorem 4.4.9. *Let R be a ring. Then the following are equivalent:*

(1) *R is uniquely clean.*
(2) *For any $a \in R$, there exists a unique idempotent $e \in R$ such that $a - e \in J(R)$.*
(3) *For any $a \in R$, there exists a central idempotent $e \in R$ such that $a - e \in J(R)$.*

Proof. (1) $\Rightarrow$ (3) According to Theorem 4.4.6, $R/J(R)$ is Boolean. Let $a \in R$. Then $a - a^2 \in J(R)$. As R is an exchange ring, every idempotent in R lifts modulo $J(R)$, and so there exists an idempotent $e \in R$ such that $a - e \in J(R)$. By using Theorem 4.4.6 again, $e \in R$ is central.

(3) $\Rightarrow$ (2) Let $a \in R$. Then there exists a central idempotent $e \in R$ such that $a - e \in J(R)$. If there exists an idempotent $f \in R$ such that $a - f \in J(R)$, then $e - f = (a - f) - (a - e) \in J(R)$. As $e \in R$ is central, $(e-f)^3 = e - f$, and so $(e-f)\big(1-(e-f)^2\big) = 0$. This implies that $e = f$, as desired.

(2) $\Rightarrow$ (1) Clearly, $R/J(R)$ is Boolean and idempotent lifts modulo $J(R)$. If $e = e^2 \in R, x \in R$, then $e+(ex-exe), e \in R$ are both idempotents. Since $(ex - exe)^2 = 0$, we see that $ex - exe \in J(R)$. By the uniqueness, we have $e + (ex - exe) = e$; hence, $ex = exe$. Likewise, $xe = exe$. Thus R is abelian. According to Corollary 4.4.7, R is uniquely clean. □

Corollary 4.4.10. *A ring R is uniquely clean if and only if R is strongly J-clean and R is abelian.*

Proof. It is immediate from Corollary 4.4.7 and Theorem 4.4.9. □

Let R be a uniquely clean ring. We claim that $R[[x]]$ is strongly J-clean. In view of Lemma 4.4.1, R is abelian. Let $a \in R$. By virtue of Corollary 4.4.10, there exists an idempotent $e \in R$ such that $a - e \in J(R)$ and $ae = ea$. For any $b \in R$, we choose $c = 0 \in eR(1-e) + (1-e)Re$. Then $[a, c] + [e, b] = 0$. In view of Lemma 4.2.9, $R[[x]]$ is strongly J-clean, and we are through.

Corollary 4.4.11. *A ring R is uniquely clean if and only if*

(1) *R is an abelian exchange ring;*
(2) *$R/P \cong \mathbb{Z}_2$ for all $P \in J\text{-}spec(R)$.*

Proof. Suppose that R is uniquely clean. Then R is an abelian exchange ring from Theorem 4.4.6. Further, R is strongly J-clean by Corollary 4.4.10. In light of Theorem 4.1.7, $R/P \cong \mathbb{Z}_2$ for all $P \in J\text{-}spec(R)$.

Conversely, assume that (1) and (2) hold. Then R is strongly clean by Corollary 1.1.9. In view of Theorem 4.1.7, R is strongly J-clean. Therefore we complete the proof by Theorem 4.4.6. □

Proposition 4.4.12. *A ring R is uniquely clean if and only if*

(1) *$R/J(R)$ is Boolean;*
(2) *$aR + bR = R$ with $a, b \in R \Longrightarrow$ $\exists e \in B(R)$ such that $a + be \in U(R)$.*

Proof. $\Longrightarrow$ In view of Theorem 4.4.6, R is an abelian exchange ring and $R/J(R)$ is Boolean. Given $aR + bR = R$ with $a, b \in R$, then there are some $x, y \in R$ such that $ax + by = 1$. In view of Lemma 4.4.5, we have an idempotent $e \in R$ such that $w := y(1 - a) - e \in J(R)$. Therefore $a + be = \big(1 - ax(1 - a)\big) - bw \in U(R)$, as desired.

$\Longleftarrow$ Given any idempotent $e \in R$, then $(-e)R + R = R$, and so there exists a central idempotent f such that $-e + f \in U(R)$. Write $e = f + u$. Then $f + u = f + fu + uf + u^2$ and $u = 1 - 2f$. Hence, $e = 1 - f$ is central. Thus, R is abelian. For any $a \in R$, $(-a)R + R = R$, and so $a = g + v$, where $g = g^2, v \in U(R)$. Thus, R is strongly clean. In light of Theorem 1.1.8, R is an exchange ring. Therefore R is uniquely clean by Theorem 4.4.6. □

Theorem 4.4.13. *A ring R is uniquely clean if and only if*

(1) *R is an abelian exchange ring;*
(2) *$R/M \cong \mathbb{Z}_2$ for all $M \in Max(R)$.*

Proof. Suppose that R is uniquely clean. Then (1) holds, by Theorem 4.4.6.

For all maximal ideals M of R, we claim that $M \supseteq J(R)$. Otherwise, $J(R) + M = R$. Write $x + y = 1, x \in J(R), y \in M$. Then $y = 1 - x \in U(R)$. Hence, $M = R$, a contradiction. Thus, $R/M \cong R/J(R)/M/J(R)$. In view of Theorem 4.4.6, $R/J(R)$ is Boolean, and then so is R/M. Hence R/M is commutative. As M is maximal, R/M is simple, and so R/M is a field. Therefore $R/M \cong \mathbb{Z}_2$.

Conversely, assume that (1) and (2) hold. By virtue of Corollary 1.1.9, R is strongly clean. Let $a \in R$. Write $a = e_1 + u_1 = e_2 + u_2, e_1 = e_1^2, e_2 = e_2^2$ and $u_1, u_2 \in U(R)$. If $R\big(1 - e_2(1 - e_1)\big)R \neq R$, then there exists a maximal ideal M of R such that $R\big(1 - e_2(1 - e_1)\big)R \subseteq M$. Hence, $\overline{a} = \overline{e_1} + \overline{u_1} = \overline{e_2} + \overline{u_2}$

in R/M. By hypothesis, $R/M \cong \mathbb{Z}_2$, whence, $\overline{u_1} = \overline{u_2} = \overline{1}$. This implies that $\overline{e_1} = \overline{e_2}$, and so $e_1 - e_2 \in M$. We infer that $e_2(1-e_1) = (e_1 - e_2)(e_1 - 1) \in M$, and so $1 = e_2(1-e_1) + \big(1 - e_2(1-e_1)\big) \in M$, a contradiction. This implies that $R\big(1 - e_2(1-e_1)\big)R = R$. As every idempotent in R is central, $e_2(1-e_1) \in R$ is an idempotent, thus, $e_2(1-e_1) = 0$. Hence $e_2 = e_2e_1$. Likewise, $e_1 = e_1e_2$. Hence, $e_1 = e_2$, and so $u_1 = u_2$. Therefore R is a uniquely clean ring. □

Let R be an abelian regular ring. Then R is uniquely clean if and only if for all $M \in Max(R)$, $R/M \cong \mathbb{Z}_2$.

Corollary 4.4.14. *Let R be a ring. Then the following are equivalent:*

(1) *R is a uniquely clean local ring.*
(2) *$R/J(R) \cong \mathbb{Z}_2$.*

Proof. (1) $\Rightarrow$ (2) This is clear.

(2) $\Rightarrow$ (1) Let M be a maximal ideal of R. Then $J(R) \subseteq M$. Hence, $R/M \cong R/J(R)/M/J(R)$, and so $R/M \cong \mathbb{Z}_2$, the result follows from Theorem 4.4.13. □

Corollary 4.4.15. *Every homomorphic image of a uniquely clean ring is uniquely clean.*

Proof. Let R be uniquely clean, and let I be an ideal of R. In light of Theorem 4.4.6, R is an abelian exchange ring. By virtue of Lemma 4.4.5, R/I is also an abelian exchange ring. Let J be a maximal ideal of R/I. Set $M = \{x \in R \mid \overline{x} \in J\}$. Then M is an ideal of R. Since $R/M \cong R/I/M/I \cong R/I/J$, M is a maximal of R. In view of Theorem 4.4.13, $R/M \cong \mathbb{Z}_2$. Therefore $R/I/J \cong \mathbb{Z}_2$, the result follows. □

A prime ideal P of a ring R is called a minimal prime ideal if it does not properly contain any other prime ideal of R.

Corollary 4.4.16. *A ring R is uniquely clean if and only if*

(1) *R is an abelian exchange ring;*
(2) *For all minimal prime ideals P of R, R/P is uniquely clean.*

Proof. Let R be uniquely clean. Then R is an abelian exchange ring by Theorem 4.4.6. In view of Corollary 4.4.15, R/P is uniquely clean for all minimal prime ideals P of R.

Conversely, assume that (1) and (2) hold. Let M be a maximal ideal of R. Then M is prime. By Zorn's Lemma, there exists a minimal prime ideal $P \subseteq M$. Thus, $R/M \cong R/P/M/P$ is uniquely clean. Clearly, 0 is the maximal ideal of R/M. It follows from Theorem 4.4.13 that $R/M \cong \mathbb{Z}_2$. By using Theorem 4.4.13 again, R is uniquely clean. □

Lemma 4.4.17. *Let R be an exchange ring. If $a^2 = 0$ and $a \notin J(R)$, then the principal ideal (a) contains a system of nonzero matrix units $\{e_{11}, e_{12}, e_{21}, e_{22}\}$.*

Proof. As $a \notin J(R)$, we can find some $r \in R$ such that $1 - ar \in R$ is not right invertible. Since R is an exchange ring, there exists an idempotent $e_{11} \in arR$ such that $1 - e_{11} \in (1 - ar)R$. If $e_{11} = 0$, then $1 - ar \in R$ is right invertible, a contradiction. Thus, $0 \neq e_{11} = ab$ with $b = be_{11}$.

Set $e_{12} = e_{11}a, e_{21} = (1 - e_{11})b$ and $e_{22} = (1 - e_{11})ba$. As $ae_{11} = 0$ and $be_{11} = b$, we directly check that

$$e_{ij}e_{kl} = 0 \ (j \neq k) \text{ and } e_{ij}e_{kl} = e_{il} \ (j = k)$$

for $1 \leq i, j, k, l \leq 2$.

If $e_{12} = 0$, then $e_{11} = e_{11}ab = e_{12}b = 0$, a contradiction. Hence, $e_{12} \neq 0$. If $e_{21} = 0$, then $b = e_{11}b$, and so $e_{11} = ae_{11}b = a^2b^2 = 0$, a contradiction. Hence, $e_{12} \neq 0$. If $e_{22} = 0$, then $ba = e_{11}ba$, and so $e_{11} = abab = a(e_{11}ba)b = 0$, a contradiction. Thus, $e_{22} \neq 0$. Accordingly, the 4 elements $\{e_{ij}\}$ constitute a system of nonzero matrix units lying in (a), as asserted. □

For any ring R, the following statements are equivalent: (1) for any $a \in R$, there exists a unique idempotent $e \in aR$ such that $1 - e \in (1 - a)R$; (2) for any $a \in R$, there exists a unique idempotent $e \in aR$ such that $e - a \in (a - a^2)R$ (see [557, Proposition 2.1.1]). We now record a result of [557, Corollary 2.2.3].

Proposition 4.4.18. *Let R be a ring, and let $S = T_2(R)$. Then the following are equivalent:*

(1) *R is Boolean.*
(2) *For any $a \in S$, there exists a unique idempotent $e \in aS$ such that $1 - e \in (1 - a)S$.*

Proof. (1) $\Rightarrow$ (2) For any $a = \begin{pmatrix} x & z \\ 0 & y \end{pmatrix} \in S$, we have an idempotent $e = \begin{pmatrix} x & z(x+y) \\ 0 & y \end{pmatrix}$ such that $a - e \in (a - a^2)S$. If there exists an idempotent

$f = \begin{pmatrix} \alpha & v \\ 0 & \beta \end{pmatrix} \in S$ such that $a - f \in (a - a^2)S$, then $\alpha = x, \beta = y$ and $v(\alpha + \beta - 1) = 0$. Further, $z - v = z(1 - x - y)r$ for some $r \in R$. Hence, $z - v = z(1 - \alpha - \beta)r$, and then $v = v(\alpha + \beta) = z((\alpha + \beta)) = z(x + y)$. Thus, $f = \begin{pmatrix} x & z(x+y) \\ 0 & y \end{pmatrix}$, as required.

(2) $\Rightarrow$ (1) Since S is an exchange ring, so is R. Let $0 \neq x \in J(R)$. Then we have

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \begin{pmatrix} x & 0 \\ 0 & 1-x \end{pmatrix} S, I_2 - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \left(I_2 - \begin{pmatrix} x & 0 \\ 0 & 1-x \end{pmatrix} \right) S;$$

$$\begin{pmatrix} 0 & x - x^2 \\ 0 & 1 \end{pmatrix} \in \begin{pmatrix} x & 0 \\ 0 & 1-x \end{pmatrix} S, I_2 - \begin{pmatrix} 0 & x - x^2 \\ 0 & 1 \end{pmatrix} \in \left(I_2 - \begin{pmatrix} x & 0 \\ 0 & 1-x \end{pmatrix} \right) S.$$

By the uniqueness, we get $x - x^2 = 0$, and so $x = 0$. This gives a contradiction; hence, $J(R) = 0$.

If $0 \neq x \in R$ with $x^2 = 0$, it follows by Lemma 4.4.17 that there exists a system of nonzero matrix units $\{e_{11}, e_{12}, e_{21}, e_{22}\}$. Set $e = e_{11} + e_{22}$. Then $0 \neq e = e^2 \in R$. Further, $e = u + v$, where $u = e_{11} + e_{12} + e_{21}, v = e_{22} - e_{12} - e_{21} \in U(R)$.

Considering $\begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} \in S$, by hypothesis, there exists a unique idempotent $g \in R$ such that $g - u \in (u - u^2)R$. Clearly, $1 - u \in (u - u^2)R$. Furthermore, $v(1-e)v^{-1} - u = (v - ve - uv)v^{-1} = (v - ve - ev + v^2)v^{-1} = -(u-u^2)v^{-1} \in (u-u^2)R$. Hence, $1 = v(1-e)v^{-1}$, and so $e = 0$. This gives a contradiction. Therefore R is reduced, and so it is abelian. According to Corollary 4.4.8, R is uniquely clean. By Lemma 4.4.4, the result follows. □

Let $S(R)$ be the nonempty set of all ideals of a ring R generated by central idempotents. By Zorn's Lemma, $S(R)$ contains maximal elements. We say that R/P is a Pierce stalk if P is a maximal element of the set $S(R)$, and call P a Pierce ideal of R. Let $Pier(R)$ be the set of all Pierce ideals of R. We have

Lemma 4.4.19. *Let R be a ring. Then $\bigcap\{P \mid P \in Pier(R)\} = 0$.*

Proof. Suppose that $0 \neq m \in \bigcap\{P \mid P \in Pier(R)\}$. Set $\Omega = \{I \lhd R \mid I \in S(R) \text{ and } m \notin I\}$. Then $\Omega \neq \emptyset$ as $0 \in \Omega$. Given $I_1 \subseteq I_2 \subseteq \cdots$ in Ω, we set $J = \bigcup_{i=1}^{\infty} I_i$. Then $J \in \Omega$. By using Zorn's Lemma, there exists an ideal

Q of R which is maximal in Ω. If Q is not a Pierce ideal, then there exists $e \in B(R)$ such that $e \notin Q$ and $Q + eR$ is an ideal of R. If $1 - e \in Q$, then $1 = e + (1 - e) \in Q + eR$, a contradiction. If $Q + (1 - e)R = R$, then $e \in eQ \subseteq Q$, a contradiction. Thus, $1 - e \notin Q$ and $Q + (1 - e)R$ is an ideal of R. Hence, we derive $m \in \big(Q + eR\big) \cap \big(Q + (1 - e)R\big)$ from the maximality of Q. Write $m = a + er = b + (1 - e)s$ with $a, b \in Q, r, s \in R$. Then $m = em + (1 - e)m = eb + (1 - e)a \in Q$, a contradiction. We have Q is a Pierce ideal. This implies that $m \in Q$, a contradiction. So the lemma is true. □

Lemma 4.4.20. *A ring R is a local ring if and only if it is an exchange ring with trivial idempotents.*

Proof. One direction is obvious. Conversely, assume that R is an exchange ring with trivial idempotents. For any $a \in R$, we can find an idempotent $e \in R$ such that $e \in aR$ and $1 - e \in (1 - a)R$. By hypothesis, $e = 0$ or 1. Thus, $a \in R$ is right invertible or $1 - a \in R$ is right invertible. If $ab = 1$, then $ba \in R$ is an idempotent; hence, $ba = 0$ or 1. If $ba = 0$, then $0 = a(ba)b = (ab)^2 = 1$, a contradiction. Thus, $ba = 1$. If $(1 - a)b = 1$, similarly, $b(1 - a) = 1$. Therefore either a or $1 - a$ is invertible, and so R is a local ring. □

Proposition 4.4.21. *Every uniquely clean ring is the subdirect product of local rings R_i, where each $R_i/J(R_i) \cong \mathbb{Z}_2$.*

Proof. Let R be a uniquely clean ring, and let $\varphi_P : R \to R/P$ be the natural epimorphism. In view of Lemma 4.4.19, $\bigcap_{P \in Pier(R)} Ker\varphi_P = \bigcap_{P \in Pier(R)} P = 0$. Hence, R is the subdirect product of all R/P, where $P \in Pier(R)$. For any $P \in Pier(R)$, it follows from Corollary 4.4.15 and Theorem 4.4.6 that R/P is an abelian exchange ring. We infer that R/P has no non-trivial idempotents; whence, R/P is local by Lemma 4.4.20. In light of Corollary 4.4.14, $R/P/J(R/P) \cong \mathbb{Z}_2$, as needed. □

It follows from Proposition 4.4.21 that every uniquely clean ring is the subring of the direct product of local rings R_i, where $R_i/J(R_i) \cong \mathbb{Z}_2$.

Let R be a ring with an identity and S be a ring (not necessary unitary), and let S be a R-R-bimodule in which $(s_1s_2)r = s_1(s_2r), r(s_1s_2) = (rs_1)s_2$ and $(s_1r)s_2 = s_1(rs_2)$ for all $s_1, s_2 \in S, r \in R$. Construct

$I(R;S) = \{(r,s) \mid r \in R, s \in S\}$. Define

$$(r_1,s_1)+(r_2,s_2) = (r_1+r_2, s_1+s_2);$$
$$(r_1,s_1)(r_2,s_2) = (r_1r_2, s_1s_2+r_1s_2+s_1r_2).$$

Then $I(R;S)$ is a ring with identity $(1,0)$. We say that $I(R;S)$ is an ideal-extension of R by S. A ring S is idempotent-free provide that S contains no nonzero idempotents. We derive

Proposition 4.4.22. *The ideal-extension $I(R;S)$ is uniquely clean and S is idempotent-free if and only if*

(1) *R is uniquely clean;*
(2) *If $e = e^2 \in R$, then $es = se$ for all $s \in S$;*
(3) *If $s \in S$, then there exists some $s' \in S$ such that $ss' = s's$ and $s+s'+ss' = 0$.*

Proof. $\Longleftarrow$ Let $g = g^2 \in S$. Then there exists some $g' \in S$ such that $gg' = g'g$ and $(-g)+g'+(-g)g' = 0$. Hence, $(1-g)(1+g') = 1$, and so $g = 0$. Thus, S is idempotent-free.

Let $(a,s) \in I(R;S)$. By (1), $a \in R$ is uniquely clean. Write $a = e+u, e = e^2 \in R, u \in U(R)$. Then $(a,s) = (e,0)+(u,s)$. Clearly, $(e,0)^2 = (e,0)$. By (3), there exists some $t \in S$ such that $u^{-1}st = tu^{-1}s, t+u^{-1}s+u^{-1}st = 0$. Hence, $(u,s)^{-1} = (u^{-1}, tu^{-1})$. Write $(a,s) = (f,y)+(v,w), (f,y)^2 = (f,y)$ and $(v,w) \in U\big(I(R;S)\big)$. Then $f = f^2 \in R, y = 0$ and $v \in U(R)$. Further, $a = f+v$ and $s = w$. This implies that $f = e, v = u$, and so $(f,y) = (e,0), (v,w) = (u,s)$. Therefore $(a,s) \in I(R;S)$ is uniquely clean, as required.

$\Longrightarrow$ It is directly verified that R is uniquely clean. Let $e = e^2 \in R$ and $s \in S$. In view of Theorem 4.4.6, $(e,0) = (e,0)^2 \in I(R;S)$ is central. Hence, $(e,0)(0,s) = (0,s)(e,0)$, and so $es = se$. For any $s \in S$, $(1,s) \in I(R;S)$ is uniquely clean. Write $(1,s) = (f,y)+(u,v)$ where $(f,y) \in I(R;S)$ is an idempotent and $(u,v) \in I(R;S)$ is a unit. From $1 = f+u = 0+1$, we deduce that $f = 0$. It follows from $(0,y)^2 = (0,y)$ that $y^2 = y$, and so $y = 0$. This implies that $s = y+v = v$; hence, $(1,s) \in I(R;S)$ is a unit. Write $(1,s)^{-1} = (1,s')$ for a $s' \in S$. Then we conclude that $ss' = s's$ and $s+s'+ss' = 0$, as asserted. □

Let $n \geq 2$. Since $T_n(R)$ contains non-central idempotents, it is always not a uniquely clean ring. Also $M_n(R)$ of all $n \times n$ matrices over a ring R is not uniquely clean. But we can derive the following.

Example 4.4.23. Let R be uniquely clean. Then the ring $T = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn}\}$ is uniquely clean.

Proof. Let $S = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn} = 0\}$. Then $T \cong I(R; S)$. One easily checks that the conditions in Proposition 4.4.22 are satisfied, hence the result. □

Let R be a ring and α be a ring endomorphism. We denote $R[[x, \alpha]]$ the skew power series ring whose elements are the power series over R, the addition is defined as usual and the multiplication subject to the relation $xr = \alpha(r)x$ for any $r \in R$.

Example 4.4.24. Let R be a ring, and let $\alpha : R \to R$ be a ring endomorphism. Then $R[[x, \alpha]]$ is uniquely clean if and only if R is uniquely clean and $\alpha(e) = e$ for all $e = e^2 \in R$.

Proof. Let $S = R[[x, \alpha]] \cdot x$. Then $R[[x, \alpha]] \cong I\big(R, S)$. Clearly, the condition (3) holds. The condition (2) is satisfied if and only if $\alpha(e) = e$ for all $e = e^2 \in R$. In addition, S is idempotent-free. Thus, the result follows from Proposition 4.4.22. □

Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \;\middle|\; a, b \in \mathbb{Z}_4 \right\}$, and let $\alpha : R \to R$, $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mapsto \begin{pmatrix} a & 2b \\ 0 & a \end{pmatrix}$. Then $R[[x, \alpha]]$ is uniquely clean. This can be seen from Proposition 4.4.44, Example 4.4.23 and Example 4.4.24.

Proposition 4.4.25. *Let R be a ring. Then the following are equivalent:*

(1) *R is uniquely clean.*
(2) *$R[[x]]/(x^2)$ is uniquely clean.*
(3) *$R[[x]]/(x^n)$ is uniquely clean for some $n \geq 2$.*
(4) *$R[[x]]/(x^n)$ is uniquely clean for all $n \geq 2$.*

Proof. (1) ⇒ (2) Choose $\alpha : R \to R$ by $\alpha(a) = a$. Then $R[[x]]$ is uniquely clean by Example 4.4.24, and then so is $R[[x]]/(x^2)$.

(2) ⇒ (4) Clearly, $R[[x]]/(x^2) \cong R[[x]]/(x^n)/(x^2)/(x^n)(n \geq 2)$. The claim is therefore established by Corollary 4.4.15.

(4) ⇒ (3) is trivial.

(3) ⇒ (1) Let $S = R[[x]]/(x^n)(n \geq 2)$. Then $S = \left\{ \sum_{i=0}^{n-1} a_i x^i \;\middle|\; x^n = 0, a_0, \ldots, a_{n-1} \in R \right\}$. Construct a map $\varphi : S \to R$ given by $\varphi\left(\sum_{i=0}^{n-1} a_i x^i \right) = a_0$.

Then φ is a ring epimorphism, and so $R \cong S/Ker\varphi$. According to Corollary 4.4.15, R is uniquely clean. □

Corollary 4.4.26. *Let R be a ring. Then the following are equivalent:*

(1) *R is uniquely clean.*
(2) *$R[x]/(x^2)$ is uniquely clean.*

Proof. One easily checks that $R[x]/(x^2) \cong R[[x]]/(x^2)$, and so the corollary is true by Proposition 4.4.25. □

Lemma 4.4.27. *Let R be uniquely clean, and let $e = e^2 \in R$. Then eRe is uniquely clean.*

Proof. For any $eae \in eRe$, then $eae + 1 - e \in R$ is uniquely clean. Thus, $eae + 1 - e = f + u, f = f^2 \in R, u \in U(R)$, and so $eae = efe + eue$ is clean in eRe. Write $eae = g + v, g = g^2 \in eRe, v \in U(eRe)$. Then $eae + 1 - e = g + (v + 1 - e)$, where $g = g^2 \in R, v + 1 - e \in U(R)$. Therefore $g = f = efe$, as desired. □

We say that B is a subring of a ring A provided that B is a non-empty subset of A such that for any $x, y \in B$, $x - y, xy \in B$ and that $1_A \in B$. Let B be a subring of A. Set $R[A, B] = \{(a_1, a_2, \ldots, a_n, b, b, \ldots) \mid a_i \in A, b \in B, n \in \mathbb{N}\}$. Then $R[A, B]$ is a ring under the general addition and multiplication.

Proposition 4.4.28. *Let B be a subring of a ring A. Then $R[A, B]$ is uniquely clean if and only if*

(1) *A is uniquely clean;*
(2) *For any $b \in B$, there exists a central idempotent $e \in A$ such that $b - e \in J(A) \cap J(B)$.*

Proof. $\Longrightarrow$ By Lemma 4.4.27, A is uniquely clean. For any $b \in B$, we see that $(0, b, b, \ldots) \in R[A, B]$. By virtue of Theorem 4.4.9, there exists a central idempotent $(f_1, \ldots, f_s, f, f, \ldots) \in R[A, B]$ and an element $(v_1, \ldots, v_t, v, v, \ldots) \in J\big(R[A, B]\big)$ such that

$$(0, b, b, \ldots) = (f_1, \ldots, f_s, f, f, \ldots) + (v_1, \ldots, v_t, v, v, \ldots).$$

This implies that $b = f + v$ where $f \in B$ is an idempotent. Further, $f \in A$ is central. One easily checks that $J\big(R[A, B]\big) = \big[J(A), J(A) \cap J(B)\big]$. Hence, $v \in J(A) \cap J(B)$, as required.

$\Longleftarrow$ Let $(a_1,\ldots,a_m,b,b,\ldots)\in R[A,B]$. Then we have strongly J-clean expressions $a_1=e_1+u_1,\ldots,a_m=e_m+u_m$, and that $e_1,\ldots,e_m\in A$ are central idempotent. Moreover, we have a central idempotent $e\in A$ and $u\in J(A)\cap J(B)$ such that $b=e+u$. This yields that $e=b-u\in B$. Thus $(a_1,\ldots,a_m,b,b,\ldots)=(e_1,\ldots,e_m,e,e,\ldots)+(u_1,\ldots,u_m,u,u,\ldots)$. One easily checks that $(e_1,\ldots,e_m,e,e,\ldots)\in R[A,B]$ is a central idempotent. In addition, $(u_1,\ldots,u_m,u,u,\ldots)\in R\big[J(A),J(A)\cap J(B)\big]=J\big(R[A,B]\big)$. Therefore $R[A,B]$ is uniquely clean by Theorem 4.4.9. □

Example 4.4.29. Let $A=\left\{\begin{pmatrix}a&b\\0&a\end{pmatrix}\;\middle|\;a,b\in\mathbb{Z}_4\right\}$, $B=\left\{\begin{pmatrix}a&2b\\0&a\end{pmatrix}\;\middle|\;a,b\in\mathbb{Z}_4\right\}$. Then $R[A,B]$ is uniquely clean.

Proof. Clearly, $\mathbb{Z}_4$ is a local ring with $J\big(\mathbb{Z}_4\big)=2\mathbb{Z}_4$. In view of Corollary 4.4.14, $\mathbb{Z}_4$ is uniquely clean, and then so is A by Example 4.4.23. Let $\begin{pmatrix}a&2b\\0&a\end{pmatrix}\in B$. If $a\in 2\mathbb{Z}_4$, then $\begin{pmatrix}a&2b\\0&a\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}+\begin{pmatrix}a&2b\\0&a\end{pmatrix}$ with $\begin{pmatrix}a&2b\\0&a\end{pmatrix}\in J(A)\cap J(B)$. If $a\notin 2\mathbb{Z}_4$, then $\begin{pmatrix}a&2b\\0&a\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}a-1&2b\\0&a-1\end{pmatrix}$ with $\begin{pmatrix}a-1&2b\\0&a-1\end{pmatrix}\in J(A)\cap J(B)$. According to Proposition 4.4.28, $R[A,B]$ is uniquely clean. □

Let R be a ring, and let $R[i]=\{a+bi\mid a,b\in R,i^2=-1\}$. Then $R[i]$ is also a ring.

Lemma 4.4.30. *Let R be a ring with $2\in J(R)$. Then $U\big(R[i]\big)=\{a+bi\mid a+b\in U(R)\}$.*

Proof. Write $(a+bi)(c+di)=1$. Then $ac-bd=1$ and $ad+bc=0$. Hence, $(a+b)(c+d)=ac+bd+ad+bc=(ac-bd)+2bd=1+2bd\in U(R)$. This implies that $a+b\in R$ is right invertible. Likewise, $a+b\in R$ is left invertible. Thus, $U\big(R[i]\big)\subseteq\{a+bi\mid a+b\in U(R)\}$.

Assume that $a+b\in U(R)$. Then $a-b=(a+b)-2b\in U(R)$. Clearly, $a(a-b)^{-1}a=(a-b+b)(a-b)^{-1}(a-b+b)=\big(1+b(a-b)^{-1}\big)(a-b+b)=a+b+b(a-b)^{-1}b$. Hence $a(a-b)^{-1}a(a+b)^{-1}-b(a-b)^{-1}b(a+b)^{-1}=1$. Likewise, $b(a-b)^{-1}a-a(a-b)^{-1}b=0$. It is easy to check that $(a+bi)(a-b)^{-1}(a-bi)(a+b)^{-1}=\big(a(a-b)^{-1}a(a+b)^{-1}+b(a-b)^{-1}b(a+b)^{-1}\big)+\big(b(a-b)^{-1}a-a(a-b)^{-1}b\big)i=1+2b(a-b)^{-1}b(a+b)^{-1}\in U(R)$. Hence, $a+bi\in R[i]$

is right invertible. Similarly, $a-b)^{-1}(a-bi)(a+b)^{-1}(a+bi) \in U(R)$. Therefore $a+bi \in U\big(R[i]\big)$, as required. □

Theorem 4.4.31. *Let R be a ring. Then $R[i]$ is uniquely clean if and only if so is R.*

Proof. Suppose that $R[i]$ is uniquely clean. Then $2 \in J(R)$, as $\overline{2}^2 = \overline{2}$ in $R/J(R)$. Let $a \in R$. Then we can find $e+fi \in B(R[i])$ and $u+vi \in J\big(R[i]\big)$ such that $a = (e+fi)+(u+vi)$. Thus, $a = e+u$ and $e \in C(R)$. As $e+fi \in R[i]$ is an idempotent, we get $e^2 - f^2 = e$ and $ef+fe = f$. This implies that $(2e-1)f = 0$, and so $f = 0$. Hence, $e \in B(R)$. It is easy to verify that $u \in J(R)$, and therefore R is uniquely clean by Theorem 4.4.9.

Conversely, assume that R is uniquely clean. Then R is an abelian exchange ring and $2 \in J(R)$. Let $a+bi \in R[i]$. By hypothesis, there exist $e, f \in B(R)$ and $u, v \in J(R)$ such that $a = e+u$ and $b = f+v$. Thus, $a+bi = (e+f)+\big(u-f+(f+v)i\big)$. Clearly, $(e+f)^2-(e+f) = 2ef \in J(R)$. As every idempotent lifts modulo $J(R)$, we can find an idempotent $g \in R$ such that $e+f = g+r$ where $r \in J(R)$. Thus, $a+bi = g+\big(r+u-f+(f+v)i\big)$.

Write $x = r+u-f$ and $y = f+v$. Then $x+y = r+u+v \in J(R)$. For any $c+di \in R[i]$, we see that $1-(x+yi)(c+di) = (1-xc+yd)-(xd+yc)i$. As $(1-xc+yd)-(xd+yc) = (1-xc-yd)-(xd+yc)+2yd = 1-(x+y)(c+d)+2yd \in U(R)$. In view of Lemma 4.4.30, $1-(x+yi)(c+di) \in U\big(R[i]\big)$, and so $x+yi \in J\big(R[i]\big)$. Therefore $R[i]$ is uniquely clean by Theorem 4.4.9. □

If $i^2 = -1$, then for any integers m, $\mathbb{Z}_m[i] = \{a+bi \mid a, b \in \mathbb{Z}_m\}$ is called the ring of Gaussian integers modulo m.

Example 4.4.32. $\mathbb{Z}_4[i]$ is uniquely clean.

Proof. Clearly, $\mathbb{Z}_4$ is a commutative local ring and $\mathbb{Z}_4/J\big(\mathbb{Z}_4\big) \cong \mathbb{Z}_2$. By virtue of Corollary 4.4.14, $\mathbb{Z}_4$ is uniquely clean, and thus completing the proof by Theorem 4.4.31. □

Lemma 4.4.33. *Let R be abelian, let G be a group and let ω be the augmentation of RG. If $Ker(\omega) \subseteq J(RG)$, then every idempotent of RG is in R.*

Proof. Let $f \in RG$ be an idempotent. Then $e := \omega(f) \in R$ is an idempotent. Set $c = f-\omega(f)$. Then $\omega(c) = 0$, and so $c \in Ker(\omega)$. Furthermore, $f = e+c$; whence, $(e+c)^2 = e+ec+ce+c^2 = e+c$. Since R is abelian, we

get $c(1-c-2e)=0$. Clearly, $(1-2e)^2=1$. By hypothesis, $c\in J(RG)$, and so $1-c-2e\in U(RG)$. It follows that $c=0$, and therefore $f=\omega(f)\in R$. □

Proposition 4.4.34. *Let R be a ring, and let G be a group. If RG is uniquely clean, then R is uniquely clean and G is a 2-group.*

Proof. Clearly, $R\cong RG/Ker(\omega)$, where ω be the augmentation of RG. In view of Corollary 4.4.15, R is uniquely clean. It follows from Theorem 4.4.13 that there exists an epimorphism $\varphi: R\to\mathbb{Z}_2$, and so $\mathbb{Z}_2G\cong RG/Ker(\psi)$, where $\psi: RG\to\mathbb{Z}_2G$, $\sum\limits_{g\in G} r_g g\mapsto \sum\limits_{g\in G}\varphi(r_g)g$. Hence, $\mathbb{Z}_2G$ is uniquely clean, by using Corollary 4.4.15 again. It follows by Theorem 4.4.6 that $\mathbb{Z}_2G/J\big(\mathbb{Z}_2G\big)$ is Boolean. Let $\omega':\mathbb{Z}_2G\to\mathbb{Z}_2$ be the augmentation of $\mathbb{Z}_2G$. Then $Ker(\omega')=\Big\{\sum\limits_{g\in G} r_g(g-1)\ \Big|\ r_g\in\mathbb{Z}_2\Big\}$. As $(1-g)^2-(1-g)\in J(\mathbb{Z}_2G)$, we get $g^2-g\in J(\mathbb{Z}_2G)$, and so $g-1\in J(\mathbb{Z}_2G)$. We infer that $Ker(\omega')\subseteq J(\mathbb{Z}_2G)$.

Let $1\neq g\in G$. Assume that $o(g)=\infty$. Then $(1-g)+(1-g^2)=2-g-g^2\in ker(\omega')$, and so $-1+g+g^2\in U(\mathbb{Z}_2G)$. Set $x=-1+g+g^2$. Write $x=\sum\limits_{k\in G}x_kk$ and $y=\sum\limits_{k\in G}y_kk$ with $xy=1$. Then

$$\sum_{l\in G}\left(\sum_{k\in G}x_ky_{k^{-1}l}\right)l=1.$$

Thus,

$$\sum_{k\in G}x_ky_{k^{-1}}=1 \text{ and } \sum_{k\in G}x_ky_{k^{-1}l}=0 \text{ for all } l\neq 1.$$

Hence, $\big(-1+g+g^2\big)\big(y_1+y_gg+y_{g^2}g^2\big)=\Big(\sum\limits_{k\in G}x_ky_{k^{-1}}\Big)+\Big(\sum\limits_{k\in G}x_ky_{k^{-1}g}\Big)+\Big(\sum\limits_{k\in G}x_ky_{k^{-1}g^2}\Big)g^2+\Big(\sum\limits_{k\in G}x_ky_{k^{-1}g^3}\Big)g^3+\Big(\sum\limits_{k\in G}x_ky_{k^{-1}g^4}\Big)g^4=1$. Thus,

$$-y_1+(-y_g+y_1)g+(-y_{g^2}+y_g+y_1)g^2+(y_{g^2}+y_g)g^3+y_{g^2}g^4=-1.$$

Hence, $y_1=-1, y_g=-1, y_{g^2}=y_g+y_1=-2$. But $y_{g^2}+y_g=-3\neq 0$ in $\mathbb{Z}_2$, contradicting the equality.

Therefore we conclude that $o(g)$ is finite. Write $o(g)=2^km$ where m is an odd number greater than 3. Choose $h=g^{2^k}$. Then $o(h)=m$. Choose $e=\sum\limits_{i=0}^{m-1}h^i$. Then $e\neq 0,1$. Observing that $(1-h)e=(1-h)\big(1+h+\cdots+h^{m-1}\big)=1-h^m=0$, we get $he=e$, and therefore $e^2=\big(1+h+\cdots+h^{m-1}\big)e=e$. This means that $0\neq 1-e\in\mathbb{Z}_2G$ is an

idempotent. But $\omega'(1-e)=0$, and so $1-e\in Ker(\omega')$. This implies that $1-e\in J(\mathbb{Z}_2G)$; hence, $1-e=0$, a contradiction. Therefore $o(g)=2^k$, i.e., G is a 2-group. □

A group G is called locally finite if every finitely generated subgroup of G is finite.

Lemma 4.4.35. *Let R be a ring, and let G be a locally finite group. Then $J(R)G\subseteq J(RG)$.*

Proof. We may assume that G is finite. Write $G=\{1=g_1,g_2,\dots,g_n\}$. Let $\alpha\in J(R)G$. Then $1-\alpha=\sum\limits_{i=1}^{n} r_{g_i}g_i$, where $r_{g_1}\in 1+J(R), r_{g_2},\dots,r_{g_n}\in J(R)$. Let $x=\sum\limits_{i=1}^{n} x_{g_i}g_i\in RG$. Then $(1-\alpha)x=1$ if and only if

$$\sum_{i=1}^{n}\left(\sum_{g_kg_l=g_i} r_{g_k}x_{g_l}\right)g_i=1,$$

i.e.,

$$\begin{aligned}
r_{g_1g_1^{-1}}x_{g_1}+r_{g_1g_2^{-1}}x_{g_2}+\cdots+r_{g_1g_n^{-1}}x_{g_n}&=1,\\
r_{g_2g_1^{-1}}x_{g_1}+r_{g_2g_2^{-1}}x_{g_2}+\cdots+r_{g_2g_n^{-1}}x_{g_n}&=1,\\
&\vdots\\
r_{g_ng_1^{-1}}x_{g_1}+r_{g_ng_2^{-1}}x_{g_2}+\cdots+r_{g_ng_n^{-1}}x_{g_n}&=1.
\end{aligned}$$

As $r_{g_ig_i^{-1}}=r_{g_1}\in 1+J(R)$, we see that

$$\begin{pmatrix}
r_{g_1g_1^{-1}} & r_{g_1g_2^{-1}} & \cdots & r_{g_1g_n^{-1}}\\
r_{g_2g_1^{-1}} & r_{g_2g_2^{-1}} & \cdots & r_{g_2g_n^{-1}}\\
\vdots & \vdots & \ddots & \vdots\\
r_{g_ng_1^{-1}} & r_{g_ng_2^{-1}} & \cdots & r_{g_ng_n^{-1}}
\end{pmatrix}\in I_n+M_n\big(J(R)\big)\subseteq GL_n(R).$$

Thus, $1-\alpha\in RG$ is right invertible. Likewise, it is left invertible. Accordingly, $1-\alpha\in RG$ is invertible. As $J(R)G$ is an ideal of RG, we deduce that $\alpha\in J(RG)$, as required. □

Theorem 4.4.36. *Let R be a ring, and let G be a locally finite group. Then RG is uniquely clean if and only if R is uniquely clean and G is a 2-group.*

Proof. One direction is clear by Proposition 4.4.34.

Conversely, assume that R is uniquely clean and G is a 2-group. Let $a\in RG$. Since R is uniquely clean, we can find a unique idempotent $e\in R$

such that $\omega(a) - e \in J(R)$. Hence, $\omega(a-e) = \omega(a) - e \in J(R)$. Write $a - e = \sum_{g\in G} r_g g$. Then $\omega(a-e) = \sum_{g\in G} r_g \in J(R)$.

Let $1 \neq g \in G$. Write $o(g) = 2^m$. For any $y \in RG$, we see that $\big((g-1)y\big)^{2^m} = (g-1)^{2^m} y^{2^m} \in J(R)G$, as $2 \in J(R)$. By virtue of Lemma 4.4.35, $\big((g-1)y\big)^{2^m} \in J(RG)$; hence, $1-(g-1)y \in U(RG)$. This shows that $g - 1 \in J(RG)$. Thus, $Ker(\omega) = \Big\{ \sum_{g\in G} r_g(g-1) \;\Big|\; r_g \in R \Big\} \subseteq J(RG)$, and so $\sum_{g\in G} r_g(g-1) \in J(RG)$. Therefore, $a - e = \sum_{g\in G} r_g(g-1) + \omega(a-e) \in J(RG)$. In light of Lemma 4.4.1, R is abelian. Therefore we complete the proof by Theorem 4.4.9. □

A group G is called solvable if it has a subnormal series whose factor groups are all abelian, that is, if there are subgroups $e = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G$ such that G_{i-1} is normal in G_i, and G_i/G_{i-1} is an abelian group for all i. For instance, all abelian groups and all nilpotent groups are solvable.

Corollary 4.4.37. *Let R be a ring, and let G be a solvable group. Then RG is uniquely clean if and only if R is uniquely clean and G is a 2-group.*

Proof. One direction is obvious by Proposition 4.4.34. Conversely, assume that R is uniquely clean and G is a 2-group. As G is solvable, there are subgroups $e = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G$ such that G_{i-1} is normal in G_i, and G_i/G_{i-1} is an abelian group for all i. Thus, each G_i/G_{i-1} is an abelian 2-group; hence, G_i/G_{i-1} is locally finite. As is well known, every extension of locally finitely group is again locally finite, we deduce that G is locally finite. Therefore RG is uniquely clean, by Theorem 4.4.36. □

A ring R is called uniquely strongly clean provided that for any $a \in R$, there exists a unique idempotent $e \in R$ such that $a-e \in U(R)$ and $ae = ea$.

Lemma 4.4.38. *Let R be a uniquely strongly clean ring. Then $2 \in J(R)$.*

Proof. Since R is uniquely strongly clean, it is an exchange ring. As in the proof of Lemma 4.4.9, we complete the proof. □

Lemma 4.4.39. *Every corner of uniquely strongly clean rings is uniquely strongly clean.*

Proof. Let R be uniquely strongly clean, and let $e = e^2 \in R$. Then eRe is strongly clean in terms of Theorem 1.1.7. Let $eae \in eRe$. Write $eae = f+u = g+v$ where $f, g \in eRe$ are idempotent, $u, v \in U(eRe)$, $fu = uf$ and $gv = vg$. Hence, $eaea + 1 - e = f + (u + 1 - e) = g + (v + 1 - e)$. Clearly, $u+1-e, v+1-e \in U(R)$, and that $f(u+1-e) = (u+1-e)f, g(v+1-e) = (v + 1 - e)g$. By the uniqueness, we get $f = g$, as required. □

We shall be concerned with systems of matrix units, i.e., systems of n^2 elements e_{ik} which are subject to the following relations: $e_{ij}e_{kl} = 0$ $(j \neq k)$ and $e_{ij}e_{kl} = e_{il}$ $(j = k)$.

Lemma 4.4.40. *Let R be a uniquely strongly clean ring. Then $R/J(R)$ is Boolean.*

Proof. Let $S = R/J(R)$. Then S is an exchange ring with $J(S) = 0$. Suppose that S is not reduced. Then there exists some $a \in S$ such that $a^2 = 0$, but $a \neq 0$. In view of Lemma 4.4.14, the principal ideal (a) contains a system of nonzero matrix units $\{e_{11}, e_{12}, e_{21}, e_{22}\}$. Set $e = e_{11}+e_{22}$. Then $0 \neq e = e^2 \in S$. As R is an exchange ring, we have an idempotent $f \in R$ such that $e = \overline{f}$. Set $u = e_{11} + e_{12} + e_{21}$ and $v = e_{12} + e_{21} - e_{22}$. Then $\big(eue\big)\big(eve\big) = \big(eve\big)\big(eue\big) = e$. Write $u = \overline{x}$ and $v = \overline{y}$. Then $f - fxf \in J(fRf)$, and so $fxf \in U(fRf)$. Likewise, $fyf \in U(fRf)$. Furthermore, $\overline{f - fxf} = \overline{fyf}$; hence, $f - fxf \in U(fRf)$. Since R is uniquely strongly clean, then so is fRf by Lemma 4.4.38. Clearly, $f - fxf = 0 + (f - fxf)$. Thus, we get $f = 0$, a contradiction. Therefore S is reduced. Hence, it is abelian. Clearly, S is strongly clean. Suppose that $a-g, a-h \in U(S)$ with idempotents $g, h \in S$. By Lemma 4.4.45, $(g-h)^2 = g-2gh+h = g-h$. Set $\alpha = g - h$. Then there exist $c, d \in U(S)$ such that $\alpha = (a - h) - (a - g) = c + d$. Write $\alpha = \overline{\beta}$ for an idempotent $\beta \in R$. Then $\overline{\beta} = \overline{\beta}c\overline{\beta} + \overline{\beta}d\overline{\beta}$ in $\alpha S\alpha \cong \beta R\beta/J\big(\beta R\beta\big)$. Hence, we have $\eta, \xi \in U(\beta R\beta)$ such that $\beta = \eta + \xi$, and so $\beta - \eta = 0 + \xi$. By using Lemma 4.4.38 again, $\beta R\beta$ is uniquely strongly clean. Hence, $\beta = 0$, and then $g = h$. We infer that S is uniquely clean. Therefore $R/J(R)$ is Boolean in terms of Theorem 4.4.11. □

Theorem 4.4.41. *Let R be a ring. Then the following are equivalent:*

(1) *R is uniquely strongly clean.*
(2) *For any $a \in R$, there exists a unique idempotent $e \in R$ such that $a - e \in J(R)$ and $ae = ea$.*

Proof. (1) $\Rightarrow$ (2) For any $a \in R$, there exists a unique idempotent $e \in R$ such that $u := a - e \in U(R)$ and $ae = ea$. In view of Lemma 4.4.46,

$R/J(R)$ is Boolean; hence, $\overline{u} = \overline{1}$. This shows that $u \in 1 + J(R)$. Hence, $a = (1-e) + (2e - 1 + u)$ with $2e - 1 + u \in J(R)$. Write $a = f + v$ with $f = f^2, v \in J(R)$ and $af = fa$. Then $a = (1-f) + (2f - 1 + v)$ with $2f - 1 + v \in U(R)$. By the uniqueness, we have $e = 1 - f$, and so $f = 1 - e$, as desired.

(2) $\Rightarrow$ (1) For any $a \in R$, there exists a unique idempotent $e \in R$ such that $w := a - e \in J(R)$ and $ae = ea$. Thus, $a = 1 - e + (2e - 1 + w)$ with $2e - 1 + w \in U(R)$. Write $a = f + v, f = f^2, v \in U(R)$ and $af = fa$. Clearly, $R/J(R)$ is Boolean; hence, $v \in 1 + J(R)$. In addition, $\overline{2}^2 = \overline{2}$, and then $2 \in J(R)$. This implies that $a = (1-f) + (2f - 1 + v)$ with $2f - 1 + v \in J(R)$. By the uniqueness, we get $1 - f = e$, and so $f = 1 - e$. Accordingly, R is uniquely strongly clean. □

Corollary 4.4.42. *Every uniquely strongly clean ring is strongly J-clean.*

Proof. It is immediate from Theorem 4.4.42. □

Lemma 4.4.43. *A ring R is uniquely clean if and only if R is abelian and uniquely strongly clean.*

Proof. It is obvious by Theorem 4.4.1. □

Proposition 4.4.44. *Let R be a ring. Then the following are equivalent:*

(1) *R is uniquely strongly clean and local.*
(2) *$R/J(R) \cong \mathbb{Z}_2$.*

Proof. As R is local, all idempotents in R are 0 and 1; hence, R is abelian. Thus, R is uniquely strongly clean if and only if R is uniquely clean, by Lemma 4.4.36. Therefore we complete the proof from Corollary 4.4.3. □

Let R be a uniquely strongly clean ring. We claim that 1 is not the sum of two units. Otherwise, $1 = u + v$ with $u, v \in U(R)$, and then $u = 1 - (-v) = 0 + u$. We infer that $1 = 0$, a contradiction. Therefore for any ring R, $M_n(R)(n \geq 2)$ is not uniquely strongly clean, as

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -1 \\ -1 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Theorem 4.4.45. *Let T be the ring of a Morita context $(A, B, M, N, \psi, \varphi)$ with zero pairings. If A and B are local, then T is uniquely strongly clean if and only if*

(1) $A/J(A) \cong B/J(B) \cong \mathbb{Z}_2$;
(2) *If $a \in 1_A + J(A), b \in J(B)$ and $v \in M$, there exists a unique $x \in M$ such that $v = ax - xb$;*
(3) *If $a \in 1_B + J(B), b \in J(A)$ and $w \in N$, there exists a unique $y \in N$ such that $w = ay - yb$.*

Proof. $\Longrightarrow$ Choose $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in T$. Then $A \cong eTe$ is uniquely strongly clean, by Lemma 4.4.38. As A is local, it follows from Proposition 4.4.37 that $A/J(A) \cong \mathbb{Z}_2$. Likewise, $B/J(B) \cong \mathbb{Z}_2$.

If $a \in 1_A + J(A), b \in J(B)$ and $v \in M$, then there exists a unique idempotent $E = \begin{pmatrix} 0 & p \\ q & 1_B \end{pmatrix}$ such that $\begin{pmatrix} a & -v \\ 0 & b \end{pmatrix} - E \in U(T)$ and $\begin{pmatrix} a & -v \\ 0 & b \end{pmatrix} E = E \begin{pmatrix} a & v \\ 0 & b \end{pmatrix}$. This implies that $ap - pb = v$.

Assume that $ax - xb = v$. Set $F = \begin{pmatrix} 0 & -x \\ 0 & 1_B \end{pmatrix}$. Then $F = F^2$, and that

$$F \begin{pmatrix} a & v \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & v \\ 0 & b \end{pmatrix} F$$

Further,

$$\begin{pmatrix} a & v \\ 0 & b \end{pmatrix} - F \in U(T).$$

The uniqueness forces that $E = F$, and so $p = x$. Therefore there exists a unique $x \in M$ such that $v = ax - xb$.

If $a \in 1_b + J(B), b \in J(A)$ and $w \in N$, then there exists a unique idempotent $E = \begin{pmatrix} 1_A & p \\ q & 0 \end{pmatrix}$ such that $\begin{pmatrix} b & 0 \\ -w & a \end{pmatrix} - E \in U(T)$ and $\begin{pmatrix} b & 0 \\ -w & a \end{pmatrix} E = E \begin{pmatrix} b & 0 \\ -w & a \end{pmatrix}$. This implies that $aq - qb = w$.

Assume that $ay - yb = w$. Set $F = \begin{pmatrix} 1_A & 0 \\ -y & 0 \end{pmatrix}$. Then $F = F^2$, and that

$$F \begin{pmatrix} b & 0 \\ -w & a \end{pmatrix} = \begin{pmatrix} b & 0 \\ -w & a \end{pmatrix} F$$

Further,

$$\begin{pmatrix} b & 0 \\ -w & a \end{pmatrix} - F \in U(T).$$

This yields that $q = y$. Accordingly, there exists a unique $y \in N$ such that $w = ay - yb$.

$\Longleftarrow$ Let $D = \begin{pmatrix} a & v \\ w & b \end{pmatrix} \in R$. If $D \in U(T)$ or $1_R - D \in U(T)$, then $D \in T$ is uniquely strongly clean, by (1). Suppose $D, \begin{pmatrix} 1_A & 0 \\ 0 & 1_B \end{pmatrix} - D \notin U(T)$. Then $a \in 1_A + J(A)$ and $b \in J(B)$, or $a \in J(A)$ and $b \in 1_B + J(B)$. If $a \in 1_A + J(A)$ and $b \in J(B)$, by hypothesis, there exists a unique $x \in M$ such that $v = ax - xb$ and a unique $y \in N$ such that $-w = (1-b)y - y(1-a)$, i.e., $ya - by = w$. Choose $E = \begin{pmatrix} 0 & -x \\ -y & 1_B \end{pmatrix}$. Then $E = E^2, DE = ED$ and $D - E \in U(T)$. Suppose that $F = F^2, DF = FD$ and $D - F \in U(T)$. Then $F = \begin{pmatrix} 0 & -s \\ -t & 1_B \end{pmatrix}$ for some $s \in M, t \in N$. Hence, $as - sb = v$ and $ta - bt = w$, and so $x = s$ and $y = t$. This implies that $E = F$.

If $a \in J(A)$ and $b \in 1 + J(B)$, then $1 - a \in 1 + J(A)$ and $1 - b \in J(B)$, there exists a unique $x \in M$ such that $-v = (1 - a)x - x(1 - b)$, i.e., $v = ax - xb$. Furthermore, there exists a unique $y \in N$ such that $ya - by = w$. Similarly to the previous discussion, we can find a unique idempotent $E \in R$ such that $DE = ED$ and $D - E \in U(T)$. Therefore R is uniquely strongly clean. □

A ring R is said to be uniquely bleached provided that for any $a \in U(R), b \in J(R)$, $l_a - r_b$ and $l_b - r_a$ are isomorphic. For instance, every commutative local ring and every local ring with nil Jacobson radical are uniquely bleached.

Corollary 4.4.46. *Let R be a local ring and ${}_RV_R$ be a bimodule. Then $\begin{pmatrix} R & V \\ 0 & R \end{pmatrix}$ if uniquely strongly clean if and only if*

(1) $R/J(R) \cong \mathbb{Z}_2$;
(2) *R is uniquely bleached.*

Proof. The assertion is directly verified by Theorem 4.4.45. □

Let $R = \mathbb{Z}_{(2)}$. Then R is a commutative local ring with $J(R) = 2R$. Hence, R is uniquely bleached and $R/J(R) \cong \mathbb{Z}_2$. Therefore $T_2(R)$ is

uniquely strongly clean, in terms of Corollary 4.4.46. But $T_2(R)$ is not uniquely clean by Theorem 4.4.6.

Theorem 4.4.47. *Let R be a uniquely bleached ring. Then the following are equivalent:*

(1) *R is uniquely clean.*
(2) *R is abelian and $T_n(R)$ is uniquely strongly clean for all $n \geq 1$.*
(3) *R is abelian and $T_n(R)$ is uniquely strongly clean for some $n \geq 1$.*

Proof. (1) $\Rightarrow$ (2) In view of Lemma 4.4.1, R is abelian. Clearly, the result holds for $n = 1$. Assume that the result holds for $n (n \geq 1)$. Let $A = \begin{pmatrix} a_{11} & \alpha \\ & A_1 \end{pmatrix} \in T_{n+1}(R)$ where $a_{11} \in R, \alpha \in M_{1\times n}(R)$ and $A_1 \in T_n(R)$. By hypothesis, we can find a unique idempotent $e_{11} \in R$ such that $u_{11} := a_{11} - e_{11} \in U(R)$ and $a_{11}e_{11} = e_{11}a_{11}$. Also we have a unique idempotent $E_1 \in T_n(R)$ such that $U_1 := A_1 - E_1 \in U\big(T_n(R)\big)$ and $A_1E_1 = E_1A_1$; hence, $U_1E_1 = E_1U_1$. Let $E = \begin{pmatrix} e_{11} & x \\ & E_1 \end{pmatrix}$ and $U = \begin{pmatrix} u_{11} & \alpha - x \\ & U_1 \end{pmatrix}$, where $x \in M_{1\times n}(R)$. Observing that

$$E^2 = E \Leftrightarrow e_{11}x + xE_1 = x;$$
$$UE = EU \Leftrightarrow u_{11}x + (\alpha - x)E_1 = e_{11}(\alpha - x) + xU_1,$$

and then

$$(u_{11} + 2e_{11} - 1)x - xU_1 = e_{11}\alpha - \alpha E_1 \quad (*).$$

In view of Theorem 4.4.6, $R/J(R)$ is Boolean, and so $2 \in J(R)$. Furthermore, $u_{11} \in 1 + J(R)$. This shows that $u_{11} + 2e_{11} - 1 \in J(R)$. Write

$$x = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}, e_{11}\alpha - \alpha E_1 = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix},$$
$$U_1 = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ & c_{22} & \cdots & c_{2n} \\ & & \ddots & \\ & & & c_{nn} \end{pmatrix}, \text{where each } c_{ii} \in U(R).$$

Then

$$(u_{11} + 2e_{11} - 1)x_1 - x_1c_{11} = v_1;$$
$$(u_{11} + 2e_{11} - 1)x_2 - x_2c_{22} = v_1 + x_1c_{12};$$
$$\vdots$$
$$(u_{11} + 2e_{11} - 1)x_n - x_nc_{nn} = v_n + x_1c_{1n} + \cdots + x_{n-1}c_{(n-1)n}.$$

By hypothesis, we have a unique x such that $(*)$ holds. As $E_1U_1 = U_1E_1$, we get

$$\begin{aligned}&(u_{11} + 2e_{11} - 1)x(e_{11}I_n + E_1) - x(e_{11}I_n + E_1)U_1\\ &= \alpha(e_{11}I_n - E_1)(e_{11}I_n + E_1)\\ &= \alpha(e_{11}I_n - E_1)\\ &= (u_{11} + 2e_{11} - 1)x - xU_1.\end{aligned}$$

Set $y = x(e_{11}I_n + E_1)$. Then $(u_{11} + 2e_{11} - 1)(y - x) - (y - x)U_1 = 0$.

Write $y - x = \begin{pmatrix} z_1 & \cdots & z_n \end{pmatrix}$. Then

$$\begin{aligned}(u_{11} + 2e_{11} - 1)z_1 - z_1c_{11} &= 0;\\ (u_{11} + 2e_{11} - 1)z_2 - z_2c_{22} &= z_1c_{12};\\ &\vdots\\ (u_{11} + 2e_{11} - 1)z_n - z_nc_{nn} &= z_1c_{1n} + \cdots + z_{n-1}c_{(n-1)n}.\end{aligned}$$

By hypothesis, we get each $z_i = 0$, and so $y = x$. This implies that $e_{11}x + xE_1 = x$. Furthermore, $u_{11}x - (\alpha - x)E_1 = e_{11}(\alpha - x) + xU_1$. Therefore, we have a uniquely strongly clean expression

$$A = \begin{pmatrix} e_{11} & x \\ & E_1 \end{pmatrix} + \begin{pmatrix} a_{11} - e_{11} & x \\ & A_1 - E_1 \end{pmatrix}.$$

By induction, $T_m(R)$ is uniquely strongly clean for all $m \in \mathbb{N}$.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) In light of Lemma 4.4.39, R is uniquely strongly clean, hence the result by Lemma 4.4.43. □

Corollary 4.4.48. *Let R be a ring with nil Jacobson radical. Then the following are equivalent:*

(1) *R is uniquely clean.*
(2) *R is abelian and $T_n(R)$ is uniquely strongly clean for all $n \geq 1$.*
(3) *R is abelian and $T_n(R)$ is uniquely strongly clean for some $n \geq 1$.*

Proof. Let $a \in U(R), b \in J(R)$. Write $b^n = 0$. Choose $\varphi = l_{a^{-1}} + l_{a^{-2}}r_b + \cdots + l_{a^{-n}}r_{b^{n-1}} : R \to R$. As in the proof of Corollary 1.4.16, $(l_a - r_b)\varphi = \varphi(l_a - r_b) = 1_R$. Thus, $l_a - r_b : R \to R$ are isomorphism. Similarly, $l_b - r_a$ is isomorphic. Therefore R is uniquely bleached, and the result follows by Theorem 4.4.47. □

Corollary 4.4.49. *Let R be a ring for which some power of each element in $J(R)$ is central. Then the following are equivalent:*

(1) *R is uniquely clean.*

(2) *R is abelian and $T_n(R)$ is uniquely strongly clean for all $n \geq 1$.*
(3) *R is abelian and $T_n(R)$ is uniquely strongly clean for some $n \geq 1$.*

Proof. Let $a \in U(R), b \in J(R)$. Write $b^n \in C(R)$. Choose $\varphi = l_{a^{-1}} + l_{a^{-2}}r_b + \cdots + l_{a^{-n}}r_{b^{n-1}} : R \to R$. Then $(l_a - r_b)\varphi = l_1 - l_{a^{-n}}r_{b^n} = l_{1-a^{-n}b^n}$. Further, we check that $\varphi(l_a - r_b) = l_{1-a^{-n}b^n}$. Thus, $l_a - r_b : R \to R$ is isomorphic. Likewise, $l_b - r_a$ is isomorphic. That is, R is uniquely bleached. Therefore we complete the proof by Theorem 4.4.47. □

Immediately, we prove that every triangular matrix ring over a Boolean ring is uniquely strongly clean. For instance, $T_n(\mathbb{Z}_2)(n \geq 2)$ is uniquely strongly clean, while it is not uniquely clean.

Corollary 4.4.50. *Let R be a commutative ring. Then the following are equivalent:*

(1) *R is uniquely clean.*
(2) *$T_n(R)$ is uniquely strongly clean for all $n \geq 1$.*
(3) *$T_n(R)$ is uniquely strongly clean for some $n \geq 1$.*

Proof. As every commutative ring is uniquely bleached and abelian, the result follows by Theorem 4.4.47. □

Lemma 4.4.51. *Let R be a ring. If $T_2(R)$ is uniquely strongly clean, then R is uniquely bleached.*

Proof. Clearly, R is uniquely strongly clean. Let $a \in U(R)$ and $b \in J(R)$, and let $r \in R$. Choose $A = \begin{pmatrix} a & -r \\ & b \end{pmatrix} \in T_2(R)$. Then we have a unique idempotent $E = (e_{ij}) \in T_2(R)$ such that $A - E \in U(T_2(R))$ and $AE = EA$. Obviously, $e_{11}, e_{22} \in R$ are idempotents, and that $a - e_{11}, b - e_{22} \in U(R)$. Since $a - 0 \in U(R)$ and $b - 1 \in U(R)$, by the uniquely strong cleanness of R, we get $e_{11} = 0$ and $e_{22} = 1$. Write $E = \begin{pmatrix} 0 & x \\ & 1 \end{pmatrix}$. As $EA = AE$, we get $ax - xb = r$. Assume that $ay - yb = r$. Then we have an idempotent $F = \begin{pmatrix} 0 & y \\ & 1 \end{pmatrix} \in T_2(R)$ such that $A - F \in U(T_2(R))$ and $AF = FA$. By the uniqueness of E, we have $x = y$. As a result, $l_a - r_b : R \to R$ is an isomorphism. Likewise, $l_b - r_a : R \to R$ is an isomorphism. Therefore R is uniquely bleached. □

Theorem 4.4.52. *Let R be an abelian ring. Then the following are equivalent:*

(1) *R is uniquely bleached, uniquely clean.*
(2) *$T_n(R)$ is uniquely strongly clean for all $n \geq 1$.*
(3) *$T_n(R)$ is uniquely strongly clean for some $n \geq 2$.*

Proof. (1) $\Rightarrow$ (2) is clear by Theorem 4.4.47.

(2) $\Rightarrow$ (3) This is trivial.

(3) $\Rightarrow$ (1) By virtue of Lemma 4.4.39, R and $T_2(R)$ are uniquely strongly clean. Since R is abelian, it is uniquely clean. Therefore we complete the proof by Lemma 4.4.51. □

Corollary 4.4.53. *Let R be a local ring. Then the following are equivalent:*

(1) *R is uniquely bleached and $R/J(R) \cong \mathbb{Z}_2$.*
(2) *$T_n(R)$ is uniquely strongly clean for all $n \geq 1$.*
(3) *$T_n(R)$ is uniquely strongly clean for some $n \geq 2$.*

Proof. (1) $\Rightarrow$ (2) In view of Corollary 4.4.14, R is uniquely clean. Thus proving (2), in terms of Theorem 4.4.52.

(2) $\Rightarrow$ (3) This is obvious.

(3) $\Rightarrow$ (1) In view of Theorem 4.4.52, R is uniquely bleached. Therefore we complete the proof, by Lemma 4.4.39 and Proposition 4.4.44. □

Let $Lat(R)$ be the lattice of all right ideals of a ring R. A ring R said to be a D_L-ring if $Lat(R)$ is distribute, i.e., $A \cap (B + C) = (A \cap B) + (A \cap C)$ for all $A, B, C \in Lat(R)$.

Corollary 4.4.54. *Let R be a D_L-ring. Then the following are equivalent:*

(1) *R is uniquely bleached, uniquely clean.*
(2) *$T_n(R)$ is uniquely strongly clean for all $n \geq 1$.*
(3) *$T_n(R)$ is uniquely strongly clean for some $n \geq 1$.*

Proof. Let $e \in R$ be an idempotent, and let $a \in R$. Set $f = e - (1 - e)ae$. Then $eR, fR \in Lat(R)$ and $eR = \big(eR \cap fR\big) + \big(eR \cap (1 - f)R\big)$. By hypothesis, $feR \subseteq eR$. This yields that $e = ef = efe = fe = f$, and so $ae = eae$. Likewise, $ea = eae$. Hence, $ea = eae = ae$, and so R is abelian. Therefore the proof is complete by Theorem 4.4.52. □

Notes. Theorem 4.3.2 was proved by Dorsey [241, Proposition 3.5.4]. Theorem 4.4.6 and Theorem 4.4.9 are due to Nicholson and Zhou [435, Theorem 20]. Theorem 4.4.36 is due to Chen et al. [179, Theorem 12]. Uniquely strong cleanness was introduced by Chen et al., who also proved Theorem 4.4.41 [181, Theorem 17] and Corollary 4.4.50 [181, Theorem 10].

This page intentionally left blank

Chapter 5

Strong Nil-cleanness

In [233], Diesl introduced the concept of strong nil-cleanness. An element $a \in R$ is strongly nil-clean provided that there exists an idempotent $e \in R$ such that $a - e \in N(R)$ and $ae = ea$. A ring R is strongly nil-clean in case every element in R is strongly nil-clean. As is well known, {strongly nil-clean rings} $\subsetneq$ {strongly π-regular rings} $\subsetneq$ {strongly clean rings}. The other motivation of studying strong nil-cleanness is derived from Lie algebra. As every idempotent matrix A over a field F admits a diagonal reduction, it follows that if $A \in M_n(F)$ has a strongly nil clean decomposition, then it satisfies the conditions: $A = E + W, E$ is similar to a diagonal matrix, $W \in M_n(F)$ is nilpotent, E and W commute. Such decomposition over a field is called the Jordan-Chevalley decomposition in Lie theory.

The aim of this chapter is to investigate strong nil-cleanness. In Section 5.1, we establish many elementary properties of strongly nil-clean rings. In Section 5.2, we construct various examples of such rings. In Section 5.3, we investigate the uniqueness for nil-clean rings. In the last section, we are concerned with the strong nil-cleanness of a single $n \times n$ matrix over commutative rings. Many strongly nil-clean matrices over projective-free rings are thereby provided.

5.1 Strongly Nil-clean Rings

In this section we establish some elementary properties of strongly nil-clean rings. We start by determining necessary and sufficient conditions on nilpotency in order that an element should be strongly nil-clean.

Theorem 5.1.1. *Let R be a ring, and let $a \in R$. Then the following are equivalent:*

(1) *$a \in R$ is strongly nil-clean.*
(2) *$a - a^2 \in N(R)$.*
(3) *For any $a \in R$, there exists $x \in R$ such that $a - ax \in N(R), ax = xa$ and $x = x^2a$.*

Proof. (1) $\Rightarrow$ (3) Let $a \in R$. Then there exists an idempotent $e \in R$ such that $w := a - e \in N(R)$ and $ae = ea$. Set $x = (1+w)^{-1}e$. Then we check that

$$ax = (1+w)^{-1}ea = (1+w)^{-1}e(a+1-e) = (1+w)^{-1}e(1+w) = e.$$

Hence, $a - ax \in N(R)$. Moreover, $ax = xa$ and $x^2a = x(ax) = xe = x$, as desired.

(3) $\Rightarrow$ (2) Let $a \in R$. Then we have $x \in R$ such that $w := a - ax \in N(R), ax = xa$ and $x = x^2a$. Thus, $a - a^2 = (ax + w) - (ax + w)^2 = (1 - w - 2ax)w \in N(R)$, as required.

(2) $\Rightarrow$ (1) Write $(a - a^2)^n = 0$ for some $n \geq 1$. Then $a^n(1-a)^n = 0$. Let $f(a) = \sum\limits_{i=0}^{n} \binom{2n}{i} a^{2n-i}(1-a)^i$. Then $f(a) \equiv 0 \,(mod\, a^n)$. Clearly,

$$f(a) + \sum_{i=n+1}^{2n} \binom{2n}{i} a^{2n-i}(1-a)^i = \big(a + (1-a)\big)^{2n} = 1.$$

Hence, $f(a) \equiv 1 \,\big(mod\,(1-a)^n\big)$. This implies that $f(a)\big(1 - f(a)\big) \equiv 0 \,\big(mod\, a^n(1-a)^n\big)$. Let $e = f(a)$. Then $e(1-e) = 0$, and so $e \in R$ is an idempotent. Further, $e - a = a^{2n} - a + g(a)(a - a^2) = \big(a^{2n} - a^{2n-1}\big) + \big(a^{2n-1)} - a^{2n-2}\big) + \cdots + \big(a^2 - a\big) + g(a)\big(a^2 - a\big) = \big(a^{2n-2} + a^{2n-3} + \cdots + a + 1\big)(a^2 - a) + g(a)\big(a^2 - a\big) = h(a)\big(a - a^2\big)$ for some $g(t), h(t) \in \mathbb{Z}[t]$. As $\big(a - a^2\big)^n = 0$ and $h(a)(a - a^2) = (a - a^2)h(a)$, we see that $(e - a)^n = 0$, and so $a = e + w$, where $w = e - a \in N(R)$. One easily checks that $ea = ae$, hence the result. $\square$

Corollary 5.1.2. *Let R be a ring. Then the following are equivalent:*

(1) *R is strongly nil-clean.*
(2) *For any $a \in R$, there exists an idempotent $e \in \mathbb{Z}[a]$ such that $a - e \in N(R)$.*
(3) *For any $a \in R$, there exists a unique idempotent $e \in R$ such that $a - e \in N(R)$ and $ae = ea$.*

Proof. (1) $\Rightarrow$ (2) Let $a \in R$. As in the proof of Theorem 5.1.1, we can find $f(t) \in \mathbb{Z}[t]$ such that $a - e \in N(R)$ where $e = f(a)$ is an idempotent.

(2) $\Rightarrow$ (3) For any $a \in R$, there exists an idempotent $e \in \mathbb{Z}[a]$ such that $a - e \in N(R)$. Then $ae = ea$. Suppose $a - f \in N(R)$ and $af = fa$, where $f \in R$ is an idempotent. Clearly, $ae = ea, af = fa$ and $ef = fe$; hence, $(a-f)(a-e) = (a-e)(a-f)$. This shows that $e - f = (a-f) - (a-e) \in N(R)$. One easily checks that $(e-f)^3 = e - f$, and so $e = f$, as desired.

(3) $\Rightarrow$ (1) This is trivial. □

Example 5.1.3. Let $\mathbb{Z}_4[i] = \{a + bi \mid a, b \in \mathbb{Z}_4\}$. Then $\mathbb{Z}_4[i]$ is strongly nil-clean.

Proof. For any $a + bi \in \mathbb{Z}_4[i]$, it is easy to check that $a + bi - (a + bi)^2 = (a - a^2) - 2abi + bi + b^2 = (a - a^2) - 2abi + (b - b^2))i + b^2(1 + i)$. As $(a-a^2)^2 = (b-b^2)^2 = 2^2 = 0$ and $(1+i)^4 = 0$, $a+bi-(a+bi)^2 \in N\big(\mathbb{Z}_4[i]\big)$. According to Theorem 5.1.1, $\mathbb{Z}_4[i]$ is strongly nil-clean. □

For further use, we record the following.

Proposition 5.1.4. *A ring R is strongly nil-clean if and only if*

(1) *R is strongly clean;*
(2) *$N(R) = \{x \in R \mid 1 - x \in U(R)\}$.*

Proof. Suppose that R is strongly nil-clean. For any $a \in R$, there exists an idempotent $e \in R$ such that $(a-1)e = e(a-1)$ and $(a-1) - e \in N(R)$, and so $ea = ae$ and $a - e = \big((a-1) - e\big) + 1 \in U(R)$. Thus, R is strongly clean. Let $x \in R$ be such that $1-x \in U(R)$. Write $1-x = f+w, f = f^2, w \in N(R)$ and $fw = wf$. Then $w(1-x) = (1-x)w$. Thus, $f = (1-x) - w \in U(R)$; hence, $f = 1$. This shows that $x = -w \in N(R)$, and so $\{x \in R \mid 1 - x \in U(R)\} \subseteq N(R)$, as desired.

Conversely, assume that (1) and (2) hold. For any $a \in R$, there exists an idempotent $e \in R$ such that $(a-1)e = e(a-1)$ and $(a-1) - e \in U(R)$. Thus $ae = ea$ and $1 - (a - e) \in U(R)$. This implies that $a - e \in N(R)$, as required. □

Theorem 5.1.5. *Let R be a ring. Then the following are equivalent:*

(1) *R is strongly nil-clean.*
(2) *$N(R)$ forms an ideal of R, and $R/N(R)$ is Boolean.*
(3) *$R/J(R)$ is Boolean and $J(R)$ is nil.*

Proof. (1) $\Rightarrow$ (2) For any $a \in J(R)$, we have an idempotent $e \in R$ and a $w \in N(R)$ such that $a = e+w$ and $ae = ea$. So $1-e = -a+(1+w) \in U(R)$, and then $e = 0$. Therefore $a = w \in N(R)$, and then $J(R) \subseteq N(R)$.

Since $R/J(R)$ is semiprimitive, there exist some primitive rings $R_1, \ldots, R_i, \ldots$ such that R is the subdirect product of them (see [377, Theorem 12.5]). That is, there exist epimorphisms $\varphi_i : R \to R_i$ such that $\bigcap\limits_i Ker(\varphi_i) = 0$. For each i, by the structure theorem of primitive rings, $R_i \cong M_{m_i}(D_i)$ for a division ring D_i, or there exists a subring S_i of R_i which admits an epimorphism $\phi_i : S_i \to M_2(D_i)$ where D_i is a division ring (see [377, Theorem 11.19]). As $R_i \cong R/Ker(\varphi_i)$ and $S_i \subseteq R_i$, we see that $a - a^2 \in N(S_i)$ for any $a \in S_i$. But $M_2(D_i) \cong S_i/Ker(\phi_i)$. Then for any $b \in M_2(D_i)$, $b-b^2 \in N\big(M_2(D_i)\big)$. Let $E_{ij} \in M_2(D_i)$ be the matrix with 1 in (i,j) entry, and 0 elsewhere. Then $(E_{11}+E_{12}+E_{21})-(E_{11}+E_{12}+E_{21})^2 = -E_{11}-E_{22} \notin N\big(M_2(D_i)\big)$, a contradiction. This implies that each $R_i \cong M_{m_i}(D_i)$ where D_i is a division ring. As in the preceding discussion, we see that $m_i = 1$. Thus, $c - c^2 \in N(D_i)$ for any $c \in D_i$, and so D_i is Boolean. This implies that $D_i \cong \mathbb{Z}_2$. Therefore $R_i \cong \mathbb{Z}_2$, and then R_i is commutative. For any $g, h \in R$, we get $\varphi_i(g)\varphi_i(h) = \varphi_i(h)\varphi_i(g)$, and then $gh - hg \in \bigcap\limits_i Ker(\varphi_i) = 0$. Hence, $gh = hg$, i.e., $R/J(R)$ is commutative. Let $x \in N(R)$, and let $r \in R$. Then there exists some $m \in \mathbb{N}$ such that $\overline{xr}^m = 0$ in $R/J(R)$, and so $(xr)^m \in J(R)$. Hence, $1-xr \in U(R)$. This implies that $x \in J(R)$. As a result, $xr \in J(R) \subseteq N(R)$, and thus, $xr \in N(R)$. Likewise, $rx \in N(R)$. For any $y, z \in N(R)$, then $\overline{y+z} \in R/J(R)$ is nilpotent. So, $(y+z)^n \in J(R)$ for some $n \in \mathbb{N}$. As $J(R) \subseteq N(R)$, we have some $s \in \mathbb{N}$ such that $(y+z)^{ns} = 0$. Then $y + z \in N(R)$, and therefore $N(R)$ forms an ideal of R. Clearly, $R/N(R)$ is Boolean, as desired.

(2) $\Rightarrow$ (3) As $N(R)$ forms an ideal of R, $N(R) \subseteq J(R)$. For any $a \in J(R)$, $a-a^2 \in N(R)$ as $R/N(R)$ is Boolean. Hence, $a(1-a) \in N(R)$. Write $\big(a(1-a)\big)^n = 0$ for some $n \in \mathbb{N}$. Then $a^n = 0$, and so $a \in N(R)$. Thus, $J(R) \subseteq N(R)$; whence, $J(R) = N(R)$. Therefore $R/J(R)$ is Boolean, as required.

(3) $\Rightarrow$ (1) For any $a \in R$, $a - a^2 \in J(R) \subseteq N(R)$. According to Theorem 5.1.1, R is strongly nil-clean. □

A ring R is called nil-clean if every element in R is the sum of an idempotent and a nilpotent. We now derive

Corollary 5.1.6. *Let R be a ring. Then the following are equivalent:*

(1) *R is strongly nil-clean.*
(2) *R is nil-clean and $N(R)$ forms an ideal of R.*
(3) *R is nil-clean and $R/J(R)$ is commutative.*

Proof. (1) $\Rightarrow$ (3) In view of Theorem 5.1.5, $R/J(R)$ is Boolean. Hence, $R/J(R)$ is commutative, as desired.

(3) $\Rightarrow$ (2) Suppose $a \in J(R)$. Then, $a = e + w, e = e^2$ and $w \in N(R)$. This implies that $(a - e)^n = 0$ for some $n \in \mathbb{N}$. Since $a \in J(R)$, we get $e \in J(R)$, and so $e = 0$. Thus, $a = w \in N(R)$; hence, $J(R) \subseteq N(R)$. Since $R/J(R)$ is commutative, we easily check that $N(R) \subseteq J(R)$, and then $N(R) = J(R)$, as required.

(2) $\Rightarrow$ (1) Let $a \in R$. Write $a = e + w$ with $e^2 = e \in R$ and $w \in N(R)$. As $N(R)$ is an ideal of R, we see that $a - a^2 \in N(R)$, hence the result. □

Corollary 5.1.7. *Let R be a ring. Then the following are equivalent:*

(1) *R is strongly nil-clean.*
(2) *R is strongly J-clean and $J(R)$ is nil.*
(3) *For any $a, b \in R$, $(a - a^2)(b - b^2) \in N(R)$.*

Proof. (1) $\Rightarrow$ (2) In view of Theorem 5.1.5, $J(R)$ is nil and $N(R)$ forms an ideal of R. Hence, $N(R) = J(R)$, and so R is strongly J-clean.

(2) $\Rightarrow$ (3) Since R is strongly J-clean, it follows from Theorem 4.1.1 that $R/J(R)$ is Boolean. For any $a, b \in R$, we have $a - a^2, b - b^2 \in J(R)$. Therefore $(a - a^2)(b - b^2) \in J(R) \subseteq N(R)$, as required.

(3) $\Rightarrow$ (1) For any $a \in R$, $(a - a^2)^2 \in N(R)$, and so $a - a^2 \in N(R)$, hence the result. □

Evidently, a ring R is strongly nil-clean if and only if every element not in $J(R)$ is the sum of an idempotent and a nilpotent that commute if and only if for any $a \notin J(R)$, $a - a^2 \in N(R)$.

We note that strong $J^{\#}$-cleanness and strong J-cleanness behave differently for matrices over commutative rings. Let $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{Z}_2)$.

It is directly verified that $A \in M_2(\mathbb{Z}_2)$ is strongly $J^{\#}$-clean, while A is not strongly J-clean. By means of strong nil-cleanness, we do have

Proposition 5.1.8. *Let R be a ring. Then the following are equivalent:*

(1) *R is strongly J-clean.*
(2) *For any $a \in R$, there exists an idempotent $e \in R$ such that $a - e \in J^{\#}(R)$ and $ae = ea$.*

Proof. (1) $\Rightarrow$ (2) This is obvious, as $J(R) \subseteq J^{\#}(R)$.

(2) $\Rightarrow$ (1) Let $a \in R$. Then there exists an idempotent $e \in R$ such that $a - e \in J^{\#}(R)$ and $ae = ea$. Let $w = a - e$. Then $a = e + w = (1 - e) + ((2e - 1) + w) = (1 - e) + (2e - 1)(1 + (2e - 1)w)$. As $(2e - 1)w = w(2e - 1)$, we see that $(2e - 1)(1 + (2e - 1)w) \in U(R)$. This shows that R is strongly clean. As $w^m \in J(R)$ for some $n \in \mathbb{N}$, we have $\overline{w} \in N(R/J(R))$. Since $\overline{a} = \overline{e} + \overline{w}$, $R/J(R)$ is strongly nil-clean. It follows by Theorem 5.1.5 that $R/J(R)$ is Boolean. By using Theorem 4.1.1, R is strongly J-clean. □

Lemma 5.1.9. *Let R be a ring, and let $a \in R$. If there exists a monic $f(t) \in \mathbb{Z}[t]$ and some $s \in \mathbb{N}$ such that $sa = 0$ and $f(a) = 0$, then there exist distinct $m, n \in \mathbb{N}$ such that $a^m = a^n$.*

Proof. Clearly, $\mathbb{Z}a \subseteq \{0, a, \ldots, (s - 1)a\}$. Write $f(t) = t^k + b_1 t^{k-1} + \cdots + b_{k-1}t + b_k \in \mathbb{Z}[t]$. Then $a^{k+1} = -b_1 a^k - \cdots - b_{k-1}a^2 - b_k a$. This implies that $\{a, a^2, a^3, \ldots, a^l, \ldots\} \subseteq \{a, a^2, a^3, \ldots, a^k, 0, a, \ldots, (s-1)a, a^2, \ldots, (s-1)a^2, \ldots, a^k, \ldots, (s-1)a^k\}$. That is, $\{a, a^2, a^3, \ldots, a^k, \ldots\}$ is a finite set. Hence, we can find distinct $m, n \in \mathbb{N}$ such that $a^m = a^n$, as desired. □

A ring R is periodic in case for any $a \in R$ there exist distinct $m, n \in \mathbb{N}$ such that $a^m = a^n$. We claim that every finite ring is periodic. Let a be an element of a finite ring R. Then $\{1, a, a^2, a^3, \ldots, a^k, \ldots\}$ is a finite set. Hence, we can find distinct $m, n \in \mathbb{N}$ such that $a^m = a^n$, and we are done. The following elementary property will be used repeatedly.

Lemma 5.1.10. *Let R be a ring. Then the following are equivalent:*

(1) *R is periodic.*
(2) *For any $a \in R$, there exists $n \in \mathbb{N}$ such that $a^n = a^{n+1}f(a)$, where $f(t) \in \mathbb{Z}[t]$.*

Proof. (1) $\Rightarrow$ (2) This is trivial.

(2) $\Rightarrow$ (1) Let $a \in R$. If a is nilpotent, then we can find some $n \in \mathbb{N}$ such that $a^n = a^{n+1} = 0$. Thus, we may assume that $a \in R$ is not nilpotent. By

hypothesis, there exist $n \in \mathbb{N}, g(t) \in \mathbb{Z}[t]$ such that $a^n = a^{n+1}g(a)$. Thus, $a^n = a^{n+1}f(a)$, where $f(t) = t\big(g(t)\big)^2 \in \mathbb{Z}[t]$. In addition, $f(0) = 0$. Let $e = a^n\big(f(a)\big)^n$. Then $0 \neq e = e^2 \in R$ and $a^n = a^n e$. Set $S = eRe$ and $\alpha = ea = ae$. Then $f(\alpha) = ef(a)$. Further,

$$\alpha^n\big(f(\alpha)\big)^n = e, \alpha^n = a^n, \alpha^n = \alpha^{n+1}f(\alpha).$$

Thus, $e = \alpha^n\big(f(\alpha)\big)^n = \alpha^{n+1}\big(f(\alpha)\big)^{n+1} = \alpha^n\big(f(\alpha)\big)^n\alpha f(\alpha) = e\alpha f(\alpha) = \alpha f(\alpha)$ in S. Write $f(t) = a_1t + \cdots + a_nt^n$. Then $\alpha\big(a_1\alpha + \cdots + a_n\alpha^n\big) = e$. This implies that $(\alpha^{-1})^{n+1} - a_1(\alpha^{-1})^{n-1} - \cdots - a_ne = 0$. Let $g(t) = t^{n+1} - a_1t^{n-1} - \cdots - a_n \in \mathbb{Z}[t]$. Then $g(t)$ is a nomic polynomial such that $g(\alpha^{-1}) = 0$.

Let $T = \{me \in S \mid m \in \mathbb{Z}\}$. Then T is a subring of S. For any $me \in T$, by hypothesis, there exists $g(t) \in \mathbb{Z}[t]$ such that $(me)^p = (me)^{p+1}g(me) \in (me)^{p+1}T$. This implies that T is strongly π-regular. Construct a map $\varphi : \mathbb{Z} \to T, m \to me$. Then $\mathbb{Z}/Ker\varphi \cong T$. As $\mathbb{Z}$ is not strongly π-regular, we see that $Ker\varphi \neq 0$. Hence, $T \cong \mathbb{Z}_q$ for some $q \in \mathbb{N}$. Thus, $qe = 0$. As a result, $q\alpha^{-1} = 0$. In view of Lemma 5.1.9, we have some $s, t \in \mathbb{N}(s \neq t)$ such that $(\alpha^{-1})^s = (\alpha^{-1})^t$, i.e., $\alpha^s = \alpha^t$. Thus, $a^{ns} = a^{st}$; hence the result follows. □

Lemma 5.1.11. *Let R be a ring. Then the following are equivalent:*

(1) *R is periodic.*
(2) *For any $a \in R$, $a - a^n \in N(R)$ for some $n \geq 2$.*

Proof. (1) ⇒ (2) For any $a \in R$, there exist distinct $m, n \in \mathbb{N}$ such that $a^m = a^n(m > n)$. If $n = 1$, then $a - a^m = 0 \in N(R)$. Suppose $n > 1$. Then $a^{n-1}(a-a^{m-n+1}) = 0$. Hence, $a^{n-2}a(a-a^{m-n+1}) = a^{n-2}a^{m-n+1}(a-a^{m-n+1}) = 0$. This implies that $a^{n-2}(a-a^{m-n+1})^2 = a^{n-2}a(a-a^{m-n+1}) - a^{n-2}a^{m-n+1}(a-a^{m-n+1}) = 0$. Further, $a^{n-3}(a-a^{m-n+1})^2 = 0$. By the obvious induction, we get $(a-a^{m-n+1})^2 = 0$, and so $a - a^{m-n+1} \in N(R)$, as required.

(2) ⇒ (1) For any $a \in R$, we have some $n \geq 2$ such that $a - a^n \in N(R)$. Write $\big(a-a^n\big)^m = 0$. Then $a^m = a^{m+1}f(a)$ for some $f(t) \in \mathbb{Z}[t]$. According to Lemma 5.1.10, R is periodic. □

We claim that a ring R is Boolean if and only if for any $a \in R$ there exists some $n \in \mathbb{N}$ such that $a^n = a^{n+1}$. As in the proof of Lemma 5.1.11, we see that $a-a^2 \in N(R)$. By hypothesis, $\big(1-(a-a^2)\big)^{n+1} = \big(1-(a-a^2)\big)^n$ for some $n \in \mathbb{N}$. Clearly, $1-(a-a^2) \in U(R)$; hence, $1-(a-a^2) = 1$. This

implies that $a = a^2$, and then R is Boolean. The converse is obvious, and we are through.

Theorem 5.1.12. *Every strongly nil-clean ring is periodic.*

Proof. This is clear from Lemma 5.1.11. □

Lemma 5.1.13. *Let R be a ring. Then the following are equivalent:*

(1) *R is Boolean.*
(2) *R is multiplicatively generated by idempotents.*
(3) *R is multiplicatively generated by idempotents and nilpotents.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (1) Given $a^2 = 0$, then a is the product of some idempotents or nilpotent $x_1, \ldots, x_n \in R$ such that $1 - a = x_1 x_2 \cdots x_n$. This implies that $x_1 \in R$ is right invertible. If $x_1 \in R$ is an idempotent, then $x_1 = 1$. If $x_1 \in N(R)$, then $x_1 = 0$, a contradiction. Thus, $1 - a = x_2 \cdots x_n$. By iteration of this process, we get $x_2 = \cdots x_n = 1$. Hence, $1 - a = 1$, and then $a = 0$. So R is reduced, and therefore R is abelian. For any $x \in R$, it is the product of some idempotents or nilpotent; whence, $x \in R$ is an idempotent. Therefore R is Boolean. □

Proposition 5.1.14. *Let R be a ring in which all units are commutative. Then the following are equivalent:*

(1) *R is strongly nil-clean.*
(2) *Every element of R is a product of elements each of which is the sum of an idempotent and a nilpotent.*

Proof. (1) $\Rightarrow$ (2) This is trivial.

(2) $\Rightarrow$ (1) Since every units in R are commutative, we easily check that all nilpotents in $N(R)$ are commutative, i.e., $N(R)$ is commutative. Let $x \in N(R)$, and let $r \in R$ be an idempotent. Clearly, $xr - rxr \in N(R)$, and hence by hypothesis, $\big(xr - rxr\big)x = x\big(xr - rxr\big)$. That is,

$$xrx - rxrx = x^2r - xrxr.$$

Multiplying the preceding by r form left and right we have $\big(rxr\big)^2 = rx^2r$. By iteration of this process, we get $\big(rxr\big)^{2^n} = rx^{2^n}r$ for all $n \in \mathbb{N}$. This implies that $rxr \in N(R)$; whence, $xr = \big(xr - rxr\big) + rxr \in N(R)$. Likewise, $rx \in N(R)$.

For any $a \in R$, write $a = (e_1 + w_1) \cdots (e_n + w_n)$ where $e_1, \ldots, e_n \in R$ are idempotents and $w_1, \ldots, w_n \in N(R)$. By the preceding discussion,

$xe_1 \in N(R)$, and so $x(e_1 + w_1) = xe_1 + xw_1 \in N(R)$. By iteration of this process, we get $x(e_1 + w_1)(e_2 + w_2), \ldots, (e_1 + w_1) \cdots (e_n + w_n) \in N(R)$. That is, $xr \in N(R)$. Similarly, $rx \in N(R)$. Hence, $N(R)$ is an ideal of R.

For any $a \in R$, we see that $\overline{a}$ is the product of finitely many idempotents in $R/N(R)$. In other words, $R/N(R)$ is multiplicatively generated by idempotents, and hence by Lemma 5.1.13, $R/N(R)$ is Boolean, and therefore R is strongly nil clean, by Theorem 5.1.5. □

We note that Proposition 5.1.14 can not be extended to strongly clean rings as the following shows.

Example 5.1.15. Let $R = \mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0, (a, b) = 1, 3 \nmid b \text{ and } 5 \nmid b\}$. Then R is a commutative ring in which every element has the form $(e_1 + u_1) \cdots (e_n + u_n)$ where $e_i \in \{0, 1\}$ and $u_i \in U(R)$, but R is not strongly clean.

Proof. For any $\frac{a}{b} \in R$, we can find some $s, t \geq 0$ and $c \in R$ such that $\frac{a}{b} = \frac{3^s 5^t c}{b}$ with $3, 5 \nmid c$. Hence, $\frac{a}{b} = (2+1)^s (4+1)^t \frac{c}{b}$, and that $2, 4, \frac{c}{b} \in U(R)$. Therefore $\frac{a}{b}$ has the form $(e_1 + u_1) \cdots (e_n + u_n)$ where $e_i \in \{0, 1\}$ and $u_i \in U(R)$.

As $3, 5 \nmid 4$, we see that $\frac{9}{4} \in R$. Observing that R is an integral domain, the set of all idempotents in R is $\{\frac{0}{1}, \frac{1}{1}\}$. One easily checks that $\frac{9}{4}, \frac{9}{4} - \frac{1}{1} = \frac{5}{4} \notin U(R)$. This shows that $\frac{9}{4} \in R$ is not strongly clean, as required. □

Theorem 5.1.16. *Let I be a nil ideal of a ring R. Then R is strongly nil-clean if and only if R/I is strongly nil-clean.*

Proof. One direction is obvious. Conversely, assume that R/I is strongly nil-clean. Let $a \in R$. Then by Theorem 5.1.1. there exists some $m \in \mathbb{N}$ such that $(\overline{a - a^2})^m = \overline{0} \in R/I$. So we have $(a - a^2)^m \in I$. As I is nil, there exists some $n \in \mathbb{N}$ such that $(a - a^2)^{mn} = 0$. This complete the proof by using Theorem 5.1.1 again. □

Corollary 5.1.17. *A ring R is strongly nil-clean if and only if*

(1) *$2 \in R$ is nilpotent;*
(2) *$R/2R$ is strongly nil-clean.*

Proof. $\Longrightarrow$ Since $\overline{2} = \overline{2^2}$ in $R/N(R)$, $2 - 2^2$ is nilpotent, and so 2 is nilpotent. As $\frac{R/2R}{N(R)/2R} \cong R/N(R)$ is Boolean, it follows by Theorem 5.1.16 that $R/2R$ is strongly nil-clean.

$\Longleftarrow$ This is obvious, by Theorem 5.1.16. □

Corollary 5.1.18. *Let I be an ideal of a ring R. Then the following are equivalent:*

(1) *R/I is strongly nil-clean.*
(2) *R/I^n is strongly nil-clean for all $n \in \mathbb{N}$.*
(3) *R/I^n is strongly nil-clean for some $n \in \mathbb{N}$.*

Proof. (1) $\Rightarrow$ (2) For any $n \in \mathbb{N}$, $\frac{R/I^n}{I/I^n} \cong R/I$. Since I/I^n is a nil ideal of R and R/I is strongly nil-clean, then by Theorem 5.1.16, R/I^n is a strongly nil-clean ring.

(2) $\Rightarrow$ (3) This is trivial.

(3) $\Rightarrow$ (1) For any ideal I of R we have $\frac{R/I^n}{I/I^n} \cong R/I$, since every homomorphic image of a strongly nil-clean ring is strongly nil-clean, we conclude that R/I is strongly nil-clean. □

Example 5.1.19. Let n be an integer and $n \geq 2$. Then $\mathbb{Z}_n$ is strongly nil-clean if and only if $n = 2^m$ for some $m \in \mathbb{N}$.

Proof. $\Longrightarrow$ As $\mathbb{Z}_n$ is strongly nil-clean, by Corollary 5.1.17, $2 \in \mathbb{Z}_n$ is nilpotent. This shows that $n = 2^m$ for some $m \in \mathbb{N}$.

$\Longleftarrow$ Since $n = 2^m$, then $2 \in \mathbb{Z}_n$ is nilpotent. Also $\mathbb{Z}_{2^n}/J(\mathbb{Z}_{2^n}) \cong \mathbb{Z}_2$, and $J(\mathbb{Z}_{2^n}) = 2\mathbb{Z}_{2^n}$ is a nil ideal. As $\mathbb{Z}_2$ is strongly nil-clean, then by Theorem 5.1.16, $\mathbb{Z}_{2^n}$ is strongly nil-clean. □

A ring R a right (left) quasi-duo ring if every maximal right (left) ideal of R is an ideal. For instance, local rings, duo rings and weakly right (left) duo rings are all right (left) quasi-duo rings. Every abelian exchange ring is a right (left) duo ring. We now derive

Lemma 5.1.20. *A ring R is strongly nil-clean if and only if*

(1) *R is right (left) quasi-duo;*
(2) *R is nil-clean.*

Proof. $\Longrightarrow$ Obviously, R is nil-clean. In light of Corollary 5.1.6, $R/J(R)$ is commutative. Let M be a right (left) maximal ideal of R. Then $M/J(R)$ is an ideal of $R/J(R)$. Let $x \in M, r \in R$. Then $\overline{rx} \in M/J(R)$, and then $rx \in M + J(R) \subseteq M$. This shows that M is an ideal of R. Therefore R is right (left) quasi-duo.

$\Longleftarrow$ In light of Theorem 5.1.5, $R/J(R)$ is nil-clean and $J(R)$ is nil. Since R is right (left) quasi-duo, then so is $R/J(R)$. It follows by [561, Corollary 2.4] that $R/J(R)$ is abelian. Hence, $R/J(R)$ is strongly nil-clean. Accordingly, R is strongly nil-clean, by Theorem 5.1.16. □

The Brown-McCoy radical of R can be defined as the intersection of the maximal two-sided ideals, and denote it by $BM(R)$.

Theorem 5.1.21. *Let R be a ring. Then the following are equivalent:*

(1) *R is strongly nil-clean.*
(2) *$R/BM(R)$ is strongly nil-clean and $BM(R)$ is nil.*

Proof. (1) $\Rightarrow$ (2) For all maximal ideals M of R, $J(R) \nsubseteq M \Longrightarrow J(R)+M = R$; hence, $x+y=1$ for some $x \in J(R), y \in M$. This shows that $y = 1-x \in U(R)$, a contradiction. Thus, $J(R) \subseteq M$, and so $J(R) \subseteq BM(R)$. In view of Lemma 5.1.20, R is a right quasi-duo ring, i.e., every maximal right ideal is an ideal. Thus, $BM(R) \subseteq J(R)$. Hence, $BM(R) = J(R)$, and then proving (2) by Theorem 5.1.5.

(2) $\Rightarrow$ (1) This is obvious, in terms of Theorem 5.1.16. □

Theorem 5.1.22. *Let R be a ring. Then the following are equivalent:*

(1) *R is strongly nil-clean.*
(2) *Each prime factor ring of R is strongly nil-clean.*

Proof. (1) $\Rightarrow$ (2) This is obvious, as every homomorphic image of a strongly nil-clean ring is strongly nil-clean ring.

(2) $\Rightarrow$ (1) Suppose that R/P is strongly nil-clean for all prime ideal P of R. We shall prove R is strongly nil-clean. If not, by virtue of Theorem 5.1.1, there exists some $a \in R$ such that $a - a^2 \notin N(R)$. Set

$$\Omega = \{I \mid I \text{ is an ideal of } R \text{ such that } \overline{a-a^2} \notin N(R/I)\}.$$

Then $\Omega \neq \emptyset$, as $0 \in \Omega$. Given $I_1 \subseteq I_2 \subseteq \cdots$ in Ω, we let $I = \bigcup_{i=1}^{\infty} I_i$. Then I is an ideal of R. If $I \notin \Omega$, then $\overline{a-a^2} \notin N(R/I)$; hence, $(a-a^2)^m \in I$ for some $m \in \mathbb{N}$. Thus, there exists some $n \in \mathbb{N}$ such that $(a-a^2)^m \in I_n$, and then $\overline{a-a^2} \notin N(R/I_n)$. This contradicts $I_n \in \Omega$. We infer that $I \in \Omega$. Therefore Ω is inductive. By using Zorn's Lemma, there exists an ideal $Q \in \Omega$ which is maximal in Ω. By hypothesis, Q is not prime. Thus, we can find two ideals K and L of R such that

$$K, L \nsubseteq Q \text{ and } KL \subseteq Q.$$

If $K+Q=R$, then $L=KL+QL\subseteq Q$, a contradiction. Hence, $K+Q$ is an ideal of R. Likewise, $L+Q$ is an ideal of R. By the maximality, we see that $K+Q, L+Q\notin\Omega$. Thus, $\overline{a-a^2}\notin N(R/(K+Q))$ and $\overline{a-a^2}\notin N(R/(L+Q))$. Hence, we have $k,l\in\mathbb{N}$ such that $(a-a^2)^k\in K+Q$ and $(a-a^2)^l\in L+Q$. This implies that $(a-a^2)^{k+l}\in(K+Q)(L+Q)\subseteq KL+Q\subseteq Q$. Thus, $\overline{a-a^2}\in N(R/Q)$, a contradiction. Therefore R is strongly nil-clean. □

Corollary 5.1.23. *A ring R is strongly nil-clean if and only if $R/P(R)$ is strongly nil-clean.*

Proof. One direction is obvious. Conversely, assume that $R/P(R)$ is strongly nil-clean. Let P be a prime ideal of R. Since $R/P\cong\frac{R/P(R)}{P/P(R)}$, we see that R/P is strongly nil-clean. This completes the proof, by Theorem 5.1.22. □

Corollary 5.1.24. *Let R be a commutative ring. Then the following are equivalent:*

(1) *R is strongly nil-clean.*
(2) *$R/P\cong\mathbb{Z}_2$ for all prime ideals P of R.*

Proof. (1) $\Rightarrow$ (2) Let $P\in R$ be a prime ideal of R. Let $\overline{a}\in R/P$. Then $a-a^2\in N(R)$; hence, $a(1-a)\in P$. Thus, $\overline{a}=\overline{0}$ or $\overline{a}=\overline{1}$, as needed.

(2) $\Rightarrow$ (1) Clearly, $\mathbb{Z}_2$ is strongly nil-clean. This completes the proof, by Theorem 5.1.22. □

We come now to investigate various element-wise characterizations of strongly nil-clean rings. We derive

Theorem 5.1.25. *A ring R is strongly nil-clean if and only if*

(1) *$2\in R$ is nilpotent;*
(2) *For all $a\in R$, there exist $n\in\mathbb{N}, k\geq 0$ (depending on a) such that $a^n-a^{n+2^k}\in N(R)$.*

Proof. $\Longrightarrow$ As $2-2^2\in N(R)$, we see that $2\in N(R)$. Let $a\in R$. In view of Theorem 5.1.1, $a-a^2\in N(R)$. By choosing $n=1$ and $k=0$, we prove (2).

$\Longleftarrow$ By the work in Corollary 5.1.2, we can immediately assume we are working in $Z[a]$, and then we can mod out by the nil radical and assume that the ring is reduced. Then, we have $a^n(1-a)^{2^k}=0$, and so $a(1-a)=0$. Therefore R is strongly nil-clean. □

Corollary 5.1.26. *Let R be a ring. Then the following are equivalent:*

(1) *R is strongly nil-clean.*
(2) *For any $a \in R$, there exists n (depending on a) such that $a^n - a^{n+1} \in N(R)$.*

Proof. (1) $\Rightarrow$ (2) This is obvious by Theorem 5.1.1.
(2) $\Rightarrow$ (1) Choosing $k = 0$ in Theorem 5.1.25, we obtain the result. □

Lemma 5.1.27. *A ring R is strongly nil-clean if and only if*

(1) *$2 \in R$ is nilpotent;*
(2) *For any $a \in R$, there exists $k \geq 0$ (depending on a) such that $a^{2^k} - a^{2^{k+1}} \in N(R)$.*

Proof. $\Longrightarrow$ By virtue of Theorem 5.1.25, $2 \in R$ is nilpotent. Now assume that $a \in R$. Then $a - a^2 \in N(R)$.
$\Longleftarrow$ This is obvious, by Theorem 5.1.25. □

Theorem 5.1.28. *A ring R is strongly nil-clean if and only if*

(1) *$2 \in R$ is nilpotent;*
(2) *For any $a \in R$, there exists $k \geq 0$ (depending on a) such that $a^{2^k} \in R$ is strongly nil-clean.*

Proof. $\Longrightarrow$ This is clear.
$\Longleftarrow$ Let $a \in R$. Then there exists $k \geq 0$ such that $a^{2^k} \in R$ is strongly nil-clean. Write $a^{2^k} = e + w$ where $e^2 = e \in R, w \in N(R)$ and $we = ew$. Then $a^{2^{k+1}} = e^2 + (2e+w)w$, and so $a^{2^k} - a^{2^{k+1}} = \big(1-(2e+w)\big)w \in N(R)$. In light of Lemma 5.1.27, R is strongly nil-clean, as asserted. □

Corollary 5.1.29. *A ring R is strongly nil-clean if and only if*

(1) *$3 \in R$ is invertible;*
(2) *For any $a \in R$, $a^2 \in R$ is strongly nil-clean.*

Proof. $\Longrightarrow$ This is obvious.
$\Longleftarrow$ By hypothesis, there exists $e^2 = e \in R$ and $w \in N(R)$ such that $2^2 = e + w$ and $ew = we$. Hence, $2^4 = e + 2ew + w^2$. It follows that $2^2 - 2^4 = (1 - 2e - w)w \in N(R)$, i.e., $12 \in N(R)$. As $3 \in U(R)$, we get $2 \in N(R)$. In light of Theorem 5.1.28, R is strongly nil-clean. □

Corollary 5.1.30. *Let R be a ring in which $a^3 = a^5$ for all $a \in R$. Then R is strongly nil-clean if and only if $3 \in R$ is invertible.*

Proof. $\Longrightarrow$ This is obvious, by Corollary 5.1.29.

$\Longleftarrow$ Since $2^3 = 2^5$, it follows from $3 \in U(R)$ that $2^3 = 0$. Thus, $2 \in N(R)$. Since $a^3 = a^5$, we have $a^4 = a^6$. It is easy to verify that $(a^8)^2 = a^{16} = a^4a^{12} = a^4a^8 = a^6a^6 = a^4a^4 = a^8$. Thus, $a^{2^3} \in R$ is an idempotent, and so it is strongly nil-clean. According to Theorem 5.1.28, we are through. □

Finally, we are concerned with a class of constant exponents.

Lemma 5.1.31. *Let R be a ring, and let $m, n \in \mathbb{N}(m < n)$. Suppose that $a - a^{2^m+2^n} \in N(R)$ for any $a \in R$. Then $a - a^{2^{m+1}} \in N(R)$ for all $a \in R$.*

Proof. By hypothesis, $(-1)-(-1)^{2^m+2^n} \in N(R)$; hence, $2 \in N(R)$. For any $a \in R, a-a^2 \equiv (a-a^2)^{2^m+2^n} \equiv (a-a^2)^{2^m}(a-a^2)^{2^n} (mod\ N(R))$, Since $2 \in N(R)$, we have $(a-a^2)^{2^m}(a-a^2)^{2^n} \equiv (a^{2^m}-a^{2^{m+1}})(a^{2^n}-a^{2^{n+1}}) \equiv a^{2^n+2^m} - a^{2^m+2^{n+1}} - a^{2^{m+1}+2^n} + a^{2(2^m+2^n)} \equiv a - a^{2^m+1} - a^{2^n+1} - a^2 (mod\ N(R))$. This implies $a^{2^m+1} \equiv a^{2^n+1} (mod\ N(R))$, and so $a^{2^{m+1}} \equiv a^{(2^m+1)+(2^m-1)} \equiv a^{(2^n+1)+(2^m-1)} \equiv a^{2^m+2^n} \equiv a (mod\ N(R))$, as desired. □

Lemma 5.1.32. *Let R be a ring, and let $n \in \mathbb{N}$. Then the following are equivalent:*

(1) *R is strongly nil-clean.*
(2) *For any $a \in R$, $a - a^{2^n+2^{n+1}} \in N(R)$.*

Proof. (1) $\Rightarrow$ (2) Let $a \in R$. Then $a - a^{2^n+2^{n+1}} = (a - a^2) + a(a - a^2) + a^2(a - a^2) + \cdots + a^{2^n+2^{n+1}-2}(a - a^2)$. Since R is strongly nil-clean ring, by Theorem 5.1.1, $a - a^2 \in N(R)$. Accordingly, $a - a^{2^n+2^{n+1}} \in N(R)$.

(2) $\Leftarrow$ (1) By Lemma 5.1.31 and its proof, we have $a - a^{2^{n+1}}, a^{2^{n+1}+1} - a^{2^n+1} \in N(R)$. This shows that $a^2 \equiv a^{2^{n+1}+1} \equiv a^{2^n+1} (mod\ N(R))$. Furthermore, $a \equiv a^{2^{n+1}} \equiv a^{2^n+1}a^{2^n-1} \equiv a^2a^{2^n-1} \equiv a^{2^n+1} (mod\ N(R))$. Therefore $a - a^2 = (a - a^{2^n+1}) - (a^2 - a^{2^n+1}) \in N(R)$. By Theorem 5.1.1, R is strongly nil-clean. □

Theorem 5.1.33. *Let R be a ring, and let $m, n \in \mathbb{N}$. Then the following are equivalent:*

(1) *R is strongly nil-clean.*
(2) *For any $a \in R$, $a - a^{2^n+2^{m(n+1)}} \in N(R)$.*

Proof. (1) $\Rightarrow$ (2) Let $a \in R$. In view of Theorem 5.1.1, $a - a^2 \in N(R)$. Therefore $a-a^{2^n+2^{m(n+1)}} = (a-a^2)+a(a-a^2)+\cdots+a^{2^n+2^{m(n+1)}-2}(a-a^2) \in N(R)$, as desired.

(2) $\Rightarrow$ (1) By virtue of Lemma 5.1.31, $a - a^{2^{n+1}} \in N(R)$. Obviously, $2^{m(n+1)} - 2^{n+1} = (2^{m(n+1)} - 1) - (2^{n+1} - 1) = k(2^{n+1} - 1)$ for some $k \geq 0$. Then $a \equiv a^{2^n + 2^{m(n+1)}} \equiv a^{2^n + 2^{n+1} + k(2^{n+1}-1)} \equiv a^{2^n + 2^{n+1} + k \times 2^{n+1} - k} = a^{2^{n+1}} a^{2^n + k \times 2^{n+1} - k} \equiv a^{(2^{n+1})^k} a^{2^n + k \times 2^{n+1} - k} = a^{(2^{n+1})^k + 2^n - k} (a^{2^{n+1}})^k \equiv a^{(2^{n+1})^k + 2^n - k} a^k = a^{(2^{n+1})^k + 2^n} = a^{(2^{n+1})^k} a^{2^n} \equiv a^{2^n + 2^{n+1}} (mod\ N(R))$. In light of Lemma 5.1.32, R is strongly nil-clean. □

Corollary 5.1.34. *Let R be a ring, and let $n \in \mathbb{N}$ be even. Then the following are equivalent:*

(1) *R is strongly nil-clean.*
(2) *For any $a \in R$, $a - a^{2^n+2} \in N(R)$.*

Proof. (1) $\Rightarrow$ (2) This is obvious, by Theorem 5.1.1.

(2) $\Leftarrow$ (1) Write $n = 2k (k \geq 0)$. Then $2^n + 2 = 2 + 2^{k(1+1)}$. In light of Theorem 5.1.33, we complete the proof. □

Corollary 5.1.34 shows that a ring R is strongly nil-clean if and only if $a - a^6 \in N(R)$ for all $a \in R$. But the previous condition "n is even" is necessary. For instance, $\mathbb{Z}_3$ is not strongly nil-clean, while $a - a^4 \in N(\mathbb{Z}_3)$ for all $a \in \mathbb{Z}_3$.

Corollary 5.1.35. *Let R be a ring, and let $n \in \mathbb{N}$ be even. Then the following are equivalent:*

(1) *R is Boolean.*
(2) *For any $a \in R$, $a = a^{2^n+2}$.*

Proof. (1) $\Rightarrow$ (2) This is obvious.

(2) $\Rightarrow$ (1) In light of Corollary 5.1.34, R is strongly nil-clean. By virtue of Theorem 5.1.5, $N(R)$ forms an ideal of R and $R/N(R)$ is Boolean. By hypothesis, $N(R) = 0$, and therefore R is Boolean. □

5.2 Examples

By virtue of Theorem 5.1.1, we easily see that any quotient of a strongly nil-clean ring is strongly nil-clean. Also we see that any finite product of strongly nil-clean rings is strongly nil-clean. However, one can construct an infinite direct product of strongly nil-clean rings that is not strongly

nil-clean, e.g., $R = \prod_{n=1}^{\infty} \mathbb{Z}_{2^n}$. Choose $a = (0, 2, 2, 2, \ldots) \in R$. Then $a - a^2 \notin N(R)$.

Theorem 5.2.1. *Every subring of strongly nil-clean rings is strongly nil-clean.*

Proof. Let S be a subring of a strongly nil-clean ring R. For any $a \in S$, we see that $a \in R$. In view of Theorem 5.1.1, there exists some $n \in \mathbb{N}$ such that $(a-a^2)^n = 0$ in R; hence, $(a-a^2)^n = 0$ in S. By using Theorem 5.1.1 again, S is strongly nil-clean. □

As a consequence of Theorem 5.2.1, every center of a strongly nil-clean ring is strongly nil-clean. Let R be strongly nil-clean, and let $e \in R$ be an idempotent. Then eRe is strongly nil-clean. Furthermore, we have

Corollary 5.2.2. *Every finite subdirect product of strongly nil-clean rings is strongly nil-clean.*

Proof. Let R be the subdirect product of strongly nil-clean rings $R_1, \ldots, R_n$. Then R is isomorphic to the subring of $R_1 \times \cdots \times R_n$. In light of Theorem 5.2.1, R is strongly nil-clean, as required. □

Example 5.2.3. Let R be a ring. Set $S = \{(x, y) \in R \times R \mid x - y \in J(R)\}$, which is a subring of $R \times R$. Then R is strongly nil-clean if and only if S is strongly nil-clean.

Proof. $\Longrightarrow$ Since S is a subring of $R \times R$, it is strongly nil-clean by Theorem 5.2.1.

$\Longleftarrow$ This is obvious, as R is a homomorphic image of S. □

Note that "strongly nil-clean" is not a Morita invariant. Let R be an arbitrary ring. Then $M_2(R)$ is not strongly nil-clean, as $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in M_2(R)$ is not strongly nil-clean.

Theorem 5.2.4. *Let R be a ring. Then the following are equivalent:*

(1) *R is strongly nil-clean.*
(2) *$T_n(R)$ is strongly nil-clean for some $n \in \mathbb{N}$.*
(3) *$T_n(R)$ is strongly nil-clean for all $n \in \mathbb{N}$.*

Proof. (1) $\Rightarrow$ (3) For any $(a_{ij}) \in T_n(R)$, we see that each $a_{ii} - a_{ii}^2 \in N(R)$. Write $\left(a_{ii} - a_{ii}^2\right)^{m_i} = 0$ for some $m_i \in \mathbb{N}$. Then $\left((a_{ij}) - (a_{ij})^2\right)^{n\left(\sum\limits_{i=1}^{n} m_i\right)} = 0$. According to Theorem 5.1.1, $T_n(R)$ is strongly nil-clean.

(3) $\Rightarrow$ (2) This is trivial.

(2) $\Rightarrow$ (1) Choose $e = diag(1, 0, \ldots, 0) \in T_n(R)$. Then $R \cong eT_n(R)e$. By virtue of Theorem 5.2.1, we complete the proof. □

Let R be a ring, and let $S_n = \{(a_{ij} \in T_n(R) \mid a_{11} = a_{22} = \cdots = a_{nn}\}$. Analogously, we deduce that R is strongly nil-clean if and only if $S_n(R)$ is strongly nil-clean for all $n \in \mathbb{N}$.

Corollary 5.2.5. *Let R be a ring. Then the following are equivalent:*

(1) *R is strongly nil-clean.*

(2) *The ring* $\left\{ \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \;\middle|\; \text{each } a_{ij} \in R \right\}$ *is strongly nil-clean.*

(3) *The ring* $\left\{ \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \;\middle|\; \text{each } a_{ij} \in R \right\}$ *is strongly nil-clean.*

Proof. This is obvious, by Theorem 5.2.4 and Theorem 5.2.1. □

Corollary 5.2.6. *Let R be a ring. Then the following are equivalent:*

(1) *R is strongly nil-clean.*
(2) *$\mathcal{T}(R)$ is strongly nil-clean.*

Proof. Construct a map

$$\varphi : \mathcal{T}(R) \to \left\{ \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \;\middle|\; \text{each } a_{ij} \in R \right\}$$

given by $\varphi(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} A \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ for any $A \in \mathcal{T}(R)$. One easily checks that φ is a ring isomorphism. Therefore we complete the proof, by Corollary 5.2.5. □

Let $\mathbb{Z}_4 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$. One directly verifies that $\mathbb{Z}_4$ is strongly nil-clean. According to Theorem 5.2.4, Corollary 5.2.5 and Corollary 5.2.6, the rings

$$\begin{pmatrix} \mathbb{Z}_4 & 0 & 0 \\ \mathbb{Z}_4 & \mathbb{Z}_4 & 0 \\ \mathbb{Z}_4 & \mathbb{Z}_4 & \mathbb{Z}_4 \end{pmatrix}, \begin{pmatrix} \mathbb{Z}_4 & 0 & 0 \\ 0 & \mathbb{Z}_4 & 0 \\ \mathbb{Z}_4 & 0 & \mathbb{Z}_4 \end{pmatrix}, \begin{pmatrix} \mathbb{Z}_4 & 0 & 0 \\ \mathbb{Z}_4 & \mathbb{Z}_4 & \mathbb{Z}_4 \\ 0 & 0 & \mathbb{Z}_4 \end{pmatrix}$$

are all strongly nil-clean, but the full matrix ring $M_3(\mathbb{Z}_4)$ is not strongly nil-clean.

Theorem 5.2.7. *Let R be a ring. Then the following are equivalent:*

(1) *R is strongly nil-clean.*
(2) *$R[[x]]/(x^n)$ is strongly nil-clean for all $n \in \mathbb{N}$.*
(3) *$R[[x]]/(x^n)$ is strongly nil-clean for some $n \in \mathbb{N}$.*

Proof. (1) $\Rightarrow$ (2) Set $S = R[[x]]/(x^n)$. Then $S = \left\{ \sum_{i=0}^{n-1} a_i x^i \;\middle|\; a_0, \ldots, a_{n-1} \in R, x^n = 0 \right\}$. For any $a(x) = \sum_{i=0}^{n-1} a_i x^i \in S$, we see that $a(x) - a^2(x) = (a_0 - a_0^2) + bx$ for some $b \in R$. As $a_0 - a_0^2 \in N(R)$, we have some $m \in \mathbb{N}$ such that $(a_0 - a_0^2)^m = 0$, and so $\big(a(x) - a^2(x)\big)^{2m+1} = \big((a_0 - a_0^2) + bx\big)^{2m+1} = 0$. Thus, $a(x) - a^2(x) \in N(S)$. This completes the proof by Theorem 5.1.1.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) For any $a \in R$, we see that $a \in S$, and so $a - a^2 \in N(S)$. This implies that $a - a^2 \in N(R)$. Therefore R is strongly nil-clean. □

Clearly, $R[[x]]/(x^n) \cong R[x]/(x^n)$, we deduce that a ring R is strongly nil-clean if and only if $R[x]/(x^n)$ is strongly nil-clean.

Set $M_0 = \{m \mid mN \subseteq J(B)$ and $N_0 = \{n \mid nM \subseteq J(A)\}$. We note that N/N_0 is an $(A/J(A), B/J(B))$-bimodule and M/M_0 is a $(B/J(B), A/J(A))$-bimodule. Further, we derive

Lemma 5.2.8. *Let T be the ring of a Morita context $\begin{pmatrix} A & N \\ M & B \end{pmatrix}$. Then* $T/J(T) \cong \begin{pmatrix} A/J(A) & N/N_0 \\ M/M_0 & B/J(B) \end{pmatrix}$.

Proof. Construct $\pi : T \to \begin{pmatrix} A/J(A) & N/N_0 \\ M/M_0 & B/J(B) \end{pmatrix}, \begin{pmatrix} a & n \\ m & b \end{pmatrix} \mapsto \begin{pmatrix} \overline{a} & \overline{n} \\ \overline{m} & \overline{b} \end{pmatrix}$. Then φ is a surjective ring morphism. Therefore the result follows, as $Ker(\varphi) = J(T)$ by Lemma 5.2.8. □

Lemma 5.2.9. *Let T be the ring of a Morita context $(A, B, N, M, \psi, \varphi)$. Then $T/J(T)$ is Boolean if and only if $A/J(A)$ and $B/J(B)$ are Boolean, $MN \subseteq J(A)$ and $NM \subseteq J(B)$.*

Proof. $\Longrightarrow$ Let $\psi(n,m) \in NM$. Then $\overline{\begin{pmatrix} 0 & n \\ m & 0 \end{pmatrix}} \in T/J(T)$ is an idempotent. Hence, $\begin{pmatrix} \psi(n,m) & 0 \\ 0 & \varphi(m,n) \end{pmatrix} \in J(T)$. This implies that $NM \subseteq J(A)$. Likewise, $MN \subseteq J(B)$. Thus, $N_0 = N$ and $M_0 = M$. Accordingly, $T/J(T) \cong A/J(A) \times B/J(B)$ is Boolean. Therefore $A/J(A)$ and $B/J(B)$ are Boolean, as desired.

$\Longleftarrow$ Since $MN \subseteq J(A)$ and $NM \subseteq J(B)$, we see that $N_0 = N$ and $M_0 = M$. Hence, $T/J(T) \cong A/J(A) \times B/J(B)$. Therefore $T/J(T)$ is Boolean, hence the result. □

Lemma 5.2.10. *Let T be the ring of a Morita context $(A, B, N, M, \psi, \varphi)$. If T is strongly nil-clean, then A and B are strongly nil-clean, $NM \subseteq J(A)$ and $MN \subseteq J(B)$.*

Proof. It follows by Theorem 5.2.1 that A and B are strongly nil-clean. By virtue of Theorem 5.1.5, $T/J(T)$ is Boolean. This completes proof by Lemma 5.2.9. □

Theorem 5.2.11. *Let T be the ring of a Morita context $(A, B, N, M, \psi, \varphi)$. If NM and MN are nilpotent ideals of A and B respectively, then the following are equivalent:*

(1) *T is strongly nil-clean.*
(2) *A and B are strongly nil-clean.*

Proof. (1) $\Rightarrow$ (2) This is obvious, by Lemma 5.2.10.

(2) $\Rightarrow$ (1) In view of Theorem 5.1.5, $A/J(A)$ and $B/J(B)$ are Boolean. As $NM \subseteq J(A)$ and $MN \subseteq J(B)$, it follows by Lemma 5.2.9, $T/J(T)$ is Boolean.

Let $\begin{pmatrix} a & n \\ m & b \end{pmatrix} \in J(T)$. Then $a \in J(A)$ and $b \in J(B)$. In view of Theorem 5.1.5, $J(A)$ and $J(B)$ are nil. Thus, we can find $q \in R$ such that $a^q = 0$ in A and $b^q = 0$ in B. Set $NM := Im(\psi)$ and $MN := Im(\varphi)$. Then

$$\begin{pmatrix} a & n \\ m & b \end{pmatrix}^{q+1} \subseteq \begin{pmatrix} NM & N \\ M & MN \end{pmatrix}.$$

Clearly,

$$\begin{pmatrix} NM & N \\ M & MN \end{pmatrix}^2 = \begin{pmatrix} NM & (NM)N \\ (MN)M & MN \end{pmatrix}.$$

For any $j \in \mathbb{N}$, one easily checks that

$$\begin{pmatrix} NM & N \\ M & MN \end{pmatrix}^{2j} = \begin{pmatrix} NM & (NM)N \\ (MN)M & MN \end{pmatrix}^{j} = \begin{pmatrix} (NM)^j & (NM)^j N \\ (MN)^j M & (MN)^j \end{pmatrix}.$$

By hypothesis, we may assume that $(NM)^p = 0$ in A and $(MN)^p = 0$ in B. Then

$$\begin{pmatrix} NM & N \\ M & MN \end{pmatrix}^{2p} = 0.$$

Therefore $\begin{pmatrix} a & n \\ m & b \end{pmatrix}^{2p(q+1)} = 0$, and so $J(T)$ is nil. In light of Theorem 5.1.5, R is strongly nil-clean, as asserted. □

Consider the Morita context $R = \begin{pmatrix} \mathbb{Z}_4 & \mathbb{Z}_4 \\ 2\mathbb{Z}_4 & \mathbb{Z}_4 \end{pmatrix}$, where the context products are the same as the product in $\mathbb{Z}_4$. Then we claim that R is strongly nil-clean. Since $\mathbb{Z}_4$ is strongly nil-clean, and so we are done by Theorem 5.2.11.

Corollary 5.2.12. *Let R be a ring, and let $s \in C(R)$. Then the following are equivalent:*

(1) *R is strongly nil-clean and $s \in J(R)$.*
(2) *$M_{(s)}(R)$ is strongly nil-clean.*

Proof. (1) $\Rightarrow$ (2) Since R is strongly nil-clean, it follows by Theorem 5.1.5 that $J(R)$ is nil. Hence, $s \in C(R) \cap N(R)$. Therefore, proving (2) by Theorem 5.1.11.

(2) $\Rightarrow$ (1) Choose $e = diag(1, 0) \in M_{(s)}(R)$. Then $R \cong eM_{(s)}(R)e$. In view of Theorem 5.2.1, R is strongly nil-clean. In light of Lemma 5.2.10, $s \in J(R)$, as required. □

As $\mathbb{Z}_4$ is strongly nil-clean, it follows from Corollary 5.2.12 that the generalized matrix ring $M_{(2)}(\mathbb{Z}_4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{Z}_4 \right\}$ is strongly nil-clean.

Corollary 5.2.13. *Let R be a ring, and let $m, n \in \mathbb{N}$. Then the following are equivalent:*

(1) *R is strongly nil-clean.*
(2) *$M_{(x^m)}\big(R[[x]]/(x^n)\big)$ is strongly nil-clean.*
(3) *$M_{(x^m)}\big(R[x]/(x^n)\big)$ is strongly nil-clean.*

Proof. Obviously, $R[[x]]/(x^n) \cong R[x]/(x^n)$, hence it suffices to show that (1) $\Leftrightarrow$ (2). One easily checks that $x^m \in N\big(R[[x]]/(x^n)\big)$. In addition, it is central. In light of Corollary 5.2.12, we complete the proof. □

For instance, $M_{(x^m)}\big(\mathbb{Z}_4[x]/(x^n)\big)(m, n \in \mathbb{N})$ is strongly nil-clean.

Let $s \in C(R)$ and $n \geq 2$. The generalized matrix ring $M_n(R; s)$ is the set of all $n \times n$ matrix over R with usual addition of matrices and with multiplication defined as follows:

$$(a_{ij})(c_{ij}) = (c_{ij}), c_{ij} = \sum_{k=1}^{n} s^{\delta_{ikj}} a_{ik} b_{kj},$$

where $\delta_{ikj} = 1 + \delta_{ik} - \delta_{kj} - \delta_{ji}$ and $\delta_{ik}, \delta_{kj}, \delta_{ji}$ are Kronecker delta symbols (see [505]).

Theorem 5.2.14. *Let R be a strongly nil-clean ring, and let $s \in C(R) \cap N(R)$. Then $M_n(R; s)$ is strongly nil-clean for all $n \in \mathbb{N}$.*

Proof. Clearly, $s^2 \in N(R) \cap C(R)$, so by Corollary 5.2.12, $M_{(s^2)}(R)$ is strongly nil-clean. It is obvious by definition that $M_{(s^2)}(R) = M_2(R; s)$, hence $M_2(R; s)$ is strongly nil-clean. Thus, the assertion is true for $n = 2$.

Now assume that $n \gneqq 2$ and the claim holds for $M_{n-1}(R; s)$ and that $M_{n-1}(R; s)$ has nil Jacobson radical. Let $A = M_{n-1}(R; s)$. Then $M_n(R; s) = \begin{pmatrix} A & M \\ N & R \end{pmatrix}$ is a Morita context, where $M = \begin{pmatrix} M_{1n} \\ \vdots \\ M_{(n-1)n} \end{pmatrix}$ and $N = (M_{n1} \cdots M_{n(n-1)})$ with $M_{in} = M_{ni} = R$ for all $i = 1, \ldots, n-1$, $\psi : N \otimes M \to R, y \otimes x \mapsto syx$ and $\varphi : M \otimes N \to A, x \otimes y \mapsto sxy$. Moreover for $x = \begin{pmatrix} x_{1n} \\ \vdots \\ x_{(n-1)n} \end{pmatrix} \in M$ and $y = (y_{n1} \cdots y_{n(n-1)}) \in N$, we have

$$xy = \begin{pmatrix} s^2 x_{1n} y_{n1} & s x_{1n} y_{n2} & \cdots & s x_{1n} y_{n,n-1} \\ s x_{2n} y_{n1} & s^2 x_{2n} y_{n2} & \cdots & s x_{2n} y_{n,n-1} \\ \vdots & \vdots & \vdots & \\ s x_{n-1,n} y_{n1} & s x_{n-1,n} y_{n2} & \cdots & s^2 x_{n-1,n} y_{n,n-1} \end{pmatrix} \in sA.$$

and

$$yx = s^2y_{n1}x_{1n} + s^2y_{n2}x_{2n} + \cdots s^2y_{n,n-1}x_{n-1,n} \in s^2R.$$

Thus, MN and NM are nilpotent.By induction and Theorem 5.2.11, $M_n(R;s)$ is strongly nil-clean, as desired. □

Corollary 5.2.15. *$M_n\big(\mathbb{Z}_4[x]/(x^n);x^m\big)$ is strongly nil-clean.*

Proof. This is obvious by Theorem 5.2.14. □

We now direct our attention to a subclass of strongly nil-clean rings which are of bounded index 2. A ring R is said to be Boolean-like provided that $(a-a^2)(b-b^2)=0$ for all $a,b\in R$.

Theorem 5.2.16. *A ring R is Boolean-like if and only if*

(1) *R is strongly nil-clean;*
(2) *$\alpha\beta=0$ for all nilpotent $\alpha,\beta\in R$.*

Proof. Suppose that R is Boolean-like. For any $a\in R$, $(a-a^2)^2=0$, and so $a-a^2\in N(R)$. Thus, R is strongly nil-clean by Theorem 5.1.1. For any nilpotents $\alpha,\beta\in R$, we can find some $m,n\in\mathbb{N}$ such that $\alpha^m=\beta^n=0$. But $\alpha^2=\alpha^3f(\alpha)$ for some $f(t)\in\mathbb{Z}[t]$. Thus, $\alpha^2=0$. Likewise, $\beta^2=0$. This shows that $\alpha\beta=(\alpha-\alpha^2)(\beta-\beta^2)=0$.

Conversely, assume that (1) and (2) hold. For any $a\in R$, $a-a^2$ is nilpotent. Hence for any $a,b\in R$, $(a-a^2)(b-b^2)=0$. Therefore R is Boolean-like. □

Corollary 5.2.17. *A ring R is Boolean-like if and only if*

(1) *R is strongly nil-clean;*
(2) *$J(R)^2=0$.*

Proof. $\Longrightarrow$ By virtue of Theorem 5.2.16, R is strongly nil-clean. It follows by Theorem 5.1.5 that $J(R)$ is nil. Let $a,b\in J(R)$. Then $ab=0$, by Theorem 5.2.16. Thus proving (2).

$\Longleftarrow$ Let $a,b\in R$. In light of Theorem 5.1.5, $R/J(R)$ is Boolean; hence, $a-a^2,b-b^2\in J(R)$. Therefore $(a-a^2)(b-b^2)\in J(R)^2=0$, as desired. □

Let M be an R-R-bimodule. The trivial extension of R by M is the ring $T(R,M)$ of pairs (r,m), where $r\in R$ and $m\in M$, and with the usual

addition and multiplication given by $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$ for $r_1, r_2 \in R$ and $m_1, m_2 \in M$. Let R be a strongly nil-clean ring, and let $a \in R$. Then there exists an idempotent $e \in R$ such that $a - e \in N(R)$ and $ae = ea$. We denote such an idempotent e by a_B and $a - a_B$ by a_N. Clearly, every Boolean ring is an abelian Boolean-like ring, but the converse is not true, e.g., $\mathbb{Z}_4$. We now give the relationship between Boolean-like rings and Boolean rings.

Corollary 5.2.18. *An abelian ring is Boolean-like if and only if it is isomorphic to a trivial extension of a Boolean ring.*

Proof. Let R be an abelian Boolean-like ring. Let $a, b \in N(R)$, and let $c \in R$. In view of Theorem 5.2.16, $a^2 = b^2 = ab = ba = 0$. Hence, $(a+b)^2 = a^2 + ab + ba + b^2 = 0$, and so $a + b \in N(R)$. Write $c = c_B + c_N$. As $c_B \in R$ is central, we see that $ac = ac_B + ac_N \in N(R)$, and thus $N(R)$ forms an ideal of R. Further, $R/N(R)$ is Boolean. For any $\overline{r} \in R/N(R), x \in N(R)$, we define $\overline{r} \cdot x = rx$. As $N^2(R) = 0$, we know that $N(R)$ is a left $R/N(R)$-module. Likewise, $N(R)$ is a right $R/N(R)$-module. Construct a map $\varphi : R \to T\big(R/N(R), N(R)\big)$ given by $\varphi(r) = \big(\overline{r}, r_N\big)$. If $r = r'$, then $r_B + r_N = r'_B + r'_N$, and so $r_B - r'_B = r'_N - r_N \in N(R)$. As R is abelian, $\big(r_B - r'_B\big)^3 = r_B - r'_B$; hence, $r_B = r'_B$. This implies that $r_N = r'_N$. Thus, φ is well defined. Further, φ is a ring morphism. If $\big(\overline{r}, r_N\big) = 0$, then $r \in N(R)$ and $r_N = 0$, and so $r = r_N = 0$. This implies that φ is injective. For any $\big(\overline{s}, d\big) \in T\big(R/N(R), N(R)\big)$, we choose $s_B + d \in R$. Then $\varphi(s_B + d) = \big(\overline{s_B + d}, (s_B + d)_N\big) = \big(\overline{s}, d\big)$, and so φ is surjective. Therefore $\varphi : R \cong T\big(R/N(R), N(R)\big)$.

Conversely, assume that $\varphi : R \cong T(S, M)$ where S is a Boolean ring and M is an S-S-bimodule. For any $(s, m), (t, n) \in T(S, M)$, we see that $\big((s, m) - (s, m)^2\big)\big((t, n) - (t, n)^2\big) = (s - s^2, *)(t - t^2, *) = (0, *)(0, *) = (0, 0)$. Accordingly, $T(S, M)$ is Boolean-like, as asserted. □

Let I be a right ideal of a ring R. We say that every nilpotent lifts modulo I in case if $x^m \in I$ for some $m \in \mathbb{N}$ then there exists $w \in N(R)$ such that $x \equiv w\big(mod\ I\big)$.

Lemma 5.2.19. *Let R be periodic. Then every nilpotent lifts modulo any right ideal of R.*

Proof. Let I be a right ideal of R, and let $x \in R$ be such that $x^m \in I (m \in \mathbb{N})$. In view of Lemma 5.1.11, we have some $n \geq 2$ such that $x - x^n \in N(R)$.

If $n \geq m$, then $x^n \in I$. Set $w = x - x^n$. Then $x - w = x^n \in I$, and $w \in N(R)$. If $n < m$, then we find some $k \geq m$ such that $x - x^{k+n} = (x - x^n) + x^{n-1}(x - x^n) + x^{2n-2}(x - x^n) + \cdots + a^k(x - x^n) \in N(R)$. Set $w = x - x^{k+n}$. Then $x \equiv w(mod\ I)$, and $w \in N(R)$, thus completing the proof. □

We next characterize Boolean-like rings by the nilpotency.

Proposition 5.2.20. *A ring R is Boolean-like if and only if*

(1) $N(R) = \{x \in R \mid 1 - x \in U(R)\ \}$;
(2) $\alpha\beta = 0$ *for all nilpotent* $\alpha, \beta \in R$;
(3) *Every nilpotent lifts modulo any right ideal of* R.

Proof. Suppose R is Boolean-like. In view of Theorem 5.2.16, R is strongly nil-clean, and $\alpha\beta = 0$ for all nilpotent $\alpha, \beta \in R$. By Proposition 5.1.4, we get $N(R) = \{x \in R \mid 1 - x \in U(R)\ \}$. According to Theorem 5.1.12, R is periodic. Therefore, it follows by Lemma 5.2.19 that every nilpotent lifts modulo any right ideal of R.

Conversely, assume that (1), (2) and (3) hold. For any $a \in R$, we see that $a^3 \in a^3R$, and then $a - w \in a^3R$ for some $w \in N(R)$. Write $a - w = a^3r$. Then $a - a^3r = w \in N(R)$. By hypothesis, $(a - a^3r)^2 = 0$. This shows that $a^2 = a^3(ar + ra + ra^3r)$. As a result, R is strongly π-regular. In light of Theorem 1.1.2, R is strongly clean. By virtue of Proposition 5.1.4, R is strongly nil-clean. Accordingly, R is Boolean-like by Theorem 5.2.16. □

Let R be Boolean-like. Then for any $a \in R$ then $a^2 \in R$ is an idempotent. For any $a \in R$, we have $a(1 - a)(-a)(1 + a) = 0$, and thus, $a^2(1 - a^2) = 0$. Therefore $a^4 = a^2$, and we are through. We also note that $N(R) = \{a - a^2 \mid a \in R\}$. For a commutative ring R, we note that R is Boolean-like if and only if there exists $b \in R$ such that $xy(1-bx)(1-by) = 0$ for all $x, y \in R$, if and only if for any $u \in U(R)$, $xy(1 - ux)(1 - uy) = 0$ for all $x, y \in R$ (see [19, Theorem 3]). Moreover, we have

Proposition 5.2.21. *Let R be a commutative ring. Then R is Boolean-like if and only if*

(1) *For any $a \in R$, $a^2 \in R$ is an idempotent*;
(2) $R/M \cong \mathbb{Z}_2$ *for all maximal ideals M of R*;
(3) $\alpha\beta = 0$ *for all nilpotent* $\alpha, \beta \in R$.

Proof. Suppose R is Boolean-like. Then (1) and (3) are obvious. In light of Theorem 5.2.16, R is strongly nil-clean. Thus, (2) follows by Corollary 5.1.24.

Conversely, assume that (1), (2) and (3) hold. For any $a \in J(R)$, we have $a^2 = a^4$, and so $a^2(1-a^2) = 0$. This shows that $a^2 = 0$. Thus, $J(R)$ is nil. Let M be a maximal of R. For any $x \in R$, $x\overline{x} = \overline{x^2}$; hence, $x - x^2 \in M$. This implies that $x - x^2 \in J(R)$, i.e., $R/J(R)$ is Boolean. In view of Theorem 5.1.5, R is strongly nil-clean. Therefore we complete the proof by Theorem 5.2.16. □

We end this section by listing more examples of Boolean-like rings.

Example 5.2.22. Let $R = \left\{ \begin{pmatrix} a & 2b \\ 0 & c \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{Z}_4 \right\}$. Then for any $x, y \in R$, $(x - x^2)(y - y^2) = 0$. Thus, R is Boolean-like.

Example 5.2.23. Let R be the ring

$$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\},$$

where $0, 1 \in \mathbb{Z}_2$. Then R is a Boolean-like ring.

Example 5.2.24. Let $R = M_{(0)}(\mathbb{Z}_2)$. For any $A, B \in R$, $(A - A^2)(B - B^2) = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} = 0$. Therefore R is Boolean-like.

Example 5.2.25. The ring H_4 is defined by the following tables:

+	0	1	p	q
0	0	1	p	q
1	1	0	q	p
p	p	q	0	1
q	q	p	1	0

,

×	0	1	p	q
0	0	0	0	0
1	0	1	p	q
p	0	p	0	p
q	0	q	p	1

.

Then H_4 is a Boolean-like ring with $char(H_4) = 2$.

5.3 Unique Decompositions

The purpose of this section is to study the unique decompositions for strongly nil-clean rings. An element $a \in R$ is called uniquely nil-clean if there exists a unique idempotent $e \in R$ such that $a - e \in R$ is a nilpotent.

A ring R is uniquely nil-clean provided that every element in R is uniquely nil-clean. It is shown that uniquely nil-clean rings forms a subclass of strongly nil-clean rings. We start by

Lemma 5.3.1. *Let R be a ring. Then the following are equivalent:*

(1) *R is strongly π-regular.*
(2) *For any $a \in R$, there exist $e = e^2 \in R, u \in U(R)$ such that $ea = eu, (1-e)a \in N(R)$ and e, u, a commutate.*
(3) *For any element $a \in R$, there exist $e = e^2 \in R, u \in U(R)$ and $w \in N(R)$ such that $a = eu + w$ and e, u, w commutate.*

Proof. (1) $\Rightarrow$ (2) Let $a \in R$. Then there exist $n \in \mathbb{N}, x \in R$ such that $a^n = a^{n+1}x, ax = xa$. Set $w = (a - a^2x) + (a - a^2x)ax + \cdots + (a - a^2x)a^{n-2}x^{n-2}$ and $z = a^n x^{n-1}$. Then $(a-a^2x)^n = a^n(1-ax)^n = a^n(1-ax)(1-ax)^{n-1} = 0$. That is, $w \in R$ is nilpotent. Since $z = zxz$ and $xz = zx$, we see that $z = zvz, vz = zv$ and $v = v^2z$, where $v = xzx$. Set $u = z + 1 - zv$. Then $u^{-1} = v + 1 - zv$. Moreover, we have $z = eu$, where $e = zv = zxzx = a^n x^n \in R$ is an idempotent. We check that $ea = a^{n+1}x^n = z = eu$ and $(1-e)a = (1-e)(eu + w) = (1-e)w \in N(R)$, as desired.

(2) $\Rightarrow$ (3) For any element $a \in R$, there exist $e = e^2 \in R, u \in U(R)$ such that $ea = eu, (1-e)a \in N(R)$ and e, u, a commutate. Set $w = (1-e)a$. Then $a = ea + (1-e)a = eu + w$, as required.

(3) $\Rightarrow$ (1) For any element $a \in R$, there exist $e = e^2 \in R, u \in U(R)$ and $w \in N(R)$ such that $a = eu + w$ and e, u, w commutate. Let $x = eu$. Then $x = x^2u^{-1}, xu^{-1} = u^{-1}x$ and $wu^{-1} = u^{-1}w$. One easily checks that $a - a^2u^{-1} = (w + eu) - (w + eu)^2u^{-1} = w - w(2e + wu^{-1})$ is nilpotent. Clearly, $au^{-1} = u^{-1}a$. Thus, we can find some $n \in \mathbb{N}$ such that $a^n \in a^{n+1}R$. Therefore R is strongly π-regular. □

We record an elementary characterization of abelian strongly π-regular rings which is due to Badawi [43, Theorem 3].

Theorem 5.3.2. *Let R be abelian. Then R is strongly π-regular if and only if*

(1) *$N(R)$ forms an ideal of R;*
(2) *$R/N(R)$ is regular.*

Proof. Suppose that R is strongly π-regular. Let $x, y \in N(R)$ and $r \in R$. Suppose $a := rx \notin N(R)$. In view of Lemma 5.3.1, we can find an idempotent $e \in R$ such that $ea = eu$ and $(1-e)a \in N(R)$, and so $erx = rex = eu$.

If $e = 0$, then $(1-e)a = a \notin N(R)$, a contradiction. Thus, $e \neq 0$. Clearly, $0 \neq ex \in N(R)$. Let n be the smallest integer such that $(ex)^n = 0$. Then $n \geq 2$. Accordingly, $u(ex)^{n-1} = (eu)(ex)^{n-1} = r(ex)^n = 0$, and then $(ex)^{n-1} = 0$. This gives a contradiction. Therefore $rx \in N(R)$. Likewise, $rx \in N(R)$. Suppose $x + y \notin N(R)$. In light of Lemma 5.3.1, there exists a nonzero idempotent $f \in R$ and a unit $v \in R$ such that $f(x+y) = fv$ and $(1-f)(x+y) \in N(R)$. Hence, $fx = fv - fy = fv\big(1 - v^{-1}y\big)$. By the consideration above, $v^{-1}y \in N(R)$, and so $1 - v^{-1}y \in U(R)$. Thus, $f = fx\big(1-v^{-1}y\big)^{-1}v^{-1} \in N(R)$; whence, $f = 0$. This gives a contradiction. Hence, $x + y \in N(R)$, and so $N(R)$ forms an ideal of R.

Let $\overline{x} \in R/N(R)$. Then there exist $n \in \mathbb{N}, y \in R$ such that $x^n = x^n y x^n$. Set $e = 1 - x^n y$. Then $e \in R$ is an idempotent. Hence, $\big((1-e)x\big)^n = (1-e)x^n = 0$, and so $(1-e)x \in N(R)$. This implies that $\overline{x} = \overline{ex} = \overline{x}(\overline{x^{n-1}y})\overline{x}$. Therefore $R/N(R)$ is regular.

Conversely, assume that (1) and (2) hold. Let $a \in R$. Then $\overline{a} \in R/N(R)$ is regular. Write $\overline{a} = \overline{a}\overline{b}\overline{a}$. Set $\overline{c} = \overline{b}\overline{a}\overline{b}$. Then $\overline{a} = \overline{aca}$ and $\overline{c} = \overline{cac}$. Set $\overline{e} = \overline{ac}$. Then $\overline{e} \in R/N(R)$ is an idempotent. As every idempotent lifts modulo $N(R)$, we may assume that $e \in R$ is an idempotent. Clearly, $\big(\overline{1} - \overline{e} + \overline{a}\big)\big(\overline{1} - \overline{e} + \overline{c}\big) = \overline{1}$. Since R is abelian, then so is $R/N(R)$, and thus $\overline{u} := \overline{1} - \overline{e} + \overline{c} \in N(R)$ is invertible. As $N(R) \subseteq J(R)$, every unit lifts modulo $N(R)$, and so we may assume that $u \in U(R)$. Set $w = a - eu$. Then $a = eu + w$ with $w \in N(R)$. Hence, $ea = e(u+w)$, and so $ea(u+w)^{-1} = e$. We infer that $ea(u+w)^{-1}ea = ea$, and then $ea = a^2e(u+w)^{-1}$. Further, $(1-e)a = (1-e)w \in N(R)$. Write $\big((1-e)a\big)^n = 0$. Then $(1-e)a^n = 0$, and so $a^n = ea^n = a^{n-1}(ea) = a^{n-1}\big(a^2e(u+w)^{-1}\big) = a^{n+1}e(u+w)^{-1}$. This establishes the proof. □

Theorem 5.3.3. *A ring R is uniquely nil-clean if and only if*

(1) *R is abelian;*
(2) *R is strongly nil-clean.*

Proof. $\Longrightarrow$ For any idempotent $e \in R$ and any $a \in R$, $e + ea(1-e) \in R$ is an idempotent. Since we see that $\big(e + ea(1-e)\big) + 0 = e + ea(1-e)$, by the uniqueness, $ea(1-e) = 0$; hence, $ea = eae$. Likewise, $ae = eae$, and so $ea = ae$. Thus, R is abelian. This implies that R is strongly nil-clean.

$\Longleftarrow$ For any $a \in R$, there exist $e^2 = e \in R, w \in N(R)$ such that $a = e + w$. Write $a = f + v, f = f^2 \in R, v \in N(R)$. In light of Theorem 5.1.5, $N(R)$ forms an ideal of R, and then $e - f = (a-w) - (a-v) = v - w \in N(R)$. As R is abelian, $(e-f)^3 = e - f$; hence, $e = f$, as desired. □

Corollary 5.3.4. *Every homomorphic image of a uniquely nil-clean ring is uniquely nil-clean.*

Proof. Let R be uniquely nil-clean, and let I be an ideal of R. In light of Theorem 5.3.3, R is abelian strongly nil-clean. This shows that R/I is strongly nil-clean. Clearly, R is strongly clean, and so it is an exchange ring. It follows from Lemma 4.4.5 that every idempotent lifts modulo I, and then R/I is abelian. By Theorem 5.3.3 again, R/I is uniquely nil-clean, as required. □

Corollary 5.3.5. *A ring R is uniquely nil-clean if and only if for any $a \in R$ there exists a central $e \in R$ such that $a - e \in N(R)$.*

Proof. One direction is obvious from Theorem 5.3.3. Conversely, letting $e \in R$ be an idempotent, we have a central idempotent $f \in R$ such that $w := e - f \in N(R)$. Thus, $(e-f)^3 = e - f$, and so $(e-f)\big(1-(e-f)^2\big) = 0$. This implies that $e = f$, and then R is abelian, hence the result by Theorem 5.3.3. □

Corollary 5.3.6. *A ring R is uniquely nil-clean if and only if*

(1) *R is abelian;*
(2) *R is periodic;*
(3) *$R/J(R)$ is Boolean.*

Proof. $\Longrightarrow$ This is obvious, by Theorem 5.3.3, Theorem 5.1.5 and Theorem 5.1.12.

$\Longleftarrow$ Let $x \in J(R)$. Since R is periodic, we have $m, n \in \mathbb{N}$ such that $x^m = x^n (m > n)$; hence, $x^n(x^{m-n} - 1) = 0$. This implies that $x^n = 0$. Thus, $J(R)$ is nil. In view of Theorem 5.1.5, R is strongly nil-clean. Therefore we complete the proof, by Theorem 5.3.3. □

We note that $\mathbb{Z}/4\mathbb{Z}$ is strongly nil-clean and $\mathbb{Z}/6\mathbb{Z}$ is not strongly nil-clean, though they are both abelian periodic.

Lemma 5.3.7. *A ring R is uniquely nil-clean if and only if R is uniquely clean and $J(R)$ is nil.*

Proof. This is obvious by Corollary 5.3.6, Theorem 4.4.6, Theorem 5.1.5 and Theorem 5.3.3. □

Lemma 5.3.8. *Every uniquely clean ring of which every prime ideal is maximal is uniquely nil-clean.*

Proof. Let R be a uniquely clean ring of which every prime ideal is maximal. Clearly, $J(R) \subseteq BM(R)$. Since R is a uniquely clean ring, it follows from Corollary 4.4.11 that R is a right quasi-duo ring. Thus, every maximal right ideal of R is two-sided. This implies that $BM(R) \subseteq J(R)$, and so $BM(R) = J(R)$. As every maximal ideal of R is prime, we see that $J(R) = P(R)$, i.e., the prime radical. Thus, $J(R)$ is nil by Lemma 4.3.1. Therefore we obtain the result in terms of Lemma 5.3.7. □

Recall that a ring R is of bounded index n if $x^n = 0$ for all nilpotent $x \in R$.

Theorem 5.3.9. *Let R be of bounded index. Then R is uniquely nil-clean if and only if*

(1) *R is uniquely clean;*
(2) *Every prime ideal of R is maximal.*

Proof. $\Longrightarrow$ In view of Lemma 5.3.7, R is uniquely clean, and so $R/J(R)$ is Boolean. Let $I/J(R)$ be a prime ideal of $R/J(R)$. Then $\big(R/J(R)\big)/\big(I/J(R)\big) \cong R/I$. We infer that R/I is a prime Boolean ring; hence, $R/I \cong \mathbb{Z}_2$. Thus, $I/J(R)$ is a maximal ideal. This means that every prime ideal of $R/J(R)$ is maximal. Clearly, $P(R) \subseteq J(R)$. For any $x \in J(R)$, we see that RxR is nil. As R is of bounded index, RxR is nilpotent. Hence, $x \in P(R)$. This shows that $J(R) = P(R)$. Let P be a prime ideal of R. Then $P/P(R)$ is a prime ideal of $R/P(R)$, and thus $P/P(R)$ is a maximal of $R/P(R)$, as $P(R) = J(R)$. That is, $R/P \cong R/P(R)/P/P(R)$ is simple. Therefore P is a maximal ideal of R, as desired.

$\Longleftarrow$ This is obvious by Lemma 5.3.8. □

Obviously, every Boolean ring is uniquely nil-clean. Moreover, we derive

Theorem 5.3.10. *Let R be a ring. Then the following are equivalent:*

(1) *R is uniquely nil-clean and $u^2 = 1$ for all $u \in U(R)$.*
(2) *R is uniquely nil-clean and $a(a+2) = 0$ for all $a \in N(R)$.*
(3) *For any $x, y \in R$, $(1 + x + xy)(1 + y + yx) = 1$ if and only if $x = y$.*
(4) *R is commutative, $2 \in N(R)$, and for each $a \in R$, there exists some $n \in \mathbb{N}$ such that $a^n = a^{n+2}$.*

Proof. (1) $\Rightarrow$ (4) Clearly, $2 \in N(R)$. Let $u, v \in U(R)$. Then $u^2 = v^2 = (uv)^2 = 1$ by hypothesis, and so $uv = (uv)^{-1} = v^{-1}u^{-1} = vu$. That is,

all units in R is commutative. This shows that $N(R)$ is commutative as $1+N(R) \subseteq U(R)$. Further, R is abelian by Theorem 5.3.3. Let $x, y \in R$. Then there exist idempotents $e, f \in R$ and nilpotents $v, w \in R$ such that $x = e+v$ and $y = f+w$. Thus, $xy = (e+v)(f+w) = (f+w)(e+v) = yx$. This means that R is commutative.

Let $a \in R$. In view of Theorem 5.3.3 and Theorem 5.1.1, $a-a^2 \in N(R)$. Hence, $\big(1-(a-a^2)\big)^2 = 1$, i.e., $(a-a^2)^2 = 2(a-a^2)$. Replacing a by $-a$, we get $(a+a^2)^2 = -2(a+a^2)$. Choose $a = 1$. Then $8 = 0$ in R. Thus,

$$4a = 2(a-a^2) - 2\big(-a-a^2\big) = (a-a^2)^2 - (a+a^2)^2 = -4a^3 = 4a^3.$$

Replacing a by $1+a$, we have $4(1+a) = 4(1+a)^3 = 4(1+a+a^2+a^3)$ as $8 = 0$. This implies that $4a^2 = -4a^3$, and so $4a = 4a^2$. But $(a-a^2)^2 = 2(a-a^2)$. Then $a^4 = 2a^3 - 3a^2 + 2a$; hence,

$$a^5 = 2\big(2a^3 - 3a^2 + 2a\big) - 3a^3 + 2a^2 = a^3 - 4a^2 + 4a = a^3,$$

as desired.

(4) $\Rightarrow$ (3) Let $x \in R$. Then $x^n = x^{n+2}$, and so $(x-x^3)^n = 0$. Hence $x - x^3 \in N(R)$. Thus, $(x-x^2)^2 = x(x-x^3) + 2(x^4-x^3) \in N(R)$, by $2 \in N(R)$. This implies that $1 - x + x^2 \in U(R)$. By hypothesis, there exists some $n \in \mathbb{N}$ such that $\big(1-x+x^2\big)^n = \big(1-x+x^2\big)^{n+2}$. Therefore $\big(1-x+x^2\big)^2 = 1$, which proves one part of (3). Assume that $(1+x+xy)(1+y+yx) = 1$. Then $(1+x+xy)^2 = 1$, and so $0 = (1+x+xy)\big(1-(1+x+xy)(1+y+yx)\big) = (1+x+xy)-(1+y+yx) = x-y$. This yields that $x = y$, as desired.

(3) $\Rightarrow$ (2) Choose $x = y = -2$. Then $8 = 0$ in R.

Let $a \in R$. Then $(1-a+a^2)^2 = 1$, and then $(a-a^2)^2 = 2(a-a^2)$. This shows that $(a-a^2)^6 = 8(a-a^2)^3 = 0$, and therefore $a - a^2 \in N(R)$. By virtue of Theorem 5.1.1, R is strongly nil-clean.

Let $e \in R$ be an idempotent, and $x \in R$. Set $a = ex(1-e)$. Then $a^2 = 0 = ae$ and $ea = a$. Clearly,

$$\big(1+(a-e)+(a-e)(-e)\big)\big(1-e-e(a-e)\big) = (1+a)(1-a) = 1-a^2 = 1.$$

By hypothesis, $a - e = -e$, and so $a = 0$. That is, $ex = exe$. Likewise, $xe = exe$. Therefore R is abelian. According to Theorem 5.3.3, R is uniquely nil-clean.

Let $x \in R$. By the discussion above, we get $(x-x^2)^2 = 2(x-x^2)$. As in the proof in "(1) $\Rightarrow$ (4)", we show that $x^5 = x^3$. If $a \in N(R)$, then $(1+a)^5 = (1+a)^3$, and so $(1+a)^2 = 1$. This yields that $a(a+2) = 0$, as asserted.

(2) $\Rightarrow$ (1) For any $u \in U(R)$, we have an idempotent $e \in R$ and a nilpotent $w \in R$ such that $u = e + w$ and $ue = eu$. This implies that $e = u - w = u(1 - u^{-1}w)$ and $uw = wu$; hence, $e \in U(R)$. Thus, $e = 1$, and so $u = 1 + w$. By hypothesis, $(u-1)\big((u-1)+2\big) = 0$. We infer that $u^2 = 1$, as required. □

Corollary 5.3.11. *Let R be a ring. Then the following are equivalent:*

(1) *R is uniquely nil-clean and $u^2 = 1$ for all $u \in U(R)$.*
(2) *R is abelian, $2 \in J(R)$ and $a^5 = a^3$ for all $a \in R$.*

Proof. (1) $\Rightarrow$ (2) is obvious as in the proof of Theorem 5.3.10.

(2) $\Rightarrow$ (1) As $2 \in J(R)$, it follows from $2^5 = 2^3$ that $2^3 = 0$ in R, and then $2 \in N(R)$. For any $u \in U(R)$, $u^5 = u^3$, and that $u^2 = 1$. Let $a \in R$. Then $a^5 = a^3$, and so $(a-a^3)^3 = 0$; hence, $(a-a^2)^2 = a(a-a^3)+2(a^4-a^3) \in N(R)$. This implies that $a - a^2 \in N(R)$. In light of Theorem 5.1.1, R is strongly nil-clean. Accordingly, R is uniquely nil-clean, by Theorem 5.3.3. □

Let $R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, 0, 1 \in \mathbb{Z}_2 \right\}$. Then R is uniquely nil-clean and $u^2 = 1$ for all $u \in U(R)$. This is an immediate consequence of Corollary 5.3.11.

The matrix and triangular matrix rings over uniquely nil-clean rings are not uniquely nil-clean as they are not abelian. We now construct certain uniquely nil-clean subrings of matrix rings.

Example 5.3.12. Let R be uniquely nil-clean. Then the ring $T = \left\{ \begin{pmatrix} a & 0 \\ c & b \end{pmatrix} \;\middle|\; a - b = 2c, a, b, c \in R \right\}$ is uniquely nil-clean.

Proof. For any $\begin{pmatrix} a & 0 \\ c & b \end{pmatrix} \in T$, it is easy to verify that

$$\begin{pmatrix} a & 0 \\ c & b \end{pmatrix} - \begin{pmatrix} a & 0 \\ c & b \end{pmatrix}^2 = \begin{pmatrix} a - a^2 & 0 \\ * & b - b^2 \end{pmatrix} \in N(T).$$

Thus, T is strongly nil-clean from Theorem 5.1.1. Suppose $\begin{pmatrix} e & 0 \\ x & f \end{pmatrix} \in T$ is an idempotent. Then $e = e^2, f = f^2$ and $e - f = 2x$. Since R is uniquely nil-clean, it is strongly nil clean, and so $2 \in N(R)$. We infer that $e - f \in N(R)$. Clearly, R is abelian; hence, $(e-f)^3 = e - f$. This

implies that $(e-f)\big(1-(e-f)^2\big)=0$. Hence, $e=f$. On the other hand, $xe+fx=x$, and so $(2e-1)x=0$. As $(2e-1)^2=1$, we get $x=0$. Therefore $\begin{pmatrix} e & 0 \\ x & f \end{pmatrix} = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} \in T$ is abelian. Accordingly, T is uniquely nil-clean by Theorem 5.3.3. □

Example 5.3.13. Let R be uniquely nil-clean. Then the ring $T=\{(a_{ij}) \in T_n(R) \mid a_{11}=\cdots=a_{nn}\}$ is uniquely nil-clean.

Proof. In view of Lemma 5.3.7, R is uniquely clean and $J(R)$ is nil. It follows from Example 4.4.23 that T is uniquely clean. Further, $J(T) = \{(a_{ij}) \in T_n(R) \mid a_{11}=\cdots=a_{nn} \in J(R)\}$ is nil. By using Lemma 5.3.7 again, T is uniquely nil-clean. □

Example 5.3.14. Let $A=\left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \;\middle|\; a,b \in \mathbb{Z}_4 \right\}$, $B=\left\{ \begin{pmatrix} a & 2b \\ 0 & a \end{pmatrix} \;\middle|\; a,b \in \mathbb{Z}_4 \right\}$. Then $R[A,B]$ is uniquely nil-clean.

Proof. In view of Example 4.4.29, $R[A,B]$ is uniquely clean. Clearly, $J(A) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \;\middle|\; a \in 2\mathbb{Z}_4, b \in \mathbb{Z}_4 \right\}$ is nil. As $J\big(R[A,B]\big) = R\big[J(A), J(A) \cap J(B)\big]$, we see that $J\big(R[A,B]\big)$ is nil. Therefore we obtain the result by Lemma 5.3.7. □

Example 5.3.15. $\mathbb{Z}_4[i]$ is uniquely nil-clean.

Proof. In view of Example 4.4.32, $\mathbb{Z}_4[i]$ is uniquely clean. Clearly, $\mathbb{Z}_4[i]$ is a finite ring, and so $J\big(\mathbb{Z}_4[i]\big)$ is nil. This completes the proof, by Lemma 5.3.7. □

Example 5.3.16. Let R be a ring. Then the following are equivalent:

(1) R is uniquely nil-clean.
(2) $R[[x]]/(x^2)$ is uniquely nil-clean.
(3) $R[[x]]/(x^n)$ is uniquely nil-clean for some $n \geq 2$.
(4) $R[[x]]/(x^n)$ is uniquely nil-clean for all $n \geq 2$.

Proof. (1) ⇒ (2) In view of Proposition 4.4.25, $R[[x]]/(x^2)$ is uniquely clean. One easily checks that $J\big(R[[x]]/(x^2)\big) = \{a+bx \mid a \in J(R), b \in R, x^2=0\}$. Since R is uniquely nil-clean, $J(R)$ is nil, and thus $J\big(R[[x]]/(x^2)\big)$ is nil. According to Lemma 5.3.7, $R[[x]]/(x^2)$ is uniquely nil-clean.

(2) ⇒ (4) Clearly, $R[[x]]/(x^2) \cong R[[x]]/(x^n)/(x^2)/(x^n) (n \geq 2)$. By Corollary 5.3.4, $R[[x]]/(x^n)$ is uniquely nil-clean for all $n \geq 2$.

(4) $\Rightarrow$ (3) This is obvious.

(3) $\Rightarrow$ (1) Let $S = R[[x]]/(x^n)(n \geq 2)$. Then R is a homomorphism image of S, and therefore R is uniquely nil-clean. □

Proposition 5.3.17. *A ring R is uniquely nil-clean if and only if*

(1) *R is abelian;*
(2) *$C(R)$ is uniquely nil-clean;*
(3) *$R = C(R) + N(R)$, where $N(R)$ is an ideal of R.*

Proof. One direction is clear from Theorem 5.3.3.

Conversely, assume that (1), (2) and (3) hold. Let $a \in R$. By (3), we have $a = c + d, c \in C(R)$ and $d \in N(R)$. By (2), we have an idempotent $e \in C(R)$ and a nilpotent $b \in C(R)$ such that $c = e + b$. Thus, $a = e + (b + d)$. Obviously, $b + d \in N(R)$. To see that R is uniquely nil-clean, let $a = f + v, f = f^2 \in R$ and $v \in N(R)$. Then $a = e + (b + d) = f + v$. By (1), we see that $(b + d)v = (a - e)(a - f) = (a - f)(a - e) = v(b + d)$. Hence, $e - f = v - (b + d) \in N(R)$. Clearly, $(e - f)\big(1 - (e - f)^2\big) = 0$. It follows that $e = f$, and therefore R is uniquely nil-clean. □

An ideal I of a ring R is primary provided that for any $x, y \in R$, $xRy \subseteq I$ implies that $x \in I$ or $y^n \in I$ for some $n \in \mathbb{N}$. For instance, in the ring of integers $\mathbb{Z}$, $\big(p^n\big)$ is a primary ideal if p is a prime. Clearly, every maximal ideal of a ring is primary.

Lemma 5.3.18. *Let R be a uniquely nil-clean ring. Then the intersection of all primary ideals of R is zero.*

Proof. Let a be in the intersection of all primary ideal of R. Assume that $0 \neq a \in R$. Let $\Omega = \{I \mid I \text{ is an ideal of } R \text{ such that } a \notin I\}$. Then $\Omega \neq \emptyset$ as $0 \in \Omega$. Given any ideals $I_1 \subseteq I_2 \subseteq \cdots$ in Ω, we set $M = \bigcup_{i=1}^{\infty} I_i$. Then $M \in \Omega$. Thus, Ω is inductive. By using Zorn's Lemma, we can find an ideal Q which is maximal in Ω. It will suffice to show that Q is primary. If not, we can find some $x, y \in R$ such that $xRy \subseteq Q$, but $x \notin Q$ and $y^n \notin Q$ for any $n \in \mathbb{N}$. Hence, $a \in Q + (x)$. Write $a = b + \sum_{i=1}^{m} c_i x d_i$ for some $b \in Q, c_i, d_i \in R$. Since R is uniquely nil-clean, it is strongly nil clean. Hence, it follows from Theorem 5.1.12 that there are some distinct $k, l \in \mathbb{N}$ such that $y^k = y^l$. Say $k > l$. For any $j \geq l$, we see that $y^j = y^l y^{j-l} = y^k y^{j-l} = y^j y^{k-l} = y^j y^{2(k-l)} = \cdots = y^j y^{l(k-l)}$. Choose $j = l(k - l)$. Then $y^{l(k-l)} = y^{l(k-l)} y^{l(k-l)}$. Choose $s = l(k - l)$. Then y^s

is an idempotent. Write $y = y_B + y_N$. Then $y^s - y_B \in N(R)$. Clearly, $y^s y_B = y_B y^s$, and thus $(y^s - y_B)^3 = y^s - y_B$. This implies that $y^s = y_B$. It follows from $xRy \subseteq Q$ that $xRy^s \subseteq Q$, and then $xRy_B \subseteq Q$. As $a = b + \sum_{i=1}^{m} c_i x d_i$, we get $ay_B = by_B + \sum_{i=1}^{m} c_i x d_i y_B \in Q$. Clearly, $y^s \notin Q$, and so $a \in Q + (y_B)$. Write $a = e + \sum_{i=1}^{k} r_i y_B s_i$ for some $e \in Q, r_i, s_i \in R$. We show that $ay_B = ey_B + \sum_{i=1}^{k} r_i y_B s_i y_B$, and so $\sum_{i=1}^{k} r_i y_B s_i y_B \in Q$. This implies that $a \in Q$, a contradiction. Therefore Q is primary, a contradiction. The lemma is therefore established. □

We shall close this section by providing the relationship between uniquely nil-clean rings and Boolean rings.

Theorem 5.3.19. *A ring R is a Boolean ring if and only if*

(1) *Every primary ideal of R is maximal;*
(2) *R is uniquely nil-clean.*

Proof. Suppose that R is a Boolean ring. Then $2^2 = 2$, and so $2 = 0$. For any $a, b \in R$, we see that $a + b = (a + b)^2 = a^2 + ab + ba + b^2 = a + ab + 2ba - ba + b$. This shows that $ab = ba$, i.e., R is commutative. Clearly, R is a uniquely nil clean ring. Let I be a primary ideal of R. If I is not maximal, by using Zorn's Lemma, there exists a maximal ideal M such that $I \subsetneqq M \subsetneqq R$. Choose $x \in M$ while $x \notin I$. As x is an idempotent, we see that $xR(1 - x) \subseteq I$, and so $(1 - x)^m \in I \subseteq M$ for some $m \in \mathbb{N}$. Thus, $1 - x \in M$. This implies that $1 = x + (1 - x) \in M$, a contradiction. Therefore I is maximal, as required.

Conversely, assume that (1) and (2) hold. Clearly, every maximal ideal of R is primary, and so $J(R) = \bigcap\{P \mid P \text{ is primary}\}$. In view of Lemma 5.3.18, $J(R) = 0$. By virtue of Corollary 5.3.6, $R/J(R)$ is Boolean, and then so is R, thus completing the proof. □

Corollary 5.3.20. *A ring R is a Boolean ring if and only if*

(1) *R is commutative;*
(2) *Every primary ideal of R is maximal;*
(3) *R is strongly nil-clean.*

Proof. This is obvious by Theorem 5.3.3 and Theorem 5.3.19. □

5.4 Strongly Nil-clean Matrices

The main purpose of this section is to characterize a single strongly nil-clean matrix over a commutative ring in terms of its factorization of the characteristic polynomial. The following elementary result on the strong nil-cleanness of an endomorphism which is an analogue of the corresponding on strong cleanness.

Theorem 5.4.1. *Let $E = End(_RM)$, and let $\alpha \in E$. Then the following are equivalent:*

(1) *$\alpha \in E$ is strongly nil-clean.*
(2) *There exists $\pi^2 = \pi \in E$ such that $\alpha\pi = \pi\alpha$, $\alpha\pi \in N\big(\pi E\pi\big)$ and $(1-\alpha)(1-\pi) \in N\big((1-\pi)E(1-\pi)\big)$.*
(3) *$M = P \oplus Q$ where P and Q are α-invariant, $\alpha|_P \in N\big(End(P)\big)$ and $(1-\alpha)|_Q \in N\big(End(Q)\big)$.*
(4) *$M = P_1 \oplus \cdots \oplus P_n$ for some $n \in \mathbb{N}$ where P_i is α-invariant and $\alpha|_{P_i}$ is strongly nil-clean in $End(P_i)$ for each i.*

Proof. (1) $\Rightarrow$ (2) Since α is strongly nil-clean in E, there exist $\pi^2 = \pi \in E, u \in N(E)$ such that $\alpha = (1-\pi) + u$ and $\pi u = u\pi$. Thus, $\pi\alpha = \pi u \in N\big(\pi E\pi\big)$. Further, $1-\alpha = \pi + (-u)$, and so $(1-\pi)(1-\alpha) = (1-\pi)(-u) \in N\big((1-\pi)E(1-\pi)\big)$.

(2) $\Rightarrow$ (3) Set $P = M\pi$ and $Q = M(1-\pi)$. Then $M = P \oplus Q$. As $\alpha\pi = \pi\alpha$, we see that P and Q are α-invariant. Since $\alpha\pi \in N\big(\pi E\pi\big)$, we can find some $m \in \mathbb{N}$ such that $(\alpha\pi)^m = 0$. For any $x\pi \in P$, it is easy to verify that $(x\pi)\big(\alpha|_P\big)^m = (x)(\pi\alpha\pi)\big(\alpha|_P\big)^{m-1} = \cdots = (x)\big(\pi\alpha\pi\big)^m = 0$. This implies that $\big(\alpha|_P\big)^m = 0$, i.e., $\alpha|_P \in N\big(End(P)\big)$. Likewise, one easily checks that $(1-\alpha)|_Q \in N\big(End(Q)\big)$.

(3) $\Rightarrow$ (4) This is obvious.

(4) $\Rightarrow$ (1) Let $\lambda_i \in End(P_i)$. We construct an R-morphism $\overline{\lambda_i} \in End(M)$ given by $\left(\sum_{j=1}^{n} p_j\right)\overline{\lambda_i} = (p_i)\lambda_i$ for any $p_j \in P_j$. By hypothesis, there exist $\pi_j^2 = \pi_j \in End(P_j), \sigma_j \in N\big(End(P_j)\big)$ such that $\alpha|_{P_j} = \pi_j + \sigma_j$ and $\pi_j\sigma_j = \sigma_j\pi_j$. Thus, $\alpha = \sum_{j=1}^{n} \overline{\alpha|_{P_j}} = \sum_{j=1}^{n} \overline{\pi_j} + \sum_{j=1}^{n} \overline{\sigma_j}$. Obviously, $\overline{\pi_j\pi_k} = \overline{\pi_k\pi_j} = 0 (j \neq k)$, we see that $\sum_{j=1}^{n} \overline{\pi_j} \in End(M)$ is an idempotent.

Moreover, $\overline{\pi_j\sigma_k} = \overline{\sigma_k\pi_j} = 0 (j \neq k)$. Hence,

$$\left(\sum_{j=1}^{n} \overline{\pi_j}\right)\left(\sum_{j=1}^{n} \overline{\sigma_j}\right) = \sum_{j=1}^{n} \overline{\pi_j\sigma_j} = \sum_{j=1}^{n} \overline{\sigma_j\pi_j} = \left(\sum_{j=1}^{n} \overline{\sigma_j}\right)\left(\sum_{j=1}^{n} \overline{\pi_j}\right).$$

Since $\sigma_j \in N\big(End(P_j)\big)$, we have $s_j \in \mathbb{N}$ such that $(\sigma_j)^{s_j} = 0$. Choose $s = max\{s_j \mid 1 \leq j \leq n\}$. Then each $(\sigma_j)^s = 0$. Clearly, $\overline{\sigma_j\sigma_k} = \overline{\sigma_k\sigma_j} = 0 (j \neq k)$. For any $\sum\limits_{j=1}^{n} p_j \in \sum\limits_{j=1}^{n} P_j$, it is easy to verify that

$$\left(\sum_{j=1}^{n} p_j\right)\left(\sum_{j=1}^{n} \overline{\sigma_j}\right)^s = \sum_{j=1}^{n} (p_j)\big(\sigma_j\big)^s = 0.$$

Thus $\sum\limits_{j=1}^{n} \overline{\sigma_j} \in N\big(End(M)\big)$, and so $\alpha \in End(M)$ is strongly nil-clean, hence the result. □

Corollary 5.4.2. *Let R be a ring, and let $\alpha \in R$. Then the following are equivalent:*

(1) *$\alpha \in R$ is strongly nil-clean.*
(2) *There exists $\pi^2 = \pi \in R$ such that $\alpha\pi = \pi\alpha$, $\alpha\pi \in N\big(\pi R\pi\big)$ and $(1-\alpha)(1-\pi) \in N\big((1-\pi)R(1-\pi)\big)$.*
(3) *There exists a decomposition $1 = e_1 + \cdots + e_n$ where $n \in \mathbb{N}$, $e_1, \ldots, e_n$ are orthogonal idempotents, and $e_i\alpha = \alpha e_i$ is strongly nil-clean in e_iRe_i for each i.*

Proof. (1) $\Leftrightarrow$ (2) This is obvious from Theorem 5.4.1.

(1) $\Rightarrow$ (3) In view of Theorem 5.4.1, $R = P_1 \oplus \cdots \oplus P_n$ for some $n \in \mathbb{N}$ where P_i is α-invariant and $\alpha|_{P_i}$ is strongly nil-clean in $End(P_i)$ for each i. Write $1 = e_1 + \cdots + e_n$ where each $e_i \in P_i$. Then $e_i - e_i^2 = \sum\limits_{j\neq i} e_je_i \in P_i \cap \Big(\sum\limits_{j\neq i} P_j\Big)$. Hence, $e_i^2 = e_i$ and $e_je_i = 0 (j \neq i)$. Further, each $P_i = Re_i$. As Re_i and $R(1-e_i)$ are α-invariant, we get $e_i\alpha = \alpha e_i$. Since $End(Re_i) \cong e_iRe_i$, we have that $e_i\alpha = \alpha e_i$ is strongly nil-clean in e_iRe_i for each i.

(3) $\Rightarrow$ (1) By hypothesis, there exists a decomposition $1 = e_1 + \cdots + e_n$ where $n \in \mathbb{N}$, $e_1, \ldots, e_n$ are orthogonal idempotents, and $e_i\alpha = \alpha e_i$ is strongly nil-clean in e_iRe_i for each i. Choose $P_i = Re_i$. Then $R = P_1 \oplus \cdots \oplus P_n$ for some $n \in \mathbb{N}$ where P_i is α-invariant and $\alpha|_{P_i}$ is strongly nil-clean in $End(P_i)$ for each i. According to Theorem 5.4.1, $\alpha \in R$ is strongly nil-clean. □

Corollary 5.4.3. *Let $E = End(_RM)$, and let $\alpha \in E$. Then the following are equivalent:*

(1) *$\alpha \in E$ is strongly nil-clean.*
(2) *There exists a left E-module decomposition $E = A \oplus B$ where A and B are α-invariant, and $\alpha|_A \in N\big(End(A)\big)$ and $(1_M - \alpha)|_B \in N\big(End(B)\big)$.*

Proof. (1) $\Rightarrow$ (2) In view of Corollary 5.4.2, there exists $\pi^2 = \pi \in E$ such that $\alpha\pi = \pi\alpha$, $\alpha\pi \in N\big(\pi E\pi\big)$ and $(1-\alpha)(1-\pi) \in N\big((1-\pi)E(1-\pi)\big)$. Set $A = E\pi$ and $B = E(1_M - \pi)$. Then $E = A \oplus B$ where A and B are α-invariant. Further, we check that $\alpha|_A \in N\big(End(A)\big)$ and $(1_M - \alpha)|_B \in N\big(End(B)\big)$.

(2) $\Rightarrow$ (1) By hypothesis, $E = A \oplus B$, where A and B are α-invariant, $\alpha|_A \in N\big(End(A)\big)$ and $(1_M - \alpha)|_B \in N\big(End(B)\big)$. Let $p : E \to A$ be the projection, and let $\pi = (1_M)p$. Then $\pi = \pi^2 \in E$. In addition, $A = E\pi$ and $B = E(1_M - \pi)$. Thus $\alpha\pi = \pi\alpha$. Furthermore, $\alpha\pi \in N\big(\pi R\pi\big)$ and $(1-\alpha)(1-\pi) \in N\big((1-\pi)R(1-\pi)\big)$. According to Corollary 5.4.2, we complete the proof. □

Let M be a left R-module, and let $\varphi \in End(M)$. If there exists a monic polynomial $h \in R[t]$ such that $h - t^{deg(h)} \in N(R)[t]$ for which $h(\varphi) = 0$, we note that $\varphi \in N\big(End(M)\big)\big)$.

For $r \in R$, define

$$\mathbb{P}_r = \{f \in R[t] \mid f \text{ monic, and } f - (t-r)^{deg(f)} \in N(R)[t]\}.$$

Theorem 5.4.4. *Let R be a commutative ring, let $\varphi \in M_n(R)$, and let $h \in R[t]$ be a monic polynomial of degree n. If $h(\varphi) = 0$ and there exists a factorization $h = h_0h_1$ such that $h_0 \in \mathbb{P}_0$ and $h_1 \in \mathbb{P}_1$, then φ is strongly nil-clean.*

Proof. Suppose that $h = h_0h_1$ where $h_0 \in \mathbb{P}_0$ and $h_1 \in \mathbb{P}_1$. Then $h_0 \equiv t^p \ \big(\mathrm{mod}N(R)\big)$ and $h_1 \equiv (t-1)^q \ \big(\mathrm{mod}N(R)\big)$, where $p = deg(h_0)$ and $q = deg(h_1)$. Hence $\overline{h_0} = t^p$ and $\overline{h_1} = (t-\overline{1})^q$ in $\big(R/N(R)\big)[t]$. Since $t^p - (t^{p-1} + \cdots + t + \overline{1})(t - \overline{1}) = \overline{1}$ in $\big(R/N(R)\big)[t]$, we see that $t^p - (t^{p-1} + \cdots + t + \overline{1})\big(t^p - (t^{p-1} + \cdots + t + \overline{1})(t - \overline{1})\big)(t - \overline{1}) = \overline{1}$. Thus, there exist $p(t), q(t) \in \big(R/N(R)\big)[t]$ such that $p(t)t^p + q(t)(t-\overline{1})^2 = \overline{1}$. By iteration of this process, we can find $\overline{u_0'}, \overline{u_1'} \in \big(R/N(R)\big)[t]$ such that $\overline{u_0'h_0} + \overline{u_1'h_1} = \overline{1}$. This implies that $1 - (u_0'h_0 + u_1'h_1) \in N(R)[t]$. Write $1-(u_0'h_0+u_1'h_1) = b_0+b_1t+\cdots+b_mt^m$ where $b_0, b_1, \ldots, b_m \in N(R)$. Then

we have $k \in \mathbb{N}$ such that $\big(1-(u_0'h_0+u_1'h_1)\big)^k = 0$ in $R[t]$, and so $u_0'h_0 + u_1'h_1 \in U\big(R[t]\big)$. Thus, there exist $u_0, u_1 \in R[t]$ such that $u_0h_0 + u_1h_1 = 1$. Let $\varphi \in M_n(R)$ with $\chi(\varphi) = h$. Then $u_0(\varphi)h_0(\varphi) + u_1(\varphi)h_1(\varphi) = 1_{nR}$. By the Cayley-Hamilton Theorem, $h(\varphi) = h_0(\varphi)h_1(\varphi) = h_1(\varphi)h_0(\varphi) = 0$. Clearly, $u_0(\varphi)h_1(\varphi) = h_1(\varphi)u_0(\varphi)$ and $h_0(\varphi)u_1(\varphi) = u_1(\varphi)h_0(\varphi)$. In light of Lemma 2.2.1, $nR = Ker\big(h_0(\varphi)\big) \oplus Ker\big(h_1(\varphi)\big)$. As $h_0t = th_0$ and $h_1t = th_1$, we see that $Ker\big(h_0(\varphi)\big)$ and $Ker\big(h_1(\varphi)\big)$ are both φ-invariant. Further, $h_0 \equiv t^p \big(\mathrm{mod} N(R)\big)$ and $h_0\big(\varphi \mid_{Ker(h_0(\varphi))}\big) = 0$. It follows from Theorem 4.3.2 that $\varphi \mid_{Ker(h_0(\varphi))} \in N\big(End(Ker(h_0(\varphi)))\big)$.

Write $h_1 = h_1(t) \in \mathbb{P}_1$, we see that $h_1(t) \equiv (t-1)^q \big(\mathrm{mod} N(R)\big)$. Set $t-1 = u$. Then $h_1(t) = h_1(u+1) := k(u)$. In addition, $k(u) \equiv u^q \big(\mathrm{mod} N(R)\big)$. Obviously, $k\big((\varphi - 1) \mid_{Ker(h_1(\varphi))}\big) = h_1\big(\varphi \mid_{Ker(h_1(\varphi))}\big) = 0$. It follows from Theorem 4.3.2 that $(\varphi - 1) \mid_{Ker(h_1(\varphi))} \in N\big(End(Ker(h_1(\varphi)))\big)$. We infer that $(1-\varphi) \mid_{Ker(h_1(\varphi))} \in N\big(End(Ker(h_1(\varphi)))\big)$. According to Theorem 5.4.1, $\varphi \in M_n(R)$ with $\chi(\varphi) = h$ is strongly nil clean. □

Let R be a commutative ring, and let $A \in M_n(R)$. The characteristic polynomial of A is the polynomial $\chi(A) = det(tI_n - A)$. If $f(t) \in R[t]$ is the characteristic polynomial of $A \in M_n(R)$, it follows by the Cayley-Hamilton Theorem that $f(A) = 0$. But the converse is not true, e.g., $A = \begin{pmatrix} \overline{2} & \overline{1} & \overline{1} \\ \overline{1} & \overline{2} & \overline{1} \\ \overline{1} & \overline{1} & \overline{2} \end{pmatrix} \in M_3(\mathbb{Z}_3), f(t) = t^3 + t^2 + t \in \mathbb{Z}_3[t]$.

Corollary 5.4.5. *Let R be a commutative ring, let $A \in M_n(R)$. If $\chi(A) = t^k(t-1)^l$ for some $k, l \in \mathbb{N}$, then A is the sum of an idempotent matrix and a nilpotent matrix that commute.*

Proof. By hypothesis, $\chi(A) = h_0h_1$, where $h_0 \in \mathbb{P}_0$ and $h_1 \in \mathbb{P}_1$. In view of Theorem 5.4.4, A is strongly nil-clean, as desired. □

For instance, choose $A = \begin{pmatrix} -2 & 2 & -1 \\ -4 & 4 & -2 \\ -1 & 1 & 0 \end{pmatrix} \in M_3(\mathbb{Z})$. Then $\chi(A) = t(t-1)^2$. Thus, A is the sum of an idempotent matrix and a nilpotent matrix that commute. In fact, we have a corresponding factorization $A = \begin{pmatrix} -1 & 1 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 1 & -1 \\ -2 & 2 & -2 \\ -1 & 1 & -1 \end{pmatrix}$.

Lemma 5.4.6. *Let R be a commutative ring, and let $h \in R[t]$ be monic. Then the following are equivalent:*

(1) *The companion matrix C_h of h is strongly nil-clean.*
(2) *There exist ideals I and J such that*

$$I + J = R[t], I \cap J = hR[t], t^k \in I, (1-t)^l \in J$$

for some $k, l \in \mathbb{N}$.

Proof. Let $h \in R[t]$ be a monic polynomial of degree n. Then the matrix of m_t is C_h with respect to the basis by the image of $\{1, t, \ldots, t^{n-1}\}$. Therefore we obtain the result as in the proof of Lemma 2.1.1. □

Theorem 5.4.7. *Let R be a commutative ring, and let $h \in R[t]$ be monic. Then the following are equivalent:*

(1) *The image of t in $R[t]/(h)$ is strongly nil-clean.*
(2) *There exist ideals I and J such that*

$$I + J = R[t], I \cap J = hR[t], t^k \in I, (1-t)^l \in J$$

for some $k, l \in \mathbb{N}$.

Proof. (1) $\Rightarrow$ (2) As in the proof of Lemma 2.1.4, we show that $m_t = \varphi(\overline{t}) \in End_R\big(R[t]/hR[t]\big)$ is strongly nil-clean, where $\varphi : R[t]/(h) \to End_R\big(R[t]/hR[t]\big)$. Hence, proving (2) by Lemma 5.4.6.

(2) $\Rightarrow$ (1) By hypothesis, $R[t]/(h) \cong R[t]/I \times R[t]/J$ as rings. Thus, the image of t in $R[t]/I$ is nilpotent and the image of $1-t$ in $R[t]/J$ is nilpotent, and so the images of t in $R[t]/I$ and $R[t]/J$ are both strongly nil-clean. We infer that the image of t in $R/(h)$ is strongly nil-clean, as desired. □

Lemma 5.4.8. *Let R be a commutative ring, and let $f \in R[t]$ be monic. If $f \mid t^n$, then $f \in \mathbb{P}_0$.*

Proof. Suppose that $t^n = fg$ for some $g \in R[t]$. Then $t^n = \overline{f}\,\overline{g}$ in $S[t]$, where $S = R/N(R)$. Write $\overline{f} = a_0 + a_1 t + \cdots + a_k t^k$ and $\overline{g} = b_0 + b_1 t + \cdots + b_l t^l$

$(k, l \in \mathbb{N}^+)$. Then $k + l = n$. We may assume that $k \geq l$. Then

$$\begin{gathered} a_0b_0 = 0 \\ a_0b_1 + a_1b_0 = 0 \\ \vdots \\ a_0b_l + a_1b_{l-1} + \cdots + a_lb_0 = 0 \\ a_1b_l + a_2b_{l-1} + \cdots + a_{l+1}b_0 = 0 \\ a_2b_l + a_3b_{l-1} + \cdots + a_{l+2}b_0 = 0 \\ \vdots \\ a_{k-1}b_l + a_kb_{1-1} = 0 \\ a_kb_l = 1. \end{gathered}$$

As S is reduced, we see that $a_1b_0 = a_2b_0 = \cdots = a_kb_0 = 0$. Hence, $b_0 = 0$, and so we have

$$\begin{gathered} a_0b_1 = 0 \\ \vdots \\ a_0b_l + a_1b_{l-1} + \cdots + a_{l-1}b_1 = 0 \\ a_1b_l + a_2b_{l-1} + \cdots + a_lb_1 = 0 \\ a_2b_l + a_3b_{l-1} + \cdots + a_{l+1}b_1 = 0 \\ \vdots \\ a_{k-1}b_l + a_kb_{l-1} = 0 \\ a_kb_l = 1. \end{gathered}$$

Similarly, $a_1b_1 = a_2b_1 = \cdots = a_kb_1 = 0$. Hence, $b_1 = 0$. Furthermore, we have $b_2 = \cdots = b_{l-1} = 0$. Thus, we get $a_0b_l = a_1b_l = a_2b_l = \cdots = a_{k-1}b_l = 0$ and $a_kb_l = 1$. It follows that $a_0 = \cdots = a_{k-1} = 0$. Thus, $\overline{f} = a_kt^k$ with $a_k \in U(S)$. As $f \in R[t]$ is monic, we see that $a_k = 1_S$. Thus, $f \equiv t^k (mod\ N(R))$. Therefore $f \in \mathbb{P}_0$, as asserted. □

Theorem 5.4.9. *Let R be a commutative ring and let $h \in R[t]$ be a monic polynomial of degree n. Then the following are equivalent:*

(1) *Every $\varphi \in M_n(R)$ with $\chi(\varphi) = h$ is strongly nil-clean.*
(2) *The companion matrix C_h of h is strongly nil-clean.*
(3) *There exists a factorization $h = h_0h_1$ such that $h_0 \in \mathbb{P}_0$ and $h_1 \in \mathbb{P}_1$.*

Proof. (1) ⇒ (2) Write $h = t^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0 \in R[t]$. Choose

$$\varphi = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix} \in M_n(R).$$

Then $\chi(\varphi) = h$, and so $\varphi \in M_n(R)$ is strongly nil-clean, as desired.

(2) $\Rightarrow$ (3) By virtue of Lemma 5.4.6, we have two ideals I and J of R such that $I + J = R[t], I \cap J = hR[t], t^k \in I, (1-t)^l \in J$ for some $k, l \in \mathbb{N}$. Then we can find some $h_0, h_1 \in R[t]$ such that $I = (h_0), J = (h_1), h = h_0h_1$ and $(h_0, h_1) = 1$, in terms of Lemma 2.1.3. As $t^k \in (h_0)$, there exists some $g \in R[t]$ such that $t^k = gh_0$. It follows by Lemma 5.4.8 that $h_0 \in \mathbb{P}_0$. Likewise, $h_1 \in \mathbb{P}_1$. Thus, proving (3).

(3) $\Rightarrow$ (1) For every $\varphi \in M_n(R)$ with $\chi(\varphi) = h$, it follows by the Cayley-Hamilton Theorem that $h(A) = 0$. Therefore φ is strongly nil-clean by Theorem 5.4.4. □

Let R be a commutative ring, and let $\varphi \in M_n(R)$. If $\chi(\varphi) = \prod_{i=1}^{n}(t - \alpha_i)$ where each $\alpha_i \in N(R) \cup (1 + N(R))$, we claim that φ is strongly nil-clean. Choose $h_0 = \prod_{\alpha_i \in N(R)} (t - \alpha_i)$ and $h_1 = \prod_{\alpha_i \in 1+N(R)} (t - \alpha_i)$. Then $\chi(\varphi) = h_0h_1$ with $h_0 \in \mathbb{P}_0$ and $h_1 \in \mathbb{P}_1$, and we are done by Theorem 5.4.9.

We say that $\varphi, \psi \in M_n(R)$ are similar in the case that there exists an invertible matrix ϕ such that $\psi = \phi^{-1}\varphi\phi$.

Corollary 5.4.10. *Let R be a projective-free ring and let $\varphi \in M_n(R)$. Then the following are equivalent:*

(1) *φ is strongly nil-clean.*

(2) *φ is similar to $\begin{pmatrix} \varphi_0 & \\ & \varphi_1 \end{pmatrix}$, where $\varphi_0 \in M_r(R)$ and $I_{n-r} - \varphi_1 \in M_{n-r}(R)$ $(0 \leq r \leq n)$ are nilpotent.*

Proof. (1) $\Rightarrow$ (2) As φ is strongly nil clean, there exists an idempotent $E \in M_n(R)$ and a nilpotent $W \in M_n(R)$ such that $\varphi = E + W$ and $EW = WE$. In view of Theorem 2.4.7, there exists an invertible $U \in M_n(R)$ such that $UEU^{-1} = \begin{pmatrix} 0_r & \\ & I_{n-r} \end{pmatrix}$; hence, $U\varphi U^{-1} = \begin{pmatrix} 0_r & \\ & I_{n-r} \end{pmatrix} + UWU^{-1}$. Set $V = UWU^{-1} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$, where $W_{11} \in M_r(R), W_{12} \in M_{r\times(n-r)}(R), W_{21} \in M_{(n-r)\times r}(R)$ and $W_{22} \in M_{n-r}(R)$. It follows from $EW = WE$ that $W_{12} = 0_{r\times(n-r)}$ and $W_{21} = 0_{(n-r)\times r}$. This implies that $UEU^{-1} = \begin{pmatrix} W_{11} & 0_{r\times(n-r)} \\ 0_{(n-r)\times r} & I_{n-r} + W_{22} \end{pmatrix}$. It is easy to check that $W_{11} \in M_r(R)$ and $W_{22} \in M_{n-r}(R)$ are nilpotent, as required.

(2) $\Rightarrow$ (1) By hypothesis, $\chi(\varphi) = \chi(\varphi_0)\chi(\varphi_1)$. Since $\varphi_0 \in M_r(R)$ and $I_{n-r} - \varphi_1 \in M_{n-r}(R)$ $(0 \leq r \leq n)$ are nilpotent, it follows from Theorem 4.3.2 that $\chi(\varphi_0) \in \mathbb{P}_0$ and $\chi(\varphi_1) \in \mathbb{P}_1$. According to Theorem 5.4.9, we obtain the result. □

For instance, $A \in M_2(\mathbb{Z}_4)$ is strongly nil clean if and only if A is nilpotent, or $I_2 - A$ is nilpotent, or A is similar to one of the matrices in the set

$$\left\{\begin{pmatrix} \overline{0} & \\ & \overline{1} \end{pmatrix}, \begin{pmatrix} \overline{0} & \\ & \overline{3} \end{pmatrix}, \begin{pmatrix} \overline{2} & \\ & \overline{1} \end{pmatrix}, \begin{pmatrix} \overline{2} & \\ & \overline{3} \end{pmatrix}\right\}.$$

Example 5.4.11. Let $\mathbb{Z}_4 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$, and let $A = \begin{pmatrix} \overline{3}\ \overline{3}\ \overline{0}\ \overline{0} \\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0} \\ \overline{0}\ \overline{0}\ \overline{3}\ \overline{3} \\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{3} \end{pmatrix} \in M_4(\mathbb{Z}_4)$.

Then $\mathbb{Z}_4$ is a commutative local ring with $N(\mathbb{Z}_4) = \{\overline{0}, \overline{2}\}$. Clearly, $\chi(A) = t(t-\overline{3})^3$. Set $h_0 = t$ and $h_1 = (t-\overline{3})^3$. Then $\chi(A) = h_0h_1$ with $h_0 \in \mathbb{P}_0$ and $h_1 \in \mathbb{P}_1$. In view of Theorem 5.4.9, $A \in M_4(\mathbb{Z}_4)$ is strongly nil-clean. In fact, we have the decomposition $A = E + W$ and $EW = WE$, where

$$E = E^2 = \begin{pmatrix} \overline{1}\ \overline{1}\ \overline{0}\ \overline{0} \\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0} \\ \overline{0}\ \overline{0}\ \overline{1}\ \overline{0} \\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{1} \end{pmatrix}$$

and

$$W = \begin{pmatrix} \overline{2}\ \overline{2}\ \overline{0}\ \overline{0} \\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0} \\ \overline{0}\ \overline{0}\ \overline{2}\ \overline{2} \\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{2} \end{pmatrix} \in N(M_4(\mathbb{Z}_4)).$$

That is, A is the sum of an idempotent matrix and a nilpotent matrix which commute.

Example 5.4.12. Let $\mathbb{Z}_4[i] = \mathbb{Z}_4[x]/(x^2+1) = \{a+bi \mid a, b \in \mathbb{Z}_4, i^2 = -1\}$. Then $\mathbb{Z}_4[i]$ is a commutative local ring and $J(\mathbb{Z}_4[i]) = \{a + bi \mid a, b \in \mathbb{Z}_4, a^2 - b^2 \in J(\mathbb{Z}_4), i^2 = -1\} = \{\overline{0}, \overline{2}i, \overline{1}+i, \overline{1}+\overline{3}i, \overline{2}, \overline{2}+\overline{2}i, \overline{3}+i, \overline{3}+\overline{3}i\}$. It is easy to check that $J(\mathbb{Z}_4[i])$ is nil, and so $N(\mathbb{Z}_4[i]) = J(\mathbb{Z}_4[i])$. Let $A = \begin{pmatrix} \overline{0}\ \overline{0}\ \overline{2}+\overline{2}i \\ \overline{1}\ \overline{0}\ \overline{1}+i \\ \overline{0}\ \overline{1}\ \overline{1}+\overline{2}i \end{pmatrix} \in M_3(\mathbb{Z}_4[i])$. Then $\chi(A) = t^3+(\overline{3}+\overline{2}i)t^2+(\overline{3}+\overline{3}i)t+(\overline{2}+\overline{2}i)$.

Set $h_0 = t^2 + (\overline{1} + \overline{3}i)t + (\overline{2} + \overline{2}i)$ and $h_1 = t + (\overline{2} + \overline{3}i)$. Then $h = h_0h_1$ with $h_0 \in \mathbb{P}_0$ and $h_1 \in \mathbb{P}_1$. According to Theorem 5.4.9, $A \in M_3\big(\mathbb{Z}_4[i]\big)$ is strongly nil-clean. That is, A is the sum of an idempotent matrix and a nilpotent matrix which commute.

For the strong nil-cleanness of 2×2 matrices over projective-free rings, we derive

Theorem 5.4.13. *Let R be projective-free. Then $A \in M_2(R)$ is strongly nil clean if and only if*

(1) $det(A) \in 1 + N(R)$ *and* $tr(A) \in 2 + N(R)$, *or*
(2) $\chi(A)$ *has a root in* $N(R)$ *and a root in* $1 + N(R)$, *or*
(3) $det(A) \in N(R)$ *and* $tr(A) \in N(R)$.

Proof. $\Longrightarrow$ Suppose that $A \in M_2(R)$ is strongly nil-clean. In view of Corollary 5.4.10, $A \in N(M_2(R))$, $I_2 - A \in N(M_2(R))$ or A is isomorphic to $diag(\alpha, \beta)$, where $\alpha \in N(R), \beta \in 1 + N(R)$. If $A \in N(M_2(R))$, then $\chi(A) = t^2 - tr(A)t + det(A) \equiv t^2 (mod\ N(R))$, and so $tr(A), det(A) \in N(R)$. If $I_2 - A \in N(M_2(R))$, then $\chi(I_2 - A) = t^2 - (2 - tr(A))t + det(I_2 - A) \equiv t^2 (mod\ N(R))$; hence, $tr(A) \in 2 + N(R)$ and $det(A) \in tr(A) - 1 + N(R) \in 1 + N(R)$; otherwise, $\chi(A)$ has a root $\alpha \in N(R)$ and a root $\beta \in 1 + N(R)$.

$\Longleftarrow$ If (1) holds, then $\chi(A) = h_0h_1$ where $h_0 = 1 \in \mathbb{P}_0$ and $h_1 = t^2 - tr(A)t + det(A) \in \mathbb{P}_1$. According to Theorem 5.4.9, $A \in M_2(R)$ is strongly nil clean. If (2) holds, then $\alpha^2 - tr(A)\alpha + det(A) = 0$ and $\beta^2 - tr(A)\beta + det(A) = 0$. As $\alpha - \beta \in U(R)$, we deduce that $tr(A) = \alpha + \beta$ and $det(A) = \alpha\beta$. Hence, $\chi(A) = (t - \alpha)(t - \beta)$ where $\alpha \in N(R)$ and $\beta \in 1 + N(R)$. Thus, $\chi(A) = h_0h_1$ where $h_0 = t - \alpha \in \mathbb{P}_0$ and $h_1 = t - \beta \in \mathbb{P}_1$. According to Theorem 5.4.9, $A \in M_2(R)$ is strongly nil-clean. If (3) holds, then $\chi(A) = h_0h_1$ where $h_0 = t^2 - tr(A)t + det(A) \in \mathbb{P}_0$ and $h_1 = 1 \in \mathbb{P}_1$. According to Theorem 5.4.9, $A \in M_2(R)$ is strongly nil-clean. Therefore $A \in M_3(R)$ is strongly nil-clean. □

Corollary 5.4.14. *Let R be a principal ideal domain. Then $A \in M_2(R)$ is strongly nil clean if and only if*

(1) $A^2 = 0$, *or*
(2) $(I_2 - A)^2 = 0$, *or*
(3) $A^2 = A$.

Proof. $\Longleftarrow$ This is clear.

$\Longrightarrow$ Since $N(R) = 0$, by Theorem 5.4.13, we have

Case 1. $det(A) = 1, tr(A) = 2$. By the Cayley-Hamilton Theorem, $A^2 - 2A + I_2 = 0$, and so $(I_2 - A)^2 = 0$.

Case 2. $\chi(A)$ has two roots 0 and 1.

Case 3. $det(A) = tr(A) = 0$. By the Cayley-Hamilton Theorem, $A^2 = 0$.

Therefore we complete the proof. □

Example 5.4.15. Let $\mathbb{Z}_{(2)} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, (m, n) = 1, 2 \nmid n\}$, and let $A = \begin{pmatrix} \frac{2}{9} & 1 \\ 1 & 0 \end{pmatrix}$. Then $A, I_2 - A \notin N(M_2(\mathbb{Z}_{(2)}))$. Moreover, the characteristic polynomial $\chi(A) = t^2 - \frac{2}{9}t - 1$ has not a root in $N\big(\mathbb{Z}_{(2)}\big)$. Thus $A \in M_2\big(\mathbb{Z}_{(2)}\big)$ is not strongly nil-clean by Theorem 5.4.13. Accordingly, $A \in M_2(\mathbb{Z}_{(2)})$ is strongly clean.

Let R be a projective-free ring. We claim that $M_2(R)$ is not strongly nil-clean. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in M_2(R)$. Then $A \in GL_2(R)$ and $I_2 - A \in GL_2(R)$. Thus, $A, I_2 - A \in M_2(R)$ are not nilpotent. Suppose that $A \in M_2(R)$ is strongly nil-clean. In view of Corollary 5.4.10, A is similar to a diagonal matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, where $\lambda \in N(R), \mu \in 1 + N(R)$. Hence, $-1 = detA = \lambda\mu \in N(R)$, a contradiction. Therefore $A \in M_2(R)$ is not strongly nil-clean.

Notes. Theorem 5.1.1 is due to Hirano et al. [317, Theorem 3] and Wang [532, Lemma 2.2]. Theorem 5.1.5 was proved by Kosan et al. [373, Theorem 2.7]. Theorem 5.2.4 was proved by Diesl [235, Theorem 4.1]. Theorem 5.2.11 and Theorem 5.2.14 are due to Kosan et al. [373, Theorem 3.4 and Theorem 3.6]. Theorem 5.3.10 is [510, Theorem] and [443, Theorem 1 and Corollary 2]. Theorem 5.4.1 was first observed by Diesl [235, Corollary 3.2].

Chapter 6

Classes of Strongly Clean Rings

Let R be an associative ring with an identity. We say that R has stable range 1 provided that $aR + bR = R$ with $a, b \in R$ implies that there exists some $y \in R$ such that $a + by \in U(R)$. One of the most important features of stable range 1 is cancellation of related modules from direct sums. A long standing question asks whether every strongly clean ring has stable range 1. The problem that motivates this chapter is to describe several subclasses of strongly clean rings with stable range 1. It is shown that every strongly π-regular ring has stable range 1. Furthermore, a ring R is strongly π-regular if and only if for any $a \in R$ there exists an idempotent $e \in R$ such that $a - e \in U(R), ae = ea$ and $eae \in N(R)$. We shall replace $N(R)$ by $J(R), J^{\#}(R)$ and R^{qnil} respectively, and introduce other variants of strongly clean rings. We shall explore various conditions under which a ring R is strongly J_{π}-clean, and determine when a ring is quasipolar.

6.1 Strongly π-Regular Rings

A ring R is strongly π-regular if for any $a \in R$, the chain $aR \supseteq a^2R \supseteq \cdots$ terminates. Strongly π-regular rings are abundant. For instance, right (left) artinian rings and periodic rings are strongly π-regular. In this section we establish elementary properties of strongly π-regular rings that are related to strong cleanness.

Theorem 6.1.1. *Let M be a right R-module, let $\alpha \in End(M)$. Then the following are equivalent:*

(1) *$\alpha \in End(M)$ is strongly π-regular.*
(2) *There exists some $n \in \mathbb{N}$ such that $M = Ker(\alpha^n) \oplus Im(\alpha^n)$.*

Proof. (1) $\Rightarrow$ (2) Since $\alpha \in End(M)$ is strongly π-regular, there exists $\beta \in End(M)$ such that $\alpha^n = \alpha^{2n}\beta$ and $\alpha\beta = \beta\alpha$ for some $n \in \mathbb{N}$. For any $m \in M$, $m = (1 - \alpha^n\beta)(m) + \alpha^n\beta(m) \in Ker(\alpha^n) \oplus Im(\alpha^n)$, and so $M = Ker(\alpha^n) + Im(\alpha^n)$. If $\alpha^n(m) \in Ker(\alpha^n)$, then $\alpha^{2n}(m) = 0$. Hence, $\beta\alpha^{2n}(m) = 0$, i.e., $\alpha^n(m) = 0$. This implies that $Ker(\alpha^n) \cap Im(\alpha^n) = 0$, as required.

(2) $\Rightarrow$ (1) By hypothesis, there exists some $n \in \mathbb{N}$ such that $M = Ker(\alpha^n) \oplus Im(\alpha^n)$. Hence, $\alpha^n M = \alpha^{2n}M$. Construct an R-morphism $\beta : M \to M$ given by $\beta(s + \alpha^{2n}(x)) = \alpha^n(x)$ for any $s \in Ker(\alpha^n)$ and any $x \in M$. If $s + \alpha^{2n}(x) = 0$, then $\alpha^{2n}(x) = 0$, and so $\alpha^n(x) \in Ker(\alpha^n)$. This implies that $\alpha^n(x) = 0$. Hence, β is well defined, and so $\beta \in End(M)$. One easily checks that $\alpha^n = \alpha^n\beta\alpha^n$ and $\alpha^n\beta = \beta\alpha^n$. Therefore $\alpha^n = \alpha^{2n}\beta = \beta\alpha^{2n}$, as desired. □

A module M is said to satisfy Fitting's Lemma if for each $f \in End_R(M)$ there exists $n \in \mathbb{N}$ such that $M = Ker(f^n) \oplus f^n(M)$. By Theorem 6.1.1, M satisfies Fitting's Lemma if and only if $End_R(M)$ is strongly π-regular. Let M be a right R-module. Given a chain of submodules of M of the form $N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_n$, we say that n is the length of the chain. The length of M is defined to be the largest length of any of its chains. If no such largest length exists, we say that M has infinite length. As is well known, A module M has finite length if and only if it is both artinian and noetherian. The length of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ is equal to the number of prime factors of n, with multiple prime factors counted multiple times. The following result is known as the Fitting's Lemma.

Corollary 6.1.2. *Let R be a ring, and let M be a finite length module over R. Then $End_R(M)$ is strongly π-regular.*

Proof. Let $f \in End_R(M)$. Then we have the descending sequence $Im(f) \supseteq Im(f^2) \supseteq Im(f^3) \supseteq \cdots$, the ascending sequence $Ker(f) \subseteq Ker(f^2) \subseteq Ker(f^3) \subseteq \cdots$. As M has finite length, we can find some n with $Ker(f^n) = Ker(f^{n+1}) = \cdots = Ker(f^{2n}) = \cdots$ and $Im(f^n) = Im(f^{n+1}) = \cdots = Im(f^{2n}) = \cdots$. Clearly, $Ker(f^n) \cap Im(f^n) = 0$. For any $x \in M$, $x = (x - f^n(x)) + f^n(x) \in Ker(f^n) + Im(f^n)$, and so $M = Ker(f^n) + Im(f^n)$.

Thus, $M = Ker(f^n) \oplus Im(f^n)$. In light of Theorem 6.1.1, we obtain the result. □

If M is indecomposable and has finite length, it follows from Corollary 6.1.2 that every endomorphism of M is either isomorphic or nilpotent.

A right R-module M is quasi-projective provided that for any R-epimorphism $g : M \to A$ and any R-morphism $f : M \to A$, there exists $h \in End_R(M)$ such that the diagram

$$\begin{array}{ccc} & & M \\ & h \swarrow & \downarrow f \\ M & \overset{g}{\twoheadrightarrow} & A \end{array}$$

commutes.

Corollary 6.1.3. *Let M be a quasi-projective right R-module. Then the following are equivalent:*

(1) *$End(M)$ is strongly π-regular.*
(2) *For any $\alpha \in End(M)$, there exists some $n \in \mathbb{N}$ such that $M = Ker(\alpha^n) + Im(\alpha)$.*

Proof. (1) $\Rightarrow$ (2) This is clear from Theorem 6.1.1.

(2) $\Rightarrow$ (1) For any $\alpha \in End(M)$, $M = Ker(\alpha^n) + Im(\alpha)$ for some $n \in \mathbb{N}$, and so $\alpha^n M = \alpha^{n+1} M$. As M is quasi-projective, there exists $\beta \in End(M)$ such that the following diagram

$$\begin{array}{ccc} & & M \\ & \beta \swarrow & \downarrow \alpha^n \\ M & \overset{\alpha^{n+1}}{\twoheadrightarrow} & \alpha^{n+1} M \end{array}$$

commutes, i.e., $\alpha^n = \alpha^{n+1}\beta$, and we therefore complete the proof. □

Furthermore, we have

Proposition 6.1.4. *Let R be a commutative ring, and let $a \in R$. Then the following are equivalent:*

(1) *$a \in R$ is strongly π-regular.*
(2) *There exists some $n \in \mathbb{N}$ such that $R = a^n R \oplus r(a^n)$.*
(3) *$R = aR + \bigcup_{n=1}^{\infty} r(a^n)$.*

Let M be a right R-module, and let $\alpha \in End(M)$. In contrast with the preceding, we claim that α is strongly clean if $M = \Big(\bigcap_{n\in\mathbb{N}} Im(\alpha^n)\Big) \oplus \Big(\bigcup_{n\in\mathbb{N}} Ker(\alpha^n)\Big)$. Clearly, $\alpha \mid_{\bigcap_{n\in\mathbb{N}} Im(\alpha^n)}$ and $(1-\alpha) \mid_{\bigcup_{n\in\mathbb{N}} Ker(\alpha^n)}$ are both automorphisms, and we are done by Theorem 1.1.14.

Lemma 6.1.5. *A ring R is strongly π-regular if and only if for any $a \in R$ there exist $n \in \mathbb{N}$ and $x \in R$ such that $a^n = a^{2n}x, ax = xa$ and $x = x^2a^n$.*

Proof. Suppose R is strongly π-regular. For any $a \in R$, there exists $n \in \mathbb{N}$ and $b \in R$ such that $a^n = a^{n+1}b$ and $ab = ba$. Choose $x = b^n a^n b^n$. Then x will work. The converse is obvious. □

Lemma 6.1.6. *Let R be a ring. For $a \in R$, the equations $a = a^2x, ax = xa$ and $x = x^2a$, have at most one solution.*

Proof. For $a \in R$, we assume that x and y are solutions. Then
$$x = x^2a = x^2(a^2y) = x(a^2x)y = (xa)(y^2a) = (xa^2)y^2 = ay^2 = y,$$
as asserted. □

Theorem 6.1.7. *Let R be a ring, and let $\{R_i\}_{i\in I}$ be a collection of strongly π-regular subrings. Then $\bigcap_{i\in I} R_i$ is strongly π-regular.*

Proof. Let $a \in \bigcap_{i\in I} R_i$. For any $i \in I$, $a \in R_i$. As R_i strongly π-regular, we can find $x \in R_i$ such that $a^n = a^{2n}x, ax = xa$ and $x = x^2a^n$ $(n \in \mathbb{N})$, by Lemma 6.1.5. For any $j \in I$, we see that $a^n \in R_j$, by using Lemma 6.1.5 again, we have $y \in R_j$ such that $a^{nm} = a^{2nm}y, a^n y = ya^n$ and $y = y^2a^{nm} (m \in \mathbb{N})$. Hence, $a^{nm}y = ya^{nm}$. But $a^{nm} = a^{2nm}x^m, a^{nm}x^m = x^m a^{nm}$ and $x^m = (x^m)^2 a^{nm}$. It follows from Lemma 6.1.6 that $y = x^m$. If $m = 1$, then $x \in R_j$; otherwise, $x = x^2a^n = x^m a^{(m-1)n} = y^{(m-1)n} \in R_j$. This shows that $x \in R$, as required. □

An integral extension of a commutative ring R is a commutative ring S containing R such that every element x of S is integral over R, that is, satisfies an equation of the form $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$, where each $a_i \in R$.

Lemma 6.1.8. *Let S be an integral extension of a commutative ring R, and let Q is a prime ideal of S. If $P := R \cap Q$ is a maximal ideal of R, then Q is a maximal ideal of S.*

Proof. Clearly, there exists a maximal ideal Q' of S such that $Q \subseteq Q'$. Set $P' = R \cap Q'$. Then $P \subseteq P'$. As P is a maximal ideal of R, we get $P = P'$. If $Q \subsetneqq Q'$, we choose $u \in Q' \setminus Q$. By hypothesis, there exists $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in R[x](n \in \mathbb{N})$ such that $p(u) = 0 \in Q$. Choose $f(x) = x^m + b_{m-1}x^{m-1} + \cdots + b_0 \in R[x](m \in \mathbb{N})$ with the least degree m such that $f(u) \in Q$. Then $b_0 \in R \cap Q' = P' = P = R \cap Q$, and so $u\big(u^{m-1} + b_{m-1}u^{m-2} + \cdots + b_1\big) = -b_0 \in Q$. As $u^{m-1} + b_{m-1}u^{m-2} + \cdots + b_1 \notin Q$, we have $u \in Q$, a contradiction. We infer that $Q = Q'$, and thus yielding the result. □

Lemma 6.1.9. *Let R be a commutative ring in which every prime ideal is maximal. If $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in R[x](n \in \mathbb{N})$, then every prime ideal of $R[x]/\big(p(x)\big)$ is maximal.*

Proof. Let $\varphi : R \to R[x]/\big(p(x)\big), r \mapsto \overline{r}$. Then φ is injective, and then $R \cong Im\varphi$. Set $T = Im\big(\varphi\big)$. Then $R[x]/\big(p(x)\big)$ is an integral extension of T. If Q is a prime ideal of $R[x]/\big(p(x)\big)$, then $P =: T \cap Q$ is a prime ideal of T. By hypothesis, P is a maximal ideal of T. In view of Lemma 6.1.8, Q is a maximal ideal of $R[x]/\big(p(x)\big)$, as required. □

Theorem 6.1.10. *Let R be a commutative ring. Then the following are equivalent:*

(1) *R is strongly π-regular.*
(2) *$R/J(R)$ is regular and $J(R)$ is nil.*
(3) *Every prime ideal of R is maximal.*
(4) *Every injective endomorphism of a finitely generated R-module is an isomorphism.*

Proof. (1) ⇒ (2) For any $x \in J(R)$, there exist $n \in \mathbb{N}$ and $y \in R$ such that $x^n = x^{n+1}y$; hence, $x^n(1 - xy) = 0$. Thus, $x^n = 0$. This shows that $J(R)$ is nil. As R is commutative, $J(R) = N(R)$, therefore $R/J(R)$ is regular, by Theorem 5.3.2.

(2) ⇒ (1) As $J(R)$ is nil, we get $J(R) = N(R)$. By Theorem 5.3.2, R is strongly π-regular.

(1) ⇒ (3) Let P be a prime ideal of R, but it is not maximal. Then there exists a maximal ideal M such that $P \subsetneqq M \subsetneqq R$. Choose $x \in M$ while $x \notin P$. Write $x^n = x^{n+1}y$ with $n \in \mathbb{N}$ and $y \in R$. Then $\overline{x^n}\big(\overline{1} - \overline{xy}\big) = 1$ in R/P. As R/P is prime, $\overline{x} = 0$ or $\overline{xy} = \overline{1}$. But $x \notin P$, we get

$1-xy \in P \subseteq M$. This implies that $1=(1-xy)+xy \in M$, a contradiction. Therefore every prime ideal of R is maximal.

(3) $\Rightarrow$ (1) If R is not strongly π-regular, there exists $a \in R$ such that $a^n \notin a^{n+1}R$ for any $n \in \mathbb{N}$.

Let $\Omega = \{I \lhd R \mid \overline{a}^n \notin \overline{a}^{n+1}\big(R/I\big) \text{ for any } n \in \mathbb{N}\}$. Clearly, Ω is a non-empty inductive set. By using Zorn's Lemma, there exists an ideal Q which is maximal in Ω. If Q is not prime, we have two ideals I and J such that

$$I \nsubseteq Q, J \nsubseteq Q \text{ and } IJ \subseteq Q.$$

By the maximality, we can find some $s,t \in \mathbb{N}$ such that $a^s - a^{s+1}y \in I$ and $a^t - a^{t+1}z \in J$ for $y,z \in R$. Thus, $a^s - a^{s+t+1}y^{t+1} \in I$ and $a^t - a^{s+t+1}z^{s+1} \in J$. Hence, there exists $x \in R$ such that $a^{s+t} - a^{s+t+1}x \in IJ \subseteq Q$; hence, $\overline{a}^{s+t} \in \overline{a}^{s+t}\big(R/Q\big)$. This contradicts the choice of Q. Accordingly, R is strongly π-regular.

(3) $\Rightarrow$ (4) Let M be a finitely generated R-module, and let $\sigma : M \to M$ be injective. For any $f(x) \in R[x]$, $m \in M$, we define $f(x) \cdot m = f(\sigma)(m)$. Then M is a finitely generated $R[x]$-module. Let $I = \{f(x) \in R[x] \mid f(x)M = 0\}$ and $S = R[x]/I$. In view of Lemma 2.2.11, we have $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in R[x]$ such that $p(\sigma) = 0$. Let $\varphi : R[x]/\big(p(x)\big) \to S, \overline{f(x)} \mapsto \overline{f(x)}$. If $f(x) \in \big(p(x)\big)$, then $f(\sigma) = 0$; hence, $f(x)M = 0$. Thus, $f(x) \in I$, and so φ is well defined. Moreover, it is surjective. This implies that $S \cong \big(R[x]/(p(x))\big)/Ker\big(\varphi\big)$. By virtue of Lemma 6.1.9, every prime ideal of $R[x]/\big(p(x)\big)$ is maximal. Accordingly, every prime ideal of S is maximal.

Let $u = x + I \in S$. If $u \in U(S)$, then we have some $g(x) \in R[x]$ such that $(x+I)\big(g(x)+I\big) = 1 + I$, i.e., $1 - xg(x) \in I$. Further, $\big(1 - xg(x)\big)M = 0$. For any $m \in M$, $\big(1-xg(x)\big)m = 0$, and so $m = xg(x)(m) = \sigma g(\sigma)(m) = \sigma\big(g(x)(m)\big)$. Thus, $\sigma : M \to M$ is surjective. Therefore it is an R-isomorphism.

If $u \notin U(S)$, then $uS \neq S$. Thus, we can find a maximal ideal P of S such that $u \in P$. Set $S_P = \{\frac{a}{b} \mid a \in S, b \in S \setminus P\}$. Then $P_P = \{\frac{a}{b} \mid a \in P, b \in S \setminus P\}$ is the unique maximal ideal of S_P. As every prime ideal of S_P is maximal, $P_P = \{\frac{a}{b} \mid a \in P, b \in S - P\}$ is the intersection of all prime ideals of S_P. By Lemma 4.3.1, P_P is nil. It follows from $\frac{u}{1} \in P_P$ that $\frac{u^k}{1} = \frac{0}{1}$ for some $k \in \mathbb{N}$. Hence, we have a $t \in S \setminus P$ such that $tu^k = 0$, and so $tu^{k-1} \in J$, where $J = \{s \in S \mid us = 0\}$. If $tu^{k-1} = 0$ and $t = \overline{g(x)}$, then $x^{k-1}g(x) \in I$, and so $\sigma^{k-1}g(\sigma)M = 0$. Since $\sigma : M \to M$ is injective, $g(x) \in I$, and then $t = 0$, a contradiction. Therefore, $tu^{k-1} \neq 0$;

whence, $J \neq 0$. Define $\overline{f(x)} \cdot m = f(x)m$. Then M is an S-module. As $uJ = 0$, we get $uJM = 0$, and thus $\sigma\big(JM\big) = 0$. Since $\sigma : M \to M$ is injective, it follows that $JM = 0$; whence, $J = 0$, a contradiction. Therefore $\sigma : M \to M$ is an R-isomorphism.

(4) $\Rightarrow$ (1) Suppose that P is a prime ideal of R, while it is not maximal. Then we can find a maximal ideal M such that $P \subsetneqq M$. Choose $x \in M$, but $x \notin P$. Construct an R-morphism $\varphi : R/P \to R/P, \overline{r} \mapsto \overline{rx}$. Then φ is injective. By hypothesis, φ is an R-isomorphism. It follows that $\overline{1} = \overline{yx}$ for $y \in R$. That is, $1 - yx \in P \subseteq M$. But $x \in M$, and so $1 \in M$, a contradiction. Accordingly, every prime ideal of R is maximal. □

We claim that every injective linear transformation of a finite-dimensional vector space over a field is an isomorphism. We are easily through by Theorem 6.1.10. Let $R = F[x_1, x_2]$ with a field F, and let (x_1) and (x_2) be principal ideals of R. Then (x_1) and (x_2) are prime. But $(x_i) \subsetneqq (x_1, x_2)$; hence, neither (x_1) nor (x_2) is maximal. Thus, R is not strongly π-regular. By Theorem 6.1.10, there exists a finitely generated R-module M and an injective R-morphism of M which is not an R-isomorphism. Contracting to this result, we record a known result which is due to Vasconcelos.

Proposition 6.1.11. *Let M be a finitely generated module over a commutative ring R. If $\sigma : M \to M$ is surjective, then σ is an R-isomorphism.*

Proof. For any $f(x) \in R[x]$, $m \in M$, we define $f(x) \cdot m = f(\sigma)(m)$. Then M is an $R[x]$-module. Let $\{m_1, \ldots, m_s\}$ be the generating set of R-module M. Then for any $m \in M$, there exist $r_1, \ldots, r_s \in R$ such that $m = \sum_{i=1}^{s} r_i \cdot m_i \in {}_{R[x]}M$. Thus, M ia finitely generated $R[x]$-module. Clearly, $(x)M \subseteq M$. For any $m \in M$, we have $z \in M$ such that $m = \sigma(z) = x \cdot z \in (x)M$, as $\sigma : M \to M$ is surjective. This implies that $(x)M = M$. By virtue of Nakayama's Lemma, $R[x] = (x) + ann_{R[x]}(M)$; hence, $\big(1 - p(x)x\big)M = 0$ for $p(x) \in R[x]$. If $\sigma(m) = 0$, then $m = m + p(\sigma)\sigma(m) = \big(1 + p(\sigma)\sigma\big)(m) = 0$, i.e., σ is injective. Therefore σ is an R-isomorphism. □

Lemma 6.1.12. *Let R be a ring, and let $a \in R$. Then the following are equivalent:*

(1) *$a \in R$ is strongly π-regular.*
(2) *There exists an idempotent $e \in R$ such that $a - e \in U(R), ae = ea$ and $eae \in N(R)$.*

(3) *There exists a unique idempotent* $e \in R$ *such that* $a-e \in U(R), ae = ea$ *and* $eae \in N(R)$.

Proof. (1) $\Rightarrow$ (2) As in the proof of Theorem 1.1.2, there exists an idempotent $e \in R$ such that $a - e \in U(R), ae = ea$. By an explicit observation, we have $(eae)^n = (1-f)a^n = (1-a^nc)a^n = 0$, as desired.

(2) $\Rightarrow$ (3) By hypothesis, there exists an idempotent $e \in R$ such that $u := a - e \in U(R), ae = ea$ and $(eae)^n = 0$ for some $n \in \mathbb{N}$. Assume that there exists an idempotent $f \in R$ such that $v := a - f \in U(R), af = fa$ and $(faf)^n = 0$ for some $m \in \mathbb{N}$. Then $(1-e)u^{mn} = \big((1-e)u\big)^{mn} = (1-e)a^{mn} = a^{mn} = (1-f)a^{mn} = \big((1-e)v\big)^{mn} = (1-f)v^{mn}$; hence, $f(1-e) = e(1-f) = 0$. Observing that $(1-e)fe = (1-e)a^n\big(u^{-1}\big)^n fe = a^n\big(u^{-1}\big)^n fe = \big(u^{-1}\big)^n a^n fe = 0$, we obtain $fe = efe$. Likewise, $ef = efe$. Hence, $ef = fe$. Therefore $e = ef = fe = f$, and the uniqueness is proved.

(3) $\Rightarrow$ (1) Suppose there exists a unique idempotent $e \in R$ such that $u := a - e \in U(R), ae = ea$ and $(eae)^n = 0$ for some $n \in \mathbb{N}$. Then $a^n = ea^n + (1-e)a^n = \big((1-e)a\big)^n = \big((1-e)u\big)^n$; hence, $a^n\big(u^{-1}\big)^n = 1-e$. Accordingly, $a^n = a^n\big(u^{-1}\big)^n a^n = a^{2n}\big(u^{-1}\big)^n = \big(u^{-1}\big)^n a^{2n}$, as required. □

Theorem 6.1.13. *Let* $E = End(_RM)$, *and let* $\alpha \in E$. *Then the following are equivalent:*

(1) α *is strongly* π*-regular in* E.
(2) *There exists* $\pi^2 = \pi \in E$ *such that* $\alpha\pi = \pi\alpha$, $\alpha\pi \in U\big(\pi E\pi\big)$ *and* $\alpha(1-\pi) \in N\big((1-\pi)E(1-\pi)\big)$.
(3) $M = P \oplus Q$ *where* P *and* Q *are* α*-invariant, and* $\alpha|_P \in Aut(P)$ *and* $\alpha|_Q \in N\big(End(Q)\big)$.

Proof. (1) $\Rightarrow$ (2) By virtue of Lemma 6.1.12, there exists an idempotent $\pi \in E$ such that $\alpha-(1-\pi) \in U(E), \alpha\pi = \pi\alpha$ and $\alpha(1-\pi) \in N\big((1-\pi)E(1-\pi)\big)$. Thus, $\alpha\pi \in U\big(\pi E\pi\big)$, as needed.

(2) $\Rightarrow$ (3) By hypothesis, there exists $\pi^2 = \pi \in E$ such that $\alpha\pi = \pi\alpha$, $\alpha\pi \in U\big(\pi E\pi\big)$ and $\alpha(1-\pi) \in N\big((1-\pi)E(1-\pi)\big)$. Set $P = M\pi$ and $Q = M(1-\pi)$. Then $M = P \oplus Q$. As $\alpha\pi = \pi\alpha$, P and Q are α-invariant. Clearly, $\pi E\pi \cong End(P)$ and $(1-\pi)E(1-\pi) \cong End(Q)$. Hence, $\alpha|_P = \alpha\pi|_P \in Aut(P)$ and $\alpha|_Q = \alpha(1-\pi)|_Q \in N\big(End(Q)\big)$, as required.

(3) $\Rightarrow$ (1) Let $\pi : M = P \oplus Q \to P, (p,q) \mapsto p$. Then $\pi = \pi^2 \in End(M)$. Since P and Q are α-invariant, we see that $\alpha\pi = \pi\alpha$. As $\alpha\pi|_P = \alpha|_P \in Aut(P)$, we get $\alpha\pi \in U\big(\pi E\pi\big)$. Similarly, $\alpha(1-\pi) \in N\big((1-\pi)E(1-\pi)\big)$.

Therefore $\alpha - (1-\pi) = \alpha\pi - \big(1 - \pi - \alpha(1-\pi)\big) \in U(E)$. This completes the proof in terms of Lemma 6.1.12. □

Stated differently, Theorem 6.1.13 says that, given a strongly π-regular endomorphism f on M, M can be decomposed into two submodules P and Q such that f restricted to P is an isomorphism, and f restricted to Q is nilpotent. As an immediate consequence of Theorem 6.1.13, we derive

Corollary 6.1.14. *Let R be a ring. Then the following are equivalent:*

(1) *R is strongly π-regular.*
(2) *For any $a \in R$, there exists $\pi^2 = \pi \in R$ such that $a\pi = \pi a$, $a\pi \in U\big(\pi R\pi\big)$ and $(1-a)(1-\pi) \in N\big((1-\pi)R(1-\pi)\big)$.*

Lemma 6.1.15. *Let R be a commutative ring, let $\varphi \in M_n(R)$, and let $h \in R[t]$ be a monic polynomial of degree n. If $h(\varphi) = 0$ and there exists a factorization $h = h_0p_0$ such that $h_0 \in \mathbb{S}_0$ and $h_1 \in \mathbb{P}_0$, then φ is strongly π-regular.*

Proof. Suppose $h = h_0p_0$ where $h_0 \in \mathbb{S}_0$ and $p_0 \in \mathbb{P}_0$, and $h(\varphi) = 0$. Write $h_0 = ta(t) + b, a(t) \in R[t]$ and $p_0 = t^k + c_1t^{k-1} + \cdots + c_{k-1}t + c_l$. Then $b \in U(R), c_1, \ldots, c_l \in N(R)$. We may assume that $deg(h_0) \geq 1$. Clearly, $h_0(-b^{-1}) + ta(t)b^{-1} = 1$, and so $h_0f + t^kg = 1$ for some $f, g \in R[t]$. We infer that $h_0f + p_0g \in 1 + N(R)[t] \subseteq U\big(R[t]\big)$. Therefore, there exist $u_0, v_0 \in R[t]$ such that $u_0h_0 + v_0p_0 = 1$. Hence, $u_0(\varphi)h_0(\varphi) + v_0(\varphi)p_0(\varphi) = 1_{nR}$. But $h(\varphi) = h_0(\varphi)p_0(\varphi) = p_0(\varphi)h_0(\varphi) = 0$. Clearly, $u_0(\varphi)p_0(\varphi) = p_0(\varphi)u_0(\varphi)$ and $h_0(\varphi)v_0(\varphi) = v_0(\varphi)h_0(\varphi)$. By virtue of Lemma 2.2.1, $nR = Ker\big(h_0(\varphi)\big) \oplus Ker\big(p_0(\varphi)\big)$. Additionally, $Ker\big(h_0(\varphi)\big)$ and $Ker\big(p_0(\varphi)\big)$ are both φ-invariant. As $h_0 \in \mathbb{S}_0$ and $h_0\big(\varphi \mid_{Ker(h_0(\varphi))}\big) = 0$, it follows from Lemma 2.2.2 that $\varphi \mid_{Ker(h_0(\varphi))} \in Aut\big(Ker(h_0(\varphi))\big)$. Since $p_0 \equiv t^l \pmod{N(R)}$ and $p_0\big(\varphi \mid_{Ker(p_0(\varphi))}\big) = 0$, by virtue of Theorem 4.3.2, $\varphi \mid_{Ker(p_0(\varphi))} \in N\big(End(Ker(p_0(\varphi)))\big)$. Therefore φ is strongly π-regular, by Theorem 6.1.13. □

We are now ready to extend [78, Proposition 44] and [241, Proposition 3.5.8], and prove the following.

Theorem 6.1.16. *Let R be a projective-free ring, and let $h \in R[t]$ be a monic polynomial of degree n. Then the following are equivalent:*

(1) *Every $\varphi \in M_n(R)$ with $\chi(\varphi) = h$ is strongly π-regular.*

(2) *There exists* $\varphi \in M_n(R)$ *with* $\chi(\varphi) = h$ *is strongly* π*-regular.*
(3) *There exist* $h_0 \in \mathbb{S}_0$ *and* $p_0 \in \mathbb{P}_0$ *such that* $h = h_0 p_0$.

Proof. (1) $\Rightarrow$ (2) Choose $\varphi = C_h$. Then C_h is strongly π-regular as $\chi(\varphi) = h$.

(2) $\Rightarrow$ (3) Let $\varphi \in M_n(R)$ with $\chi(\varphi) = h$ be strongly π-regular. In view of Theorem 6.1.13, we have a decomposition $nR = P \oplus Q$ such that $\varphi\,|_P$ is invertible on P, and $\varphi\,|_Q$ is nilpotent on Q. As R is projective-free, P and Q are both free. Thus, $\chi(\varphi) = \chi\big(\varphi\,|_P\big)\chi\big(\varphi\,|_Q\big)$. Set $h_0 = \chi\big(\varphi\,|_P\big)$ and $p_0 = \chi\big(\varphi\,|_Q\big)$. Then $h_0 \in \mathbb{S}_0$. In view of Lemma 4.3.2, $p_0 \in \mathbb{P}_0$. Finally, $h = h_0 p_0$, as desired.

(3) $\Rightarrow$ (1) This is clear from Lemma 6.1.15. □

Corollary 6.1.17. *Let* R *be a commutative local ring. Then the following are equivalent:*

(1) $M_n(R)$ *is strongly* π*-regular.*
(2) *For any monic* h *of degree* n *in* $R[t]$*, there exist* $h_0 \in \mathbb{S}_0$ *and* $p_0 \in \mathbb{P}_0$ *such that* $h = h_0 p_0$.

Proof. (1) $\Rightarrow$ (2) For any monic h of degree n in $R[t]$, $C_h \in M_n(R)$ is strongly π-regular. In light of Theorem 6.1.16, there exist $h_0 \in \mathbb{S}_0$ and $p_0 \in \mathbb{P}_0$ such that $h = \chi\big(C_h\big) = h_0 p_0$.

(2) $\Rightarrow$ (1) For any $A \in M_n(R)$, we see that $\chi(A) = h_0 p_0$ where $h_0 \in \mathbb{S}_0$ and $p_0 \in \mathbb{P}_0$. By Theorem 6.1.16, $A \in M_n(R)$ is strongly π-regular, hence the result. □

Lemma 6.1.18. *Let* R *be a Hensel local ring. If* $J(R)$ *is nil, then* $M_n(R)$ *is strongly* π*-regular for all* $n \in \mathbb{N}$.

Proof. Clearly, $M_n\big(R/J(R)\big)$ is strongly π-regular. Let $A \in M_n(R)$. Then $\overline{A} \in M_n\big(R/J(R)\big)$ is strongly π-regular. By Theorem 6.1.16, $\overline{\chi(A)} = \chi\big(\overline{A}\big) = GH$ with $G, H \in \big(R/J(R)\big)[t]$ monic, $G \in \mathbb{S}_0, H \in \mathbb{P}_0$. Similarly to the discussion in Lemma 6.1.15, $(G, H) = \overline{1}$. By hypothesis, $\chi(A) = gh$ for some monic $g, h \in R[t]$ with $\overline{g} = G$ and $\overline{h} = H$. Clearly, $g \in \mathbb{S}_0$. Write $H = t^m + \overline{b_1}t^{m-1} + \cdots + \overline{b_m} \in \mathbb{P}_0$. Then each $\overline{b_i}^{k_i} = \overline{0}$ for some $k_i \in \mathbb{N}$, and so $b_i^{k_i} \in J(R)$. As $J(R)$ is nil, we can find some $l_i \in \mathbb{N}$ such that $b_i^{k_i l_i} = 0$. This implies that $h \in \mathbb{P}_0$. According to Theorem 6.1.16, $A \in M_n(R)$ is strongly π-regular, as needed. □

Theorem 6.1.19. *Let* R *be a commutative local ring. Then the following are equivalent:*

(1) $J(R)$ *is nil.*
(2) R *is strongly* π*-regular.*
(3) $M_n(R)$ *is strongly* π*-regular for all* $n \in \mathbb{N}$.

Proof. (1) $\Rightarrow$ (3) Let $f \in R[t]$ be a monic polynomial and $\overline{f} = GH$ in $\big(R/J(R)\big)[t]$ where G and H are monic polynomials with $(G, H) = \overline{1}$. Write $\overline{k} = G$ and $\overline{l} = H$, where $k, l \in R[t]$ are monic. Then $f - kl := \sum_{i=0}^{s} a_i t^i \in J(R)[t]$. Write $I = (a_0, \ldots, a_s)$. As $J(R)$ is nil, I is nilpotent. Say $I^m = 0$. In addition, $\overline{f} = \overline{kl}$ in $\big(R/I\big)[t]$. As in the proof of Theorem 2.2.13, $(k, l) = 1$. Thus, we have some polynomials $\alpha, \beta \in R[t]$ such that $\alpha k + \beta l = 1$. Choose $g_1 = k + \beta(f - kl), h_1 = l + \alpha(f - kl) \in R[t]$. Then $f - g_1 h_1 = -\alpha\beta(f - kl)^2 \in I^2$ and $\overline{g_1} = G, \overline{h_1} = H$. Thus, $\overline{f} = \overline{g_1 h_1}$ and $\big(\overline{g_1}, \overline{h_1}\big) = \overline{1}$ in $\big(R/I^2\big)[t]$, and that $\overline{g_1} = G, \overline{h_1} = H$ in $R/J(R)$. Likewise, we have polynomials $g_2, h_2 \in R[t]$ such that $\overline{f} = \overline{g_2 h_2}$ and $\big(\overline{g_2}, \overline{h_2}\big) = \overline{1}$ in $\big(R/I^4\big)[t]$, and that $\overline{g_2} = G, \overline{h_2} = H$ in $R/J(R)$. By iteration of this process, we have polynomials $g_m, h_m \in R[t]$ such that $\overline{f} = \overline{g_m h_m}$ and $\big(\overline{g_m}, \overline{h_m}\big) = \overline{1}$ in $\big(R/I^{2m}\big)[t]$, and that $\overline{g_m} = G, \overline{h_m} = H$ in $R/J(R)$. Let the leading coefficients of g_m and h_m are c and d respectively. Then $c, d \in 1 + J(R)$, and so $c^{-1}, d^{-1} \in 1 + J(R)$. Choose $g = c^{-1} g_m$ and $h = d^{-1} h_m$. Then $f = gh$ and $\overline{g} = G$ and $\overline{h} = H$ in $\big(R/J(R)\big)[t]$. In addition, $g, h \in R[t]$ are monic. Therefore R is a Hensel local ring. According to Lemma 6.1.18, $M_n(R)$ is strongly π-regular for all $n \in \mathbb{N}$.

(3) $\Rightarrow$ (2) This is trivial.

(2) $\Rightarrow$ (1) This is proved as in the proof of Theorem 6.1.10. □

[111, Example 1] provided a local ring R whose Jacobson radical is nil, but $M_2(R)$ is not strongly π-regular. This shows that the hypothesis that R is commutative in Theorem 6.1.19 is necessary.

Proposition 6.1.20. *Let* R *be a projective-free ring, and let* $A \in M_n(R)$. *Then* $A \in M_n(R)$ *is strongly* π*-regular if and only if* A *is similar to a matrix* $\begin{pmatrix} \beta & \\ & \gamma \end{pmatrix}$, *where* $\beta \in M_k(R)$ *is invertible and* $\gamma \in M_{n-k}(R) (0 \leq k \leq n)$ *is nilpotent.*

Proof. Let $A \in M_n(R)$ be strongly π-regular, and let $\alpha \in End(nR)$ be the corresponding endomorphism if A. By Theorem 6.1.13, $nR = P \oplus Q$ where P and Q are α-invariant, and $\alpha|_P \in Aut(P)$ and $\alpha|_Q \in N\big(End(Q)\big)$. Since R is projective-free, P and Q are free. Let $\{\varepsilon_1, \ldots, \varepsilon_k\}$ and $\{\varepsilon_{k+1}, \ldots, \varepsilon_n\}$ be bases of P and Q. Then $\{\varepsilon_1, \ldots, \varepsilon_k, \varepsilon_{k+1}, \ldots, \varepsilon_n\}$ is a basis of nR.

Let β and γ be the corresponding matrices of $\alpha|_P$ and $\alpha|_Q$, respectively. We conclude that A is similar to $\begin{pmatrix} \beta & \\ & \gamma \end{pmatrix}$, where β is invertible and γ is nilpotent.

Conversely, assume that there exists $P \in GL_2(R)$ such that $PAP^{-1} = \begin{pmatrix} \beta & \\ & \gamma \end{pmatrix}$, where $\beta \in M_k(R)$ is invertible and $\gamma \in M_{n-k}(R)(0 \leq k \leq n)$ is nilpotent. If $k = n$, then A is invertible. If $k = 0$, then A is nilpotent. If $0 < k < n$, we write $\gamma^m = 0$ for some $m \in \mathbb{N}$. Then

$$PA^mP^{-1} = \begin{pmatrix} \beta^m & \\ & 0 \end{pmatrix}, \text{ and } PA^{m+1}P^{-1} = \begin{pmatrix} \beta^{m+1} & \\ & 0 \end{pmatrix}.$$

It follows that

$$A^m = A^{m+1}P^{-1}\begin{pmatrix} \beta^{-1} & \\ & 0 \end{pmatrix}P, \text{ and } AP^{-1}\begin{pmatrix} \beta^{-1} & \\ & 0 \end{pmatrix}P = P^{-1}\begin{pmatrix} \beta^{-1} & \\ & 0 \end{pmatrix}PA,$$

and we therefore complete the proof. □

Corollary 6.1.21. *Let R be a commutative local ring, and let $A \in M_2(R)$. Then $A \in M_2(R)$ is strongly π-regular if and only if*

(1) *$A \in M_2(R)$ is invertible, or*
(2) *$A \in M_2(R)$ is nilpotent, or*
(3) *the equation $x^2 - tr(A)x + det(A) = 0$ has a root in $U(R)$ and a root in $N(R)$.*

Proof. Let $A \in M_2(R)$ be strongly π-regular. By Proposition 6.1.20, A is similar to a matrix $\begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}$, where $\alpha \in M_k(R)$ is invertible and $\beta \in M_{n-k}(R)(0 \leq k \leq n)$ is nilpotent. If $k = n$, then $A \in M_2(R)$ is invertible. If $k = 0$, then $A \in M_2(R)$ is nilpotent. Otherwise, there exists $U \in GL_2(R)$ such that $UAU^{-1} = \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}$ with $\alpha \in U(R)$ and $\beta \in N(R)$. Hence, $\chi(A) = x^2 - tr(A)x + det(A) = x^2 - (\alpha + \beta)x + \alpha\beta$ has a root $\alpha \in U(R)$ and a root $\beta \in N(R)$.

Conversely, if $A \in M_2(R)$ is invertible or nilpotent, the it is strongly π-regular. Now assume that the equation $x^2 - tr(A)x + det(A) = 0$ has a root $\alpha \in U(R)$ and a root $\beta \in N(R)$. Then $tr(A) = \alpha + \beta \in U(R)$ and $det(A) = \alpha\beta \in N(R)$.

Case I. $a_{21} \in U(R)$. Set $U = \begin{pmatrix} \alpha - a_{22} & \beta - a_{22} \\ a_{21} & a_{21} \end{pmatrix}$. Then $U^{-1} = a_{21}^{-1}(\alpha - \beta)^{-1} \begin{pmatrix} a_{21} & a_{22} - \beta \\ -a_{21} & \alpha - a_{22} \end{pmatrix}$. Therefore $U^{-1}AU = \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}$; hence $A \in M_2(R)$ is strongly π-regular by Proposition 6.1.20.

Case II. $a_{21} \in J(R)$.

(i) $a_{12} \in U(R)$. Applying Case I to $A^T \in M_2(R)$, then A is strongly π-regular.

(ii) $a_{12} \in J(R)$. Then $a_{11}+a_{22} \in U(R)$ and $a_{11}a_{22} \in J(R)$. This implies that $(a_{11}-a_{22})^2 = (a_{11}+a_{22})^2 - 4a_{11}a_{22} \in U(R)$; hence, $a_{11}-a_{22} \in U(R)$. Clearly,

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} A \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} - a_{12} & a_{12} \\ a_{11} + a_{21} - a_{12} - a_{22} & a_{12} + a_{22} \end{pmatrix},$$

and that $a_{11} + a_{21} - a_{12} - a_{22} \in U(R)$. Applying Case I, A is strongly π-regular.

So the corollary is true. □

Theorem 6.1.22. *Let R be a ring. Then the following are equivalent:*

(1) *$M_n(R)$ is strongly π-regular for all $n \in \mathbb{N}$.*
(2) *For any finitely generated left R-module M, injective endomorphisms of M are isomorphism.*

Proof. (1) $\Rightarrow$ (2) Let M be a cyclic left R-module. Write $M = R/I$ for a left ideal I. Let $f \in End(M)$ be injective. Then $(\overline{1})f = \overline{x}$ for an $x \in R$. Since R is strongly π-regular, there exists $y \in R$ and $m \in \mathbb{N}$ such that $x^m = yx^{m+1}$. Hence, $(\overline{1})f^m = \overline{x^m} = \overline{yx^{m+1}} = y(\overline{1})f^{m+1} = (\overline{y})f^{m+1}$. It follows that $Mf^m \subseteq Mf^{m+1}$, and then $Mf^m = Mf^{m+1}$. For any $\overline{z} \in M$, there exists $\overline{v} \in M$ such that $(\overline{v})f^{m+1} = (\overline{z})f^m$. As f is injective, $(\overline{v})f = \overline{z}$. Accordingly, f is surjective; hence, it is isomorphic.

Let M be a finitely generated left R-module. Say $M = \sum_{i=1}^{n} Ra_i$. Then $nM = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \;\middle|\; x_1, \ldots, x_n \in M \right\}$ is a left $M_n(R)$-module. Clearly, $nM = M_n(R) \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$. Let $f \in End_R(M)$ be injective. Then it deduces

$f^* \in End_{M_n(R)}(nM)$, where $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} f^* = \begin{pmatrix} (x_1)f \\ \vdots \\ (x_n)f \end{pmatrix}$. Clearly, f^* is injective. By the preceding discussion, f^* is surjective, and then so is f. Therefore f is an R-isomorphism.

(2) $\Rightarrow$ (1) Let $0 \neq f \in End(nR)$. Set $N = \bigcup_{n=1}^{\infty} Ker(f^n)$. Then $N \subseteq nR$ and $Nf \subseteq N$. This induces $\overline{f} : nR/N \to nR/N$. If $(\overline{x})\overline{f} = \overline{0}$, then $(x)f \in N$, and so $((x)f)f^m = 0$ for some $m \in \mathbb{N}$. It follows that $x \in N$, i.e., $\overline{x} = \overline{0}$. This means that $\overline{f}$ is injective. By hypothesis, $\overline{f}$ is isomorphic. Let $\{e_1, \ldots, e_n\}$ be a basis of nR. Write $\overline{e_i} = (\overline{u_i})\overline{f}$ for each i. Then $e_i - (u_i)f \in N$. Hence, we can find a $k \in \mathbb{N}$ such that $e_i - (u_i)f \in Ker(f^k)$ for all i. Construct a left R-morphism $g : nR \to nR$ given by $(e_i)g = u_i$ for each i. Then $(e_i)(f^k - gf^{k+1}) = (e_i - (u_i)f)f^k = 0$. This forces that $f^k = gf^{k+1}$, and therefore $End(nR)$ is strongly π-regular, as desired. □

Recall that a ring R is left perfect ring if R satisfies the minimum condition for principal right ideals. Every matrix ring over a left perfect ring is left perfect. If R is left perfect, then for each finitely generated left R-module M, injective endomorphisms of M are isomorphism. Since every perfect ring is strongly π-regular, we are through by Theorem 6.1.22.

Corollary 6.1.23. *Let R be a commutative ring. Then the following are equivalent:*

(1) *R is strongly π-regular.*
(2) *$M_n(R)$ is strongly π-regular for some $n \in \mathbb{N}$.*
(3) *$M_n(R)$ is strongly π-regular for all $n \in \mathbb{N}$.*

Proof. (1) $\Rightarrow$ (3) By Theorem 6.1.10, every injective endomorphism of a finitely generated R-module is an isomorphism. For each $n \in \mathbb{N}$, $M_n(R)$ is strongly π-regular in terms of Theorem 6.1.22.

(3) $\Rightarrow$ (2) This is trivial.

(2) $\Rightarrow$ (1) This is clear as $R \cong EM_n(R)E$ for some idempotent $E \in M_n(R)$. □

It follows that every matrix ring over a Boolean ring is strongly π-regular because, any Boolean ring is a commutative strongly π-regular ring.

Corollary 6.1.24. *Let R be strongly π-regular. Then $M_n(C(R))$ is strongly π-regular for all $n \in \mathbb{N}$.*

Proof. By virtue of Corollary 6.1.23, it will suffice to prove that $C(R)$ is strongly π-regular. Let $x \in C(R)$. Then there exists $n \in \mathbb{N}$ and $y \in R$ such that $x^n = x^{n+1}y$ and $xy = yx$. Let $z = x^n y^{n+1}$. Then $x^n = x^{n+1}y = x^{n+2}y^2 = \cdots = x^{2n+1}y^{n+1} = x^{n+1}z$ and $xz = zx$. For any $r \in R$, we check that $zr = (x^n y^{n+1})r = (y^{n+1}r)x^n = y^{n+1}r(x^{2n+1}y^{n+1}) = x^{2n+1}(y^{n+1}ry^{n+1}) = x^n(ry^{n+1}) = r(x^n y^{n+1}) = rz$. Therefore $z \in C(R)$, as required. □

Example 6.1.25. Let R be a principal ideal domain, and let $0 \neq a \notin U(R)$. Then $M_n\big(R/(a)\big)$ is strongly π-regular for all $n \in \mathbb{N}$.

Proof. Write $a = p_1^{s_1} \cdots p_m^{s_m}$, where the s_i's $\in \mathbb{N}$ and the p_i's are distinct primes in R. Let $I = (p_1 \cdots p_s)$. Then $N(R/(a)) = I/(a)$. Since R/I is isomorphic to a finite product of fields, it is regular. Thus, $R/(a)/N(R/(a)) \cong R/I$ is regular. In light of Theorem 5.3.2, $R/(a)$ is strongly π-regular, and then we are through by Corollary 6.1.23. □

Lemma 6.1.26. *Let $R = S + K$ where S is a subring of R and K is a nilpotent ideal of R. Then R is strongly π-regular if and only if so is S.*

Proof. Write $K^m = 0 (m \in \mathbb{N})$. Suppose R is strongly π-regular. For any $x \in S$, there exist $n \in \mathbb{N}$ and $y \in R$ such that $x^n = x^{nm+1}y$. Write $y = z + b$ with $z \in S$ and $b \in K$. Then $x^n - x^{nm+1}z = x^{nm+1}b \in K$; hence, $\big(x^n - x^{nm+1}z\big)^m = 0$. We infer that $x^{nm} \in x^{nm+1}S$. Therefore S is strongly π-regular.

Conversely, assume that S is strongly π-regular. For any $a \in R$, there exist $c \in S$ and $d \in K$ such that $a = c + d$. Say $c^n = c^{nm+1}s$ for $n \in \mathbb{N}$ and $s \in S$. Then $a^n - a^{nm+1}s \in K$, and so $\big(a^n - a^{nm+1}s\big)^m = 0$. Thus $a^{nm} \in a^{nm+1}R$, and we therefore complete the proof. □

The following fact was observed by Tang et al. in [503, Theorem 3.5].

Theorem 6.1.27. *Let T be the ring of a Morita context $(A, B, M, N, \psi, \varphi)$. If $Im(\psi)$ and $Im(\varphi)$ are nilpotent, then A and B are strongly π-regular if and only if so is T.*

Proof. $\Longrightarrow$ Let $S = \begin{pmatrix} A & \\ & B \end{pmatrix}$ and $K = \begin{pmatrix} Im(\psi) & N \\ M & Im(\varphi) \end{pmatrix}$. Then $T = S + K$. By hypothesis, S is strongly π-regular. Write $\big(Im(\psi)\big)^s = 0$ and $\big(Im(\varphi)\big)^t = 0$ for some $s, t \in \mathbb{N}$. Choose $p = \max(s, t)$. Set $NM := Im(\psi)$ and $MN := Im(\varphi)$. Analogously to Theorem 5.2.11, we see that

$\begin{pmatrix} NM & N \\ M & MN \end{pmatrix}^{2p} = 0$, i.e., K is nilpotent. Therefore T is strongly π-regular, by Lemma 6.1.26.

$\Longleftarrow$ This is clear. □

Corollary 6.1.28. *Let R be strongly π-regular, and let $s \in N(R) \cap C(R)$. Then $M_{(s)}(R)$ is strongly π-regular.*

Proof. Let $\psi : R \otimes R \to R, n \otimes m \mapsto snm$ and $\varphi : R \otimes R \to R, m \otimes n \mapsto smn$. Then we may regard $M_s(R)$ as the Morita context $(R, R, R, R, \psi, \varphi)$. We see that $Im(\varphi)$ and $Im(\psi)$ are nilpotent as $s \in N(R) \cap C(R)$. Therefore the result follows by Theorem 6.1.27. □

Corollary 6.1.29. *A ring R is strongly π-regular if and only if so is $T_n(R)$.*

Proof. This is obvious, by Theorem 6.1.27 and induction. □

We come now to discuss stable range 1 for strongly π-regular rings. We have

Lemma 6.1.30. *Let R be a ring. Suppose that $ax + b = 1$ in R. Then the following are equivalent:*

(1) *There exists $y \in R$ such that $a + by \in U(R)$.*
(2) *There exists $z \in R$ such that $x + zb \in U(R)$.*

Proof. (1) $\Rightarrow$ (2) Let $u = (a+by)^{-1}$. Then $\big(x+(1-xy)ub\big)\big(a+y(1-xa)\big) = xa+xy-xyxa+uba-xyuba+uby-xyuby-ubyxa+xyubyxa = xa+xy-xyxa+uba-xyuba+(1-ua)-xy(1-ua)-(1-ua)xa+xy(1-ua)xa = uba-xyuba+1-ua+xyua+uaxa-xyuaxa = u(1-ax)a-xyu(1-ax)a+1-ua+xyua+uaxa-xyuaxa = 1$. Likewise, $\big(a+y(1-xa)\big)\big(x+(1-xy)ub\big) = 1$, as desired.

(2) $\Rightarrow$ (1) Applying the preceding discussion to the opposite ring R^{op}, we obtain the result. □

Lemma 6.1.31. *Let e be an idempotent in R. If R has stable range 1, then so does eRe.*

Proof. Given $ax+b = e$ with $a, x, b \in eRe$, we have $(a+1-e)(x+1-e)+b = 1$. Thus, there exists $y \in R$ such that $u := (a+1-e)+by \in U(R)$. Hence, $\big((a+1-e)+by\big)u^{-1} = 1$, and so $(1-e)u^{-1} = (1-e)\big((a+1-e)+by\big)u^{-1} = 1-e$. This implies that $(1-e)u^{-1}e = 0$; whence, $u^{-1}e = eu^{-1}e$.

Then $\big(a + b(eye)\big)(eu^{-1}e) = e$. Furthermore, $(eu^{-1}e)\big(a + b(eye)\big) = e$. Accordingly, $a + b(eye) \in U(eRe)$, hence the result. □

Theorem 6.1.32. *Let A be a right R-module, and let $E = End_R(A)$. Then the following are equivalent:*

(1) *E has stable range* 1.
(2) *Given any right R-module decomposition $M = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1 \cong A \cong A_2$, there exists $C \subseteq M$ such that $M = C \oplus B_1 = C \oplus B_2$.*
(3) *Given any right R-module decomposition $M = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1 \cong A \cong A_2$, there exists $C \subseteq M$ such that $M = A_1 \oplus C = A_2 \oplus C$.*

Proof. (1)⇒(2) Given $M = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1 \cong A \cong A_2$, then $M = A_1 \oplus B_1 \cong A \oplus B_1$. Thus, we have projections $p_1 : M \to A_1 \cong A, p_2 : M \to B_1$ and injections $q_1 : A \cong A_1 \to M, q_2 : B_1 \to M$ such that $p_1q_1 = 1_A, q_1p_1 + q_2p_2 = 1_M$. As $M = A_2 \oplus B_2 \cong A \oplus B_2$, we have a projection $f : M \to A_2 \cong A$ and an injection $g : A \cong A_2 \to M$ such that $fg = 1_A$. Since $(fq_1)(p_1g) + fq_2p_2g = 1_A$, there exists some $y \in E$ such that $fq_1 + fq_2p_2gy \in U(E)$. That is, $f\big(q_1 + fq_2p_2gy\big) \in U(E)$. One easily checks that $M = Ker(f) \oplus C$, where $C = Im(q_1 + q_2p_2gy)$. As $p_1(q_1 + q_2p_2gy) = 1_A$, $M = Ker(p_1) \oplus C$. Accordingly, $M = C \oplus B_1 = C \oplus B_2$.

(2) ⇒ (1) Suppose $ax + b = 1_A$ with $a, x, b \in E$. Set $M = 2A$, and let $p_i : M \to A$, $q_i : A \to M$ (for $i = 1, 2$) denote the projections and injections of this direct sum. Set $A_1 = q_1(A)$ and $B_1 = q_2(A)$, so that $M = A_1 \oplus B_1$ with $A_1 \cong A$. Define $f = ap_1 + bp_2$ from M to A and $g = q_1x + q_2$ from A to M. As $fg = 1_A$, $M = Ker(f) \oplus g(A)$. Set $A_2 = g(A)$ and $B_2 = Ker(f)$, so that $M = A_2 \oplus B_2$ and $A_2 \cong A$. By hypothesis, $M = C \oplus B_1 = C \oplus B_2$ for some $C \subseteq M$.

Let $h : A \cong A_1 \cong C \to M$ be the natural injection. Then $C = h(A)$. So $M = Ker(p_1) \oplus h(A)$, we infer that p_1h is an isomorphism. Furthermore, $M = Ker(f) \oplus h(A)$; whence, fh is an isomorphism. Clearly, $fh = (ap_1 + bp_2)h = \big(a + bp_2h(p_1h)^{-1}\big)p_1h \in U(E)$, and then $a + bp_2h(p_1h)^{-1} \in U(E)$, as desired.

(1) ⇒ (3) As in "(1) ⇒ (2)", we have $(fq_1)(p_1g) + fq_2p_2g = 1_A$. In view of Lemma 6.1.30, there exists $z \in E$ such that $p_1g + zfq_2p_2g \in U(E)$. Hence, $M = Im(g) \oplus C$, where $C = Ker(p_1 + zfq_2p_2)$. Since $(p_1 + zfq_2p_2)q_1 = 1_A$, we have $M = Im(q_1) \oplus C$. Therefore $M = A_1 \oplus C = A_2 \oplus C$.

(3) ⇒ (1) As in "(2) ⇒ (1)", we have $M = A_1 \oplus B_1$ with $A_1 \cong A$. Define $f = ap_1 + p_2$ from M to A and $g = q_1x + q_2b$ from A to M. Then $fg = 1_A$;

hence that $M = Ker(f) \oplus g(A)$. Set $A_2 = g(A)$ and $B_2 = Ker(f)$. Then $M = A_2 \oplus B_2$ with $A_2 \cong A$. Thus, $M = A_1 \oplus C = A_2 \oplus C$ for some $C \subseteq M$.

Let $h : M \to A_1$ be the projection. Then $C = Ker(h)$, and so $M = Ker(h) \oplus q_1(A)$. This shows that hq_1 is an isomorphism. Clearly, $M = Ker(h) \oplus g_2(A)$, and so hg_2 is an isomorphism. Observe that $hg_2 = h(q_1x + q_2b) = (hq_1)\big(x + (hq_1)^{-1}(hq_2)b\big) \in U(E)$. Thus $x + (hq_1)^{-1}(hq_2)b \in U(E)$. Therefore E has stable range 1, by Lemma 6.1.30. □

Corollary 6.1.33. *Let A be a finitely generated projective right module over a ring R. If R has stable range 1, then so does $End_R(A)$.*

Proof. Let A be a finitely generated projective right R-module. Then we have an idempotent $E \in M_n(R)$ $(n \in \mathbb{N})$ such that $A \cong E(nR)$. Hence $End_R(A) \cong EM_n(R)E$. Suppose $M = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1 \cong nR \cong A_2$. Then $A_1 = A_{11} \oplus A_{12} \oplus \cdots \oplus A_{1n}$ and $A_2 = A_{21} \oplus A_{22} \oplus \cdots \oplus A_{2n}$ with each $A_{1i} \cong R \cong A_{2i}$. By virtue of Theorem 6.1.32, there exists $D_1 \subseteq M$ such that $M = D_1 \oplus A_{12} \oplus \cdots \oplus A_{1n} \oplus B_1 \cong D_1 \oplus A_{22} \oplus \cdots \oplus A_{2n} \oplus B_2$. Similarly, we have $D_2, \ldots, D_n \subseteq M$ such that $M = D_1 \oplus D_2 \oplus \cdots \oplus D_n \oplus B_1 = D_1 \oplus D_2 \oplus \cdots \oplus D_n \oplus B_2$. By Theorem 6.1.32 again, $End_R(nR)$ has stable range 1, i.e., $M_n(R)$ has stable range 1. Therefore we complete the proof by Lemma 6.1.31. □

Lemma 6.1.34. *Let $a \in R$ be regular, let $K = r(a)$ and $R = aR \oplus E$. If K has the finite exchange property, then there exist right ideals $A_i, A_i', B_i, B_i', C_i$ and C_i' for $i \geq 1$ such that the following conditions are satisfied:*

(1) $R = K \oplus \left(\bigoplus_{j=1}^{i} (A_j \oplus B_j) \right) \oplus C_i$ *for all* $i \geq 1$.

(2) $E \cong (A_i \oplus B_i) \oplus (A_i' \oplus B_i')$ *for all* $i \geq 1$.

(3) $K \cong A_i' \oplus B_i' \oplus C_i'$ *for all* $i \geq 1$.

(4) $A_i' \oplus B_i' \cong A_{i+1} \oplus A_{i+1}'$ *for all* $i \geq 1$.

(5) $aR = C_1 \oplus C_1'$ *and* $aA_i \oplus aB_i = B_{i+1} \oplus B_{i+1}'$ *and* $aC_i = C_{i+1} \oplus C_{i+1}'$ *for all* $i \geq 1$.

Proof. Write $R = K \oplus L = E \oplus aR$. Since K has the finite exchange property, we have A_1, A_1', C_1, C_1' such that

$$R = K \oplus A_1 \oplus C_1, E = A_1 \oplus A_1' \text{ and } aR = C_1 \oplus C_1'.$$

Set $B_1 = B_1' = 0$. Then

$$\begin{aligned} R &= K \oplus (A_1 \oplus B_1) \oplus C_1 \\ &= E \oplus aR \\ &= (A_1 \oplus A_1') \oplus (aA_1 \oplus aB_1) \oplus aC_1 \\ &= (A_1 \oplus B_1) \oplus (A_1' \oplus B_1') \oplus (aA_1 \oplus aB_1) \oplus aC_1. \end{aligned}$$

It follows from $R = K \oplus A_1 \oplus C_1 = A_1 \oplus A_1' \oplus C \oplus C_1'$ that

$$K \cong A_1' \oplus B_1' \oplus C_1'.$$

By [582, Lemma 2], we have right R-modules A_2, A_2', B_2, B_2, C_2 and C_2' such that

$$R = K \oplus (A_1 \oplus B_1) \oplus (A_2 \oplus B_2) \oplus C_2, A_2 \oplus A_2' = A_1' \oplus B_1', B_2 \oplus B_2' = aA_1 \oplus aB_1$$

and $C_2 \oplus C_2' = aC_1$. Thus result holds for $i = 1$.

Now assume that, for some $n \in \mathbb{N}$, we have constructed $A_i, A_i', B_i, B_i', C_i$ and C_i', with $1 \leq i \leq n$, satisfying the preceding conditions. Then

$$\begin{aligned} R &= K \oplus \left(\bigoplus_{i=1}^{n} (A_i \oplus B_i) \right) \oplus C_n \\ &= E \oplus aR \\ &= (A_1 \oplus A_1') \oplus \left(\bigoplus_{i=1}^{n-1} (aA_i \oplus aB_i) \right) \oplus aA_n \oplus aB_n \oplus aC_n \\ &= (A_1 \oplus A_1') \oplus \left(\bigoplus_{i=1}^{n-1} (B_{i+1} \oplus B_{i+1}') \right) \oplus aA_n \oplus aB_n \oplus aC_n \\ &= \left(\bigoplus_{i=1}^{n} (A_i \oplus B_i) \right) \oplus (A_n' \oplus B_n') \oplus aA_n \oplus aB_n \oplus aC_n. \end{aligned}$$

By using [582, Lemma 2], we get

$$R = K \oplus \left(\bigoplus_{i=1}^{n} (A_i \oplus B_i) \right) \oplus A_{n+1} \oplus B_{n+1} \oplus C_{n+1}$$

such that

$$\begin{aligned} A_{n+1} \oplus A_{n+1}' &= A_n' \oplus B_n', \\ B_{n+1} \oplus B_{n+1}' &= aA_n \oplus aB_n, \\ C_{n+1} \oplus C_{n+1}' &= aC_n. \end{aligned}$$

Moreover,

$$\begin{aligned} E &= (A_n \oplus B_n) \oplus (A_n' \oplus B_n') \\ &\cong aA_n \oplus aB_n \oplus A_{n+1} \oplus A_{n+1}' \\ &= B_{n+1} \oplus B_{n+1}' \oplus A_{n+1} \oplus A_{n+1}' \\ &= (A_{n+1} \oplus B_{n+1}) \oplus (A_{n+1}' \oplus B_{n+1}'). \end{aligned}$$

Therefore $K \cong A_2 \oplus B_2 \oplus C_2$. Clearly, $A_1 \cong aA_1$, and so $E \cong (A_2 \oplus B_2) \oplus (A_2' \oplus B_2')$. By induction, we complete the proof. □

An element $a \in R$ is unit-regular if $a = aua$ for a unit $u \in R$. Obviously, $a \in R$ is unit-regular if and only if a is the product of an idempotent and a unit in R.

Lemma 6.1.35. *Let $a \in R$ be a nilpotent, regular element. If $K = r(a)$ has the finite exchange property, then a is unit-regular.*

Proof. Write $a^{n+2} = 0 (n \in \mathbb{N})$. As $a \in R$ is regular, we have $A_i, A_i', B_i, B_i', C_i$ and C_i' for $i \geq 1$ such that the conditions (1)–(5) in Lemma 6.1.34 hold. By Condition (5), we have $C_{n+1} \subseteq a^{n+1}R$; hence, $C_{n+1} \subseteq K$. Since $K \cap C_{n+1} = 0$, we get $C_{n+1} = 0$, and then $C_{n+1}' = aC_n$. Let $\psi : C_n \to aC_n$ given by $\psi(r) = ar$ for any $r \in C_n$. Then ψ is an R-epimorphism. On the other hand, $Ker(\psi) = \{r \in C_n \mid ar = 0\} \subseteq C_n \cap K = 0$; hence, $\psi : aC_n \cong C_n$. Since $R = K \oplus \left(\bigoplus_{i=1}^{n} (A_i \oplus B_i) \right) \oplus C_n = K \oplus \left(\bigoplus_{i=1}^{n} (A_i \oplus B_i) \right) \oplus A_{n+1} \oplus B_{n+1} \oplus C_{n+1}$, we have $C_{n+1}' \cong C_n \cong A_{n+1} \oplus B_{n+1} \oplus C_{n+1} = A_{n+1} \oplus B_{n+1}$. Therefore, $K \cong A_{n+1}' \oplus B_{n+1}' \oplus C_{n+1}' \cong A_{n+1}' \oplus B_{n+1}' \oplus A_{n+1} \oplus B_{n+1} \cong E$. That is, $r(a) \cong R/aR$. Accordingly, $a \in R$ is unit-regular. □

We have at our dispose all the information necessary to prove the following result.

Theorem 6.1.36. *Every strongly π-regular ring has stable range 1.*

Proof. Let R be a strongly π-regular ring. Given $ax+b = 1$ with $a, x, b \in R$, then we can find an idempotent $e \in bR$ such that $1 - e \in (1 - b)R$, by Theorem 1.1.2 and Theorem 1.1.8. Write $e = bs$ and $1 - e = axt$ for some $s, t \in R$. Then $(1 - e)axt + e = 1$. Set $c = (1 - e)a$. We have $n \in \mathbb{N}$ such that $c^n = c^{2n}d$ and $cd = dc$. Let $f = c^n d$. Then $f^2 = f \in R$. Further, $\big(fc\big)\big(fc^{2n-1}d^2\big) = \big(fc^{2n-1}d^2\big)\big(fc\big) = f$. That is, $fc \in fRf$ is invertible. Clearly, $\big((1 - f)c\big)^n = 0$. As $cxtc = c$, we have $(1 - f)cxt(1 - f)c = (1 - f)c$, i.e., $(1 - f)c \in (1 - f)R(1 - f)$ is regular and nilpotent. Clearly, $(1 - f)R(1 - f)$ is an exchange ring. According to Lemma 6.1.35, $(1 - f)c$ is unit-regular in $(1 - f)R(1 - f)$. So $c = fc + (1 - f)c \in R$ is unit-regular. Write $c = gv$ with $g = g^2 \in R$ and $v \in U(R)$. Then $gvxt + e = 1$; hence, $gvxt(1 - g) + e(1 - g) = 1 - g$. Thus, $g + e(1 - g) = 1 - gvxt(1 - g)$, and so $(1-e)a+e(1-g)v = \big(1+gvxt(1-g)\big)^{-1}v$. We see that $a+bs\big((1-g)v-a\big) \in U(R)$, the result follows. □

Corollary 6.1.37. *Let R be a strongly π-regular ring, and let A be a finitely generated projective right R-module. Then for any right R-modules B and C, $A \oplus B \cong A \oplus C$ implies that $B \cong C$.*

Proof. By Theorem 6.1.36 and Corollary 6.1.33, $End_R(A)$ has stable range 1. If $\varphi : A \oplus B \cong A \oplus C$, then $A \oplus C = \varphi(A) \oplus \varphi(B)$ with $A \cong \varphi(A)$. In light of Theorem 6.1.32, $A \oplus C = D \oplus C = D \oplus \varphi(B)$ for some $D \subseteq A \oplus C$. Consequently, $C \cong \varphi(B) \cong B$, as desired. □

6.2 Strongly J_n-Clean Rings

Our main interest here is in studying certain strongly clean rings related to their Jacobson radicals. Let R be a ring, and let n be a fixed natural number. Following Koliha and Patricio, the commutant of $a \in R$ is defined by $comm(a) = \{x \in R \mid xa = ax\}$. We say that $a \in R$ is strongly J_n-clean provided that there exists an idempotent $e \in comm(a)$ such that $a - e \in U(R)$ and $(eae)^n \in J(R)$. A ring R is strongly J_n-clean provided that every element in R is strongly J_n-clean. We now characterize strongly J_n-clean rings by means of an analog of strongly π-regularity.

Theorem 6.2.1. *Let R be a ring, and let $n \in \mathbb{N}$. Then the following are equivalent:*

(1) *R is strongly J_n-clean.*
(2) *For any $a \in R$, there exists an idempotent $e \in comm(a)$ and a unit $u \in R$ such that $a^n \equiv eu \equiv ue\big(mod\ J(R)\big)$.*

Proof. (1) $\Rightarrow$ (2) For any $a \in R$, there exists an idempotent $f \in comm(a)$ and a unit $v \in R$ such that $a = f + v$ and $fa^n \in J(R)$. Clearly, $a^n = fa^n + (1 - f)a^n$. It is easy to verify that $(1 - f)a = (1 - f)v$, and so $(1 - f)a^n = (1 - f)v^n$. Let $e = 1 - f$ and $u = v^n$. Then $a^n \equiv eu \equiv ue\big(mod\ J(R)\big)$ with $e \in comm(a)$, proving (2).

(2) $\Rightarrow$ (1) Let $a \in R$. Then there exists an idempotent $f \in comm(a)$ and a unit $v \in R$ such that

$$a^n \equiv fv \equiv vf\big(mod\ J(R)\big).$$

Let $e = 1 - f$ and $u = a - e$. Then $e \in comm(a)$. Choose

$$z = a^{n-1}v^{-1}f - (1 + a + \cdots + a^{n-1})e.$$

As in the proof of Theorem 1.1.2, we check that $uz = (a-e)\big(a^{n-1}v^{-1}f - (1+a+\cdots+a^{n-1})e\big) \equiv 1\big(mod\ J(R)\big)$; hence, $u \in R$ is right invertible. Similarly, we have $zu \equiv 1\big(mod\ J(R)\big)$, and so $u \in U(R)$. Thus, $a = e+u$ and $ea^n \in J(R)$, hence the result. □

For a fixed natural number n, a ring R is n-regular provided that for any $a \in R$ there exists $b \in R$ such that $a^n = a^n b a^n$. A ring R is regular if R is 1-regular. Let $R = T_n(\mathbb{Z}_2)$, the ring of all $n \times n$ upper triangular matrix over $\mathbb{Z}_2$. Then R is n-regular. For instance, $\begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$ is 2-regular, while it is not regular. Let $R = \left\{ \begin{pmatrix} A_m & & \\ & 1 & \\ & & 1 \\ & & & \ddots \end{pmatrix} \middle| \begin{array}{l} A_m \text{ is } m \times m \\ \text{upper triangular matrix over } \mathbb{Z}_2, m = 1,2,3,\ldots \end{array} \right\}$. Then R is π-regular, while it is not n-regular for all $n \in \mathbb{N}$. In fact, we have

$$\{\text{regular rings}\} \subsetneq \{n\text{-regular rings}\} \subsetneq \{\pi\text{-regular rings}\}$$

for all $n \geq 2$.

Corollary 6.2.2. *Let R be abelian, and let $n \in \mathbb{N}$. Then the following are equivalent:*

(1) *R is strongly J_n-clean.*
(2) *$R/J(R)$ is n-regular and every idempotent lifts modulo $J(R)$.*

Proof. (1) $\Rightarrow$ (2) Since R is strongly J_n-clean, it is strongly clean. In view of Theorem 1.1.8, R is an exchange ring. Hence, every idempotent lifts modulo $J(R)$, by Lemma 4.4.5. Let $a \in R$. According to Theorem 6.2.1, there exists an idempotent $e \in comm(a)$ and a unit $u \in R$ such that $a^n \equiv eu \equiv ue\big(mod\ J(R)\big)$. This implies that $a^n \equiv ea^n \equiv u^{-1}a^{2n}\big(mod\ J(R)\big)$. Therefore $a^n u^{-1} a^n \equiv a^n e \equiv (ue)e = ue \equiv a^n\big(mod\ J(R)\big)$, and so $R/J(R)$ is n-regular.

(2) $\Rightarrow$ (1) As every idempotent lifts modulo $J(R)$, $R/J(R)$ is abelian. For any $a \in R$, there exists $b \in R$ such that $a^n \equiv a^n b a^n\big(mod\ J(R)\big)$. Set $c = ba^n b$. Then $a^n \equiv a^n c a^n\big(mod\ J(R)\big)$ and $c \equiv ca^n c\big(mod\ J(R)\big)$. Set

$u = c + 1 - a^n c$. Since R is abelian, it is easy to verify that

$$(a^n + 1 - a^n c)u \equiv u(a^n + 1 - a^n c) \equiv 1\big(mod\ J(R)\big).$$

Hence, $u \in U(R)$. Let $f = a^n c$. Then we can find an idempotent $e \in R$ such that $e \equiv f\big(mod\ J(R)\big)$. Additionally, $a^n \equiv eu \equiv ue\big(mod\ J(R)\big)$. Clearly, $e \in comm(a)$, and therefore we complete the proof by Theorem 6.2.1. □

All abelian n-regular rings are exchange rings; hence, it follows that every abelian n-regular ring is strongly J_n-clean. Obviously, $\mathbb{Z}_4$ is 2-regular, while it is not regular. $\mathbb{Z}_8$ is 3-regular, while it is not 2-regular. In light of Corollary 6.2.2, $\mathbb{Z}_4$ is strongly J_2-clean and $\mathbb{Z}_8$ is strongly J_3-clean. Let $R = F[x]/(x^2) = \{a + bt \mid a, b \in F, t^2 = 0\}$, where F is a field of characteristic 2. Then for any $a \in R$, there exists $b \in R$ such that $a^2 = a^3 b$. Hence, $a^2 = a^2 b^2 a^2$. That is, R is an abelian 2-regular ring. Thus, R is strongly J_2-clean. In this case, R is not regular.

Lemma 6.2.3. *Let R be a strongly π-regular ring of bounded index n. Then $a^n R = a^{n+1} R$ for all $a \in R$.*

Proof. Let $a \in R$. By virtue of Lemma 6.1.12, there exist $e = e^2 \in R, u \in U(R)$ and $w \in N(R)$ such that $a = eu + w$ and e, u, w commutate. Explicitly, $e = a^m x^m (m \in \mathbb{N})$ and $ax = xa$. Hence, $e = a^{(n+1)m} x^{(n+1)m}$. By hypothesis, $\big(a - a^{(n+1)m} x^{(n+1)m}\big)^n = 0$, and so $a^n \in a^{n+1} R$. Therefore $a^n R = a^{n+1} R$. □

Theorem 6.2.4. *Every strongly π-regular ring of bounded index n is strongly J_n-clean.*

Proof. Let R be a strongly π-regular ring of bounded index n, and let $a \in R$. According to Lemma 6.2.3, we have $b, c \in R$ such that $a^n = a^{n+1} b$ and $b^n = b^{n+1} c$. As in the proof of Lemma 1.1.1, we get $a^n \in a^{n+1} R \cap R a^{n+1}$.

Write $a^n = a^{n+1} b = r a^{n+1}$. Then $a^n b = r a^{n+1} b = r a^n$. By induction, we get $a^n b^k = r^k a^n$ for any $k \in \mathbb{N}$. Let $t = a^n b^{n+1}$. Then $t = r^{n+1} a^n$. It is easy to verify that $at = a^{n+1} b^{n+1} = (a^{n+1} b) b^n = a^n b^n = r^n a^n = r^n (r a^{n+1}) = (r^{n+1} a^n) a = ta; a^{n+1} t = a^{2n+1} b^{n+1} = a^n (a^{n+1} b) b^n = a^{2n} b^n = \cdots = a^{n+1} b = a^n; t^2 a = t(at) = t(a^{n+1} b^{n+1}) = a^{n+1} t b^{n+1} = a^n b^{n+1} = t$. Let $s = t^n$. Then $a^n = a^{2n} s, as = sa, s = s^2 a^n$. Set $u = s + 1 - a^n s$. Then $a^n u = a^n s + a^n (1 - a^n s) = a^n s$ is an idempotent. Moreover, $u^{-1} = a^n + 1 - a^n s$. Let $e = a^n s$. Then $a^n = e u^{-1} = u^{-1} e$ and $ea = ae$. Accordingly, the result follows by Theorem 6.2.1. □

Corollary 6.2.5. *Every π-regular ring of bounded index n is strongly J_n-clean.*

Proof. Let R be a π-regular ring of bounded index n. In view of [509, Theorem 2], R is a strongly π-regular ring, and the result follows from Theorem 6.2.4. □

As consequences, we deduce that every finite ring R is strongly J_n-clean for all $n \geq |R|$; every regular ring of bounded index n is strongly J_n-clean, and that every right P-exchange ring of bounded index n is strongly J_n-clean (see [120, Corollary 15.3.8]).

By the decompositions of 2×2 matrices, Ying considered the strong J_2-cleanness of 2×2 matrix ring over a division ring (see [557, Theorem 3.3.10]). As a consequence of Theorem 6.2.4, we extend Ying's result to the general case.

Example 6.2.6. Let D be a division ring, and let $n \geq 2$. Then $M_n(D)$ is strongly J_n-clean, while it is not strongly J_{n-1}-clean.

Proof. Since $M_n(D)$ is artinian, it is strongly π-regular. Let $A \in M_n(R)$ be nilpotent. Then there exist $U, V \in GL_n(D)$ such that $UAV = diag\big(I_r, 0_{n-r}\big)(0 \leq r < n)$. If $r = 0$, then $A = 0$, and so it is of bounded index n. Otherwise, $UAU^{-1} = diag\big(I_r, 0_{n-r}\big)W$ with $0 < r < n$. Write $W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$. Then

$$UA^mU^{-1} = \begin{pmatrix} W_{11}^m & W_{11}^{m-1}W_{12} \\ 0 & 0 \end{pmatrix} \text{ for any } m \in \mathbb{N}.$$

Here, $W_{11} \in M_r(D)$. By induction, one easily checks that A is of bounded index n. Thus, $M_n(D)$ is strongly J_n-clean by Theorem 6.2.4. Choose $X = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in M_n(D)$. Then X is not strongly J_{n-1}-clean. Otherwise, we have some $E = E^2 \in M_n(D)$ and $U \in GL_n(D)$ such that $EX = XE$ and $EX^{n-1} \in J\big(M_n(D)\big)$. Write $E = (e_{ij})$. As $EX^{n-1} = X^{n-1}E$, we get $e_{ni} = 0$ $(i = 1, \ldots, n-1)$. It follows from $EX^{n-1} \in J\big(M_n(D)\big)$ that $e_{nn} = 0$. So every entry at the last row of U are zero, a contradiction. Accordingly, $M_n(D)$ is not strongly J_{n-1}-clean. □

Lemma 6.2.7. *Let R be a ring, let $n \in \mathbb{N}$, and let $e \in R$ be an idempotent. Then $a \in eRe$ is strongly J_n-clean in R if and only if a is strongly J_n-clean in eRe.*

Proof. Suppose that $a \in eRe$ is strongly J_n-clean in eRe. Then $a = f + u, f = f^2 \in eRe, u \in U(eRe), af = fa$ and $fa^n \in J(fRf)$. Thus, $a = (f + 1 - e) + (u - 1 + e)$, where $f + 1 - e \in R$ is an idempotent, $u - 1 + e \in U(R)$ and $(f + 1 - e)a^n = fa^n \in J(R)$. Therefore $a \in eRe$ is strongly J_n-clean in R.

Conversely, suppose that $a = f + u, f = f^2 \in R, u \in U(R), af = fa$ and $fa^n = 0$. As in the proof of Lemma 1.1.11, we get $fe = ef$. Accordingly, $a = efe + eue$, $(fef)^2 = fef, eue \in U(eRe)$ and $a(efe) = (efe)a$. Furthermore, $(efe)a^n = e(fa^n)e \in J(eRe)$, hence the result. □

As a consequence, every corner of a strongly J_n-clean ring is strongly J_n-clean.

Theorem 6.2.8. *Let B be a subring of a ring A, and let $n \in \mathbb{N}$. Then $R[A, B]$ is strongly J_n-clean if and only if*

(1) *A is strongly J_n-clean;*
(2) *For any $a \in B$, there exists an idempotent $e \in comm(a)$ such that $a - e \in U(B)$ and $(eae)^n \in J(A) \cap J(B)$.*

Proof. Suppose $R[A, B]$ is strongly J_n-clean. Choose $f = (1, 0, 0, \ldots) \in R[A, B]$. Then $A \cong fR[A, B]f$ with $f = f^2$. By virtue of Lemma 6.2.7, A is strongly J_n-clean. For any $a \in B$, we see that $(0, a, a, \ldots) \in R[A, B]$. By hypothesis, there exists an idempotent $(e_1, \ldots, e_s, e, e, \ldots) \in R[A, B]$ and a unit $(u_1, \ldots, u_t, u, u, \ldots) \in R[A, B]$ such that

$$(0, a, a, \ldots) = (e_1, \ldots, e_s, e, e, \ldots) + (u_1, \ldots, u_t, u, u, \ldots)$$

is a strongly clean expression and $(e_1, \ldots, e_s, e, e, \ldots)(0, a, a, \ldots)^n \in J\big(R[A, B]\big)$. This implies that $a = e + u$ is a strongly clean expression in B. One easily checks that $J\big(R[A, B]\big) = R\big[J(A), J(A) \cap J(B)\big]$. As a result, we see that $ea^n \in J(A) \cap J(B)$, as required.

Conversely, assume that (1) and (2) hold. Let $(a_1, \ldots, a_m, a, a, \ldots) \in R[A, B]$. Then we have strongly clean expressions $a_1 = e_1 + u_1, \ldots, a_m = e_m + u_m, a = e + u$, and that $e_1 a_1^n, \ldots, e_m a_m^n \in J(A), ea^n \in J(A) \cap J(B)$. Thus, $(a_1, \ldots, a_m, a, a, \ldots) = (e_1, \ldots, e_m, e, e, \ldots) + (u_1, \ldots, u_m, u, u, \ldots)$. Additionally,

$$(e_1, \ldots, e_m, e, e, \ldots)(a_1, \ldots, a_m, a, a, \ldots)^n \in R\big[J(A), J(A) \cap J(B)\big].$$

Therefore $R[A, B]$ is strongly J_n-clean. □

Example 6.2.9. Let D be a division ring, and let $A = M_2(D)$, $B = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \;\middle|\; a, b \in D \right\}$. Then $R[A, B]$ is strongly J_2-clean.

Proof. By virtue of Example 6.2.6, A is strongly J_2-clean. Let $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in B$. If $a \neq 0$, then $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ is a strongly clean expression. Obviously, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in J(A) \cap J(B)$. If $a = 0$, then $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}$ is a strongly clean expression. In addition, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}^2 \in J(A) \cap J(B)$. According to Theorem 6.2.8, $R[A, B]$ is strongly J_2-clean. In this case, $R[A, B]$ is not strongly J_1-clean. □

Example 6.2.10. Let D be a division ring, and let $A = M_3(D)$, $B = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \;\middle|\; a, b, c \in D \right\}$. Then $R[A, B]$ is strongly J_3-clean.

Proof. In view of Example 6.2.6, A is strongly J_3-clean. Let $\begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \in B$. If $a \neq 0$, then $\begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$ is a strongly clean expression. Obviously, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \in J(A) \cap J(B)$. If $a = 0$, then $\begin{pmatrix} 0 & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & b & c \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ is a strongly clean expression. Additionally, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^3 \in J(A) \cap J(B)$. By virtue of Theorem 6.2.8, $R[A, B]$ is strongly J_3-clean. In this case, $R[A, B]$ is not strongly J_2-clean. □

Clearly, a ring R is strongly J_1-clean if and only if it is strongly J_n-clean for all $n \in \mathbb{N}$.

Example 6.2.11. Let D be a division ring, and let $R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \;\middle|\; a, b \in D \right\}$. Then R is strongly J_1-clean.

Proof. Let $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in R$. If $a \neq 0$, then $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in R$ is invertible. If $a = 0$, then $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}$. Accordingly, R is strongly J_1-clean. □

Theorem 6.2.12. *A ring R is strongly J-clean and only if*

(1) *R is strongly J_1-clean;*
(2) *$R/J(R)$ is Boolean.*

Proof. Suppose R is strongly J-clean. Then $R/J(R)$ is Boolean, by Theorem 4.1.1. For any $a \in R$, we have an idempotent $e \in R$ such that $w := a - e \in J(R)$ and $ae = ea$. Hence, $a = (1 - e) + (2e - 1 + w)$ with $2e - 1 + w \in U(R)$. Furthermore, $(1 - e)a = (1 - e)w \in J(R)$. Therefore R is strongly J_1-clean.

Conversely, assume that (1) and (2) hold. Let $a \in R$. Then there exists an idempotent $e \in R$ such that $u := a - e \in U(R), ae = ea$ and $eae \in J(R)$. As $R/J(R)$ is Boolean, $\overline{2}^2 = \overline{2}$; hence, $2 \in J(R)$. Further, $\overline{u} = \overline{1}$, and then $u \in 1 + J(R)$. Write $u = 1 + w$ for some $w \in J(R)$. Then $a = e + u = (1 - e) + (2e + w)$ with $2e + w \in J(R)$, hence the result. □

In what follows, we explore the strong J_1-cleanness of a single 2×2 matrix over local rings.

Theorem 6.2.13. *Let R be a local ring. Then $A \in M_2(R)$ is strongly J_1-clean if and only if $A \in GL_2(R)$, or $A \in J\big(M_2(R)\big)$, or A is similar to a matrix $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $\alpha \in J(R), \beta \in U(R)$.*

Proof. If $A \in GL_2(R)$, then it is strongly J_1-clean. If $A \in J\big(M_2(R)\big)$, then $A = I_2 + (A - I_2)$ where $A - I_2 \in GL_2(R)$, and so $A \in M_2(R)$ is strongly J_1-clean. If $\alpha \in J(R)$ and $\beta \in U(R)$, then $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \alpha - 1 & 0 \\ 0 & \beta \end{pmatrix}$ where $\begin{pmatrix} \alpha - 1 & 0 \\ 0 & \beta \end{pmatrix} \in GL_2(R)$ and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in J\big(M_2(R)\big)$. Therefore $A \in M_2(R)$ is strong J_1-clean.

Conversely, assume that $A \in M_2(R)$ is strongly J_1-clean . Then we can find $E^2 = E \in M_2(R)$ and $W \in GL_2(R)$ such that $A = E + W, EW = WE$ and $EAE \in J\big(M_2(R)\big)$. In light of Lemma 1.2.12, there exists $U \in GL_2(R)$ such that $UEU^{-1} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$. If $r = 0$, then $A \in GL_2(R)$. If $r = 2$, then $A \in J\big(M_2(R)\big)$. If $r = 1$, then $UAU^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + UWU^{-1}$. Write $UWU^{-1} = (w_{ij})$. Since $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}(w_{ij}) = (w_{ij})\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, we get $w_{12} = w_{21} = 0$. Thus,

$$UAU^{-1} = \begin{pmatrix} 1 + w_{11} & 0 \\ 0 & w_{22} \end{pmatrix}.$$

Obviously, $w_{22} \in U(R)$. As $EAE \in J\big(M_2(R)\big)$, we see that

$$\big(UEU^{-1}\big)\big(UAU^{-1}\big)\big(UEU^{-1}\big) \in J\big(M_2(R)\big).$$

That is,

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\big(UAU^{-1}\big)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in J\big(M_2(R)\big),$$

and therefore $1 + w_{11} \in J(R)$, as desired. □

Corollary 6.2.14. *Let R be a commutative local ring, and let $A \in M_2(R)$. Then the following are equivalent:*

(1) *$A \in M_2(R)$ is strongly clean.*
(2) *$I_2 - A \in GL_2(R)$ or $A \in M_2(R)$ is strongly J_1-clean .*

Proof. (2) $\Rightarrow$ (1) This is trivial.

(1) $\Rightarrow$ (2) In view of Lemma 1.2.1, $A \in GL_2(R)$, or $I_2 - A \in GL_2(R)$ or A is similar to a matrix $\begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}$, where $\alpha \in J(R), \beta \in U(R)$. Thus, $I_2 - A \in GL_2(R)$, or $A \in M_2(R)$ is strongly J_1-clean by Theorem 6.2.13. □

We ask whether $M_2(R)$ is strongly clean if and only if $M_2(R)$ is strongly J_1-clean. The answer is negative. For instance, letting $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_2\big(\mathbb{Z}_2\big)$, one easily checks that A is not strongly J_1-clean, and so $M_2\big(\mathbb{Z}_2\big)$ is not strongly J_1-clean, but $M_2\big(\mathbb{Z}_2\big)$ is strongly clean.

Corollary 6.2.15. *Let R be a commutative local ring. If $A \in M_2(R)$ is strongly J_1-clean , then $A \in GL_2(R)$, or $A \in J\big(M_2(R)\big)$, or A has an invertible trace.*

Proof. If $A \notin GL_2(R)$ and $A \notin J\big(M_2(R)\big)$, it follows from Theorem 6.2.13 that A is similar to a matrix $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ where $\alpha \in J(R), \beta \in U(R)$. Therefore $tr(A) = \alpha + \beta \in U(R)$, as asserted. □

Lemma 6.2.16. *Let R be a ring, and let $a \in R$. Then the following are equivalent:*

(1) *$a \in R$ is strongly J_1-clean.*
(2) *There exists an idempotent $e \in comm(a)$ such that $eae \in J(eRe)$ and $(1-e)a(1-e) \in U\big((1-e)R(1-e)\big)$.*

Proof. (1) $\Rightarrow$ (2) Since $a \in R$ is strongly J_1-clean, we have an idempotent $e \in comm(a)$ such that $a - e \in U(R)$ and $eae \in J(R)$. Thus, $eae \in J(eRe)$ and $(1-e)a(1-e) = (1-e)(a-e)(1-e) \in U\big((1-e)R(1-e)\big)$.

(2) $\Rightarrow$ (1) By hypothesis, there exists an idempotent $e \in comm(a)$ such that $eae \in J(eRe)$ and $(1-e)a(1-e) \in U\big((1-e)R(1-e)\big)$. Hence, $a = e + eae + (1-e)a(1-e) - e$. Here, $eae + (1-e)a(1-e) - e \in U(R)$. Furthermore, $eae \in J(R)$, proving (1). □

Lemma 6.2.17. *Let M be a left R-module, let $E = End(M)$, and let $\alpha \in E$. Then the following are equivalent:*

(1) *$\alpha \in E$ is strongly J_1-clean.*
(2) *There exists a direct sum decomposition $M = P \oplus Q$ where P and Q are α-invariant, $\alpha|_P \in Aut(P)$ and $\alpha|_Q \in J\big(End(Q)\big)$.*

Proof. (1) $\Rightarrow$ (2) By virtue of Lemma 6.2.16, we have an idempotent $e \in comm(\alpha)$ such that $e\alpha e \in J(eEe)$ and $(1_E - e)\alpha(1_E - e) \in U\big((1_E - e)E(1_E - e)\big)$. Set $P = M(1_E - e)$ and $Q = Me$. Then P and Q are α-invariant as $e\alpha = \alpha e$. Clearly, $M = P \oplus Q$. Moreover, we have $\alpha|_P \in Aut(P)$ and $\alpha|_Q \in J\big(End(Q)\big)$, as desired.

(2) $\Rightarrow$ (1) By hypothesis, there exists a direct sum decomposition $M = P \oplus Q$ where P and Q are α-invariant, $\alpha|_P \in Aut(P)$ and $\alpha|_Q \in J\big(End(Q)\big)$. Let $e : M \to M$ be the projection onto P. Then $e \in E$ is an idempotent.

For any $m \in M$, we write $(m)e\alpha = (r)e$ for some $r \in M$. Then $(m)e\alpha = \big((r)e\big)e = (me)\alpha e = \big(me + m(1_E - e)\big)\alpha e = (m)\alpha e$. This shows that $e\alpha = \alpha e$.

It follows from $\alpha|_P \in J\big(End(P)\big)$ that $e\alpha e \in J(eEe)$. Furthermore, $(1_E - e)\alpha(1_E - e) \in U\big((1_E - e)E(1_E - e)\big)$. Therefore we complete the proof, by Lemma 6.2.16. □

Theorem 6.2.18. *Let R be a commutative local ring. Then $A \in M_2(R)$ is strongly J_1-clean if and only if*

(1) $A \in GL_2(R)\big)$, *or*
(2) $A \in J\big(M_2(R)\big)$, *or*
(3) $\chi(A)$ *has a root in* $J(R)$ *and a root in* $U(R)$.

Proof. Suppose that $A \in M_2(R)$ is strongly J_1-clean. By virtue of Theorem 6.2.13, $A \in GL_2(R)$, or $A \in J\big(M_2(R)\big)$, or A is similar to a matrix $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $\alpha \in J(R), \beta \in U(R)$. In the last case, $\chi(A) = (x-\alpha)(x-\beta)$ has a root $\alpha \in J(R)$ and a root $\beta \in U(R)$, as required.

Conversely, if (1) or (2) holds, then $A \in M_2(R)$ is strongly J_1-clean. If (3) holds, we may assume that $\alpha \in J(R)$ and $\beta \in U(R)$ are two roots of $\chi(A)$. Write $\chi(A) = t^2 + at + b$. Then $\alpha^2 + a\alpha + b = 0$, and so $\chi(A) = (t-\alpha)(t+\alpha+a)$. Thus, $(\beta-\alpha)(\beta+\alpha+a) = 0$. Obviously, $\beta - \alpha \in U(R)$. This implies that $\beta = -a - \alpha$. Therefore $\chi(A) = (t-\alpha)(t-\beta)$. Clearly, $(t-\alpha) - (t-\beta) = \beta - \alpha \in U(R)$, and so

$$(t-\alpha)(\beta-\alpha)^{-1} - (t-\beta)(\beta-\alpha)^{-1} = 1.$$

Choose $X = A - \alpha I_2$ and $Y = A - \beta I_2$. Then

$$X(\beta-\alpha)^{-1}I_2 - Y(\beta-\alpha)^{-1}I_2 = I_2.$$

In addition, $XY = YX = 0$, $X(\beta-\alpha)^{-1}I_2 = (\beta-\alpha)^{-1}I_2X$ and $(\beta-\alpha)^{-1}I_2Y = Y(\beta-\alpha)^{-1}I_2$. In view of Lemma 2.2.1, we have a left R-module decomposition $2R = Ker(X) \oplus Ker(Y)$. For any $x \in X$, $(x)AX = (x)XA = 0$, and so $(x)A \in Ker(X)$. This implies that $Ker(X)$ is A-invariant. Likewise, $Ker(Y)$ is A-invariant. For any $x \in Ker(X)$, it is easy to verify that $0 = (x)X = (x)\big(A - \alpha I_2\big)$; hence, $(x)A = (x)\alpha I_2$. As $\alpha \in J(R)$, we show that $A|_{Ker(X)} \in J\big(End(Ker(X))\big)$. For any $y \in Ker(Y)$, we see that

$$0 = (y)Y = (y)\big(A - \beta I_2\big).$$

Thus, $(y)A = (y)\big(\beta I_2\big)$. As $\beta \in U(R)$, we know that $A|_{Ker(Y)} \in Aut\big(Ker(Y)\big)$. Therefore $A \in M_2(R)$ is strongly J_1-clean, by Lemma 6.2.17. □

Let $\mathbb{Z}_{(2)} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, (m,n) = 1, 2 \nmid n\}$. Then $\mathbb{Z}_{(2)}$ is a commutative local ring. Clearly, the equation $x^2 - x = -\frac{2}{9}$ has a root $\frac{2}{3} \in J\big(\mathbb{Z}_{(2)}\big)$ and a root $\frac{1}{3} \in U\big(\mathbb{Z}_{(2)}\big)$. It follows from Theorem 6.2.18 that $\begin{pmatrix} 1 & 1 \\ -\frac{2}{9} & 0 \end{pmatrix} \in M_2\big(\mathbb{Z}_{(2)}\big)$ is strongly J_1-clean.

Corollary 6.2.19. *Let R be a commutative local ring, and let $A \in M_2(R)$. Then the following are equivalent:*

(1) *$A \in M_2(R)$ is strongly J_1-clean .*
(2) *$A \in GL_2(R)$ or $A \in J\big(M_2(R)\big)$, or $trA \in U(R)$ and the quadratic equation $x^2 - x = -\frac{detA}{tr^2A}$ has a root in $J(R)$.*
(3) *$A \in GL_2(R)$ or $A \in J\big(M_2(R)\big)$, or $trA \in U(R), detA \in J(R)$ and the quadratic equation $x^2 - x = \frac{detA}{tr^2A - 4detA}$ is solvable.*

Proof. (1) $\Rightarrow$ (2) Assume that $A \notin GL_2(R)$ and $A \notin J\big(M_2(R)\big)$. By virtue of Theorem 6.2.13, A is similar to $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, where $\lambda \in J(R), \mu \in U(R)$. Clearly, $tr(A) = \lambda + \mu \in U(R)$. Furthermore, $y^2 - (\lambda + \mu)y + \lambda\mu = 0$ has a root λ in $J(R)$, and then so does the equation

$$(\lambda + \mu)^{-1}y^2 - y = -(\lambda + \mu)^{-1}\lambda\mu.$$

Set $x = (\lambda + \mu)^{-1}y$. Then

$$(\lambda + \mu)x^2 - (\lambda + \mu)x = -(\lambda + \mu)^{-1}\lambda\mu.$$

That is, $x^2 - x = -(\lambda + \mu)^{-2}\lambda\mu$. Therefore, $x^2 - x = -\frac{detA}{tr^2A}$ has a root in $J(R)$.

(2) $\Rightarrow$ (3) Assume that $trA \in U(R)$ and the equation $x^2 - x = -\frac{detA}{tr^2A}$ has a root $a \in J(R)$. Then $detA \in J(R)$. Hence $\big(a(2a-1)^{-1}\big)^2 - \big(a(2a-1)^{-1}\big) = \frac{detA}{tr^2A \cdot \big(4(a^2-a)+1\big)} = \frac{detA}{tr^2A \cdot \big(-4(trA)^{-2}detA+1\big)} = \frac{detA}{tr^2A - 4detA}$. Therefore the equation $x^2 - x = \frac{detA}{tr^2A - 4detA}$ is solvable.

(3) $\Rightarrow$ (1) Suppose $A \notin GL_2(R)$ and $A \notin J\big(M_2(R)\big)$. Then $trA \in U(R), detA \in J(R)$ and the equation $x^2 - x = \frac{detA}{tr^2A - 4detA}$ has a root $a \in R$. Clearly, $b := 1 - a \in R$ is a root of this equation. As $a^2 - a \in J(R)$, we see that either $a \in J(R)$ or $1 - a \in J(R)$. Thus, $2a - 1 = 1 - 2(1 - a) \in U(R)$.

It is easy to verify that $\big(a(2a-1)^{-1}trA\big)^2 - trA\cdot\big(a(2a-1)^{-1}trA\big) + detA = -\frac{tr^2A\cdot(a^2-a)}{4(a^2-a)+1} + detA = 0$. Hence, the equation $x^2 - trA\cdot x + detA = 0$ has roots $a(2a-1)^{-1}trA$ and $b(2b-1)^{-1}trA$. Since $ab \in J(R)$, we see that $a+b=1$ and either $a \in J(R)$ or $b \in J(R)$. Thus, $x^2 - tr(A)x + det(A) = 0$ has a root in $U(R)$ and a root in $J(R)$. In light of Theorem 6.2.18, we complete the proof. □

Lemma 6.2.20. *Let R be a commutative local ring, and let $A \in M_2(R)$. If A has an invertible trace, then $A \in GL_2(R)$ or A is similar to a matrix $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in J(R), \mu \in U(R)$.*

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $tr(A) \in U(R)$. Case I. $c \in U(R)$. Then

$$[c,1]B_{12}(-ac^{-1})AB_{12}(ac^{-1})[c^{-1},1] = \begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$$

for some $\lambda, \mu \in R$. If $\lambda \in U(R)$, then $A \in GL_2(R)$. If $\lambda \in J(R)$, it follows from $tr(A) \in U(R)$ that $\mu \in U(R)$, and so the result follows. Case II. $b \in U(R)$. Then

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix},$$

and the result follows from Case I. Case III. $c, b \in J(R), a-d \in U(R)$. Then

$$B_{21}(-1)AB_{21}(1) = \begin{pmatrix} a+b & b \\ a-d+b-c & b-d \end{pmatrix},$$

where $a-d+b-c \in U(R)$; hence the result follows from Case I. Case IV. $c, b \in J(R), a, d \in U(R)$. Then

$$B_{21}(-ca^{-1})A = \begin{pmatrix} a & b \\ 0 & d-ca^{-1}b \end{pmatrix};$$

hence, $A \in GL_2(R)$. Case V. $c, b, a, d \in J(R)$. Then $A \in J\big(M_2(R)\big)$, and so $tr(A) \in J(R)$, a contradiction. So the lemma is true. □

Theorem 6.2.21. *Let R be a commutative local ring. Then the following are equivalent:*

(1) *Every $A \in M_2(R)$ with invertible trace is strongly J_1-clean.*
(2) *For any $\lambda \in J(R), \mu \in U(R)$, the quadratic equation $x^2 + \mu x + \lambda = 0$ has a root in $J(R)$ and a root in $U(R)$.*

Proof. (1) $\Rightarrow$ (2) Let $\lambda \in J(R), \mu \in U(R)$. Choose $A = \begin{pmatrix} 0 & \lambda \\ 1 & -\mu \end{pmatrix}$. Then $A \in M_2(R)$ is strongly J_1-clean. Clearly, $A \notin GL_2(R)$ and $A \notin J\big(M_2(R)\big)$. In view of Theorem 6.2.18, the quadratic polynomial $\chi(A) = x^2 + \mu x + \lambda$ has a root in $J(R)$ and a root in $U(R)$.

(2) $\Rightarrow$ (1) Let $A \in M_2(R)$ with $tr(A) \in U(R)$. In light of Lemma 6.2.20, $A \in GL_2(R)$, or A is similar to a matrix $\begin{pmatrix} 0 & \lambda \\ 1 & -\mu \end{pmatrix}$, where $\lambda \in J(R), \mu \in U(R)$. If $A \notin GL_2(R)$, then $\chi(A) = x^2 + \mu x + \lambda$, and so $\chi(A)$ has a root in $J(R)$ and a root in $U(R)$. By virtue of Theorem 6.2.18, the proof is complete. □

Example 6.2.22. Let $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in M_2\big(\mathbb{Z}_4\big)$. Then $A \notin GL_2\big(\mathbb{Z}_4\big)$ and $A \notin J\big(M_2(\mathbb{Z}_4)\big)$. Since $tr(A) = 0 \notin U\big(\mathbb{Z}_4\big)$, it follows from Corollary 6.2.15 that A is not strongly J_1-clean. For any $\lambda \in J\big(\mathbb{Z}_4\big), \mu \in U\big(\mathbb{Z}_4\big)$, we directly check that the quadratic equation $x^2 + \mu x + \lambda = 0$ has a root in $J\big(\mathbb{Z}_4\big)$ and a root in $U\big(\mathbb{Z}_4\big)$. Therefore every 2×2 matrix over $\mathbb{Z}_4$ with invertible trace is strongly J_1-clean.

We now investigate a single strongly J_1-clean matrix over commutative power series rings.

Lemma 6.2.23. *Let R be a commutative ring, and let $A(x) \in M_2\big(R[[x]]\big)$. Then the following hold:*

(1) $A(x) \in GL_2\big(R[[x]]\big)$ *if and only if* $A(0) \in GL_2(R)$.
(2) $A(x) \in J\big(M_2(R)\big)$ *if and only if* $A(0) \in J\big(M_2(R)\big)$.

Proof. As $M_2\big(R[[x]]\big) \cong M_2(R)[[x]]$, we easily obtain the result. □

Theorem 6.2.24. *Let R be a commutative local ring, and let $A(x) \in M_2\big(R[[x]]\big)$. Then the following are equivalent:*

(1) $A(x) \in M_2\big(R[[x]]\big)$ *is strongly J_1-clean.*
(2) $A(x) \in M_2\big(R[[x]]/(x^m)\big)$ *is strongly J_1-clean.*
(3) $A(0) \in M_2(R)$ *is strongly J_1-clean.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) These are obvious.

(3) $\Rightarrow$ (1) Clearly, $R[[x]]$ is a commutative local ring. Since $A(0)$ is strongly J_1-clean in $M_2(R)$, it follows from Theorem 6.2.18 that $A(0) \in GL_2(R)$, or $A(0) \in J\big(M_2(R)\big)$, or $\chi\big(A(0)\big)$ has a root $\alpha \in J(R)$ and

a root $\beta \in U(R)$. If $A(0) \in GL_2(R)$ or $A(0) \in J\big(M_2(R)\big)$, in light of Lemma 6.2.23, $A(x) \in GL_2\big(R[[x]]\big)$ or $A(x) \in J\big(M_2(R)\big)$, and then $A(x) \in M_2\big(R[[x]]\big)$ is strongly J_1-clean. Thus, we may assume that $\chi\big(A(0)\big) = t^2 + \mu t + \lambda$ has a root $\alpha \in J(R)$ and a root $\beta \in U(R)$.

Write $\chi\big(A(x)\big) = t^2 + \mu(x)t + \lambda(x)$ where $\mu(x) = \sum\limits_{i=0}^{\infty} \mu_i x^i, \lambda(x) = \sum\limits_{i=0}^{\infty} \lambda_i x^i \in R[[x]]$ and $\mu_0 = \mu, \lambda_0 = \lambda$. Write $y = \sum\limits_{i=0}^{\infty} b_i x^i$. Then $y^2 = \sum\limits_{i=0}^{\infty} c_i x^i$ where $c_i = \sum\limits_{k=0}^{i} b_k b_{i-k}$. Clearly, $y^2 + \mu(x)y + \lambda(x) = 0$ holds in $R[[x]]$ if $\big(b_0^2 + 2b_0b_1x + (b_1^2 + 2b_0b_2)x^2 + \cdots\big) + \big(\mu_0b_0 + (\mu_0b_1 + \mu_1b_0)x + (\mu_0b_2 + \mu_1b_1 + \mu_2b_0)x^2 + \cdots\big) + \big(\lambda_0 + \lambda_1x + \lambda_2x^2 + \cdots\big) = 0$. That is,

$$\begin{aligned} b_0^2 + \mu b_0 + \lambda &= 0, \\ (2b_0 + \mu_0)b_1 &= -\lambda_1 - \mu_1 b_0, \\ (2b_0 + \mu_0)b_2 &= -\lambda_2 - \mu_1 b_1 - \mu_2 b_0 - b_1^2, \\ &\vdots \end{aligned}$$

Let $b_0 = \alpha$. It is easy to verify that $\mu_0 = \alpha + \beta \in U(R)$. Hence, $2b_0 + \mu_0 \in U(R)$. Choose

$$\begin{aligned} b_1 &= (2b_0 + \mu_0)^{-1}(-\lambda_1 - \mu_1 b_0), \\ b_2 &= (2b_0 + \mu_0)^{-1}(-\lambda_2 - \mu_1 b_1 - \mu_2 b_0 - b_1^2), \\ &\vdots \end{aligned}$$

Then $y = \sum\limits_{i=0}^{\infty} b_i x^i \in R[[x]]$ is a root of $\chi\big(A(x)\big)$. In addition, $y \in J\big(R[[x]]\big)$ as $b_0 \in J(R)$.

Since $y^2 + \mu(x)y + \lambda(x) = 0$, we see that $\chi\big(A(x)\big) = (t - y)(t + y) + \mu(t - y) = (t - y)(t + y + \mu)$. Set $z = -y - \mu$. Then $z \in U\big(R[[x]]\big)$ as $\mu \in U\big(R[[x]]\big)$. Therefore $\chi\big(A(x)\big)$ has a root in $J\big(R[[x]]\big)$ and a root in $U\big(R[[x]]\big)$, and proving (1) by Theorem 6.2.18. □

Corollary 6.2.25. *Let R be a commutative local ring. Then the following are equivalent:*

(1) *Every $A \in M_2(R)$ with invertible trace is strongly J_1-clean.*
(2) *Every $A(x) \in M_2\big(R[[x]]\big)$ with invertible trace is strongly J_1-clean.*

Proof. (1) ⇒ (2) Let $A(x) \in M_2\big(R[[x]]\big)$ with invertible trace. Then $trA(0) \in U(R)$. By hypothesis, $A(0) \in M_2(R)$ is strongly J_1-clean. In light of Theorem 6.2.24, $A(x) \in M_2\big(R[[x]]\big)$ is strongly J_1-clean.

(2) ⇒ (1) This is clear. □

6.3 Strongly J_π-Clean Rings

A ring R is strongly J_π-clean in case for any $a \in R$, there exists an idempotent $e \in comm(a)$ such that $a - e \in U(R)$ and $eae \in J^{\#}(R)$. This class of strongly clean rings was introduced by Diesl [233]. Clearly, every strongly J_n-ring is a strongly J_π-ring. But the converse is not true as the following example shows.

Example 6.3.1. Let D be a division ring, and let $R = \bigoplus_{i=1}^{\infty} M_i(D)$. Then R is strongly J_π-clean , while R is not strongly J_n-clean for all $n \in \mathbb{N}$.

Proof. Let $x \in R$. Write $x = (x_1, x_2, \ldots, x_m, 0, 0, \ldots) \in R$ for some $m \in \mathbb{N}$. In view of Example 6.2.6, all $x_i \in M_i(D)(i = 1, \ldots, m)$ are strongly J_m-clean. Therefore $x \in R$ is strongly J_m-clean; hence, it is strongly J_π-clean. If R is strongly J_n-clean, then so is $M_{n+1}(D)$ by Lemma 6.2.7. This contradicts to the result of Example 6.2.6, and we are done. □

For a commutative ring R, R is strongly J_n-clean if and only if R is strongly J_π-clean. But such rings are very different in the nonocommutative case. We easily see that {strongly J_1-clean rings} $\subsetneqq$ {strongly J_2-clean rings} $\subsetneqq$ {strongly J_3-clean rings} $\subsetneqq \cdots \subsetneqq$ {strongly J_π-clean rings}.

Proposition 6.3.2. *Let R be a ring.*

(1) *If R is strongly π-regular, then R is strongly J_π-clean.*
(2) *If R is strongly J_π-clean, then $R/J(R)$ is strongly π-regular.*

Proof. (1) This is proved as in Theorem 6.2.1.

(2) For any $a \in R$, there exists an idempotent $f \in comm(a)$ and a unit $v \in R$ such that $a = f + v$ and $fa^n \in J(R)$. Analogously to Theorem 6.2.1, there exists an idempotent $e \in comm(a)$ and a unit $u \in R$ such that $a^n - eu \in J(R)$. Hence, $a^n - a^{2n}u^{-1} \in J(R)$, and we therefore complete the proof. □

In fact, we have the inclusions {strongly π-regular ring} $\subsetneq$ {strongly J_π-clean rings } $\subsetneq$ {strongly clean rings}.

Theorem 6.3.3. *Let R be a local ring, and let $n \in \mathbb{N}$. Then the following are equivalent:*

(1) $T_2(R)$ *is strongly* J_π*-clean.*
(2) $T_2(R)$ *is strongly* J_1*-clean.*
(3) R *is bleached.*

Proof. (1) $\Rightarrow$ (3) Let $a \in J(R), b \in U(R)$ and $v \in R$. Then we can find an idempotent $E = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}$ such that $\begin{pmatrix} a & v \\ 0 & b \end{pmatrix} - E \in U\big(T_2(R)\big), E\begin{pmatrix} a & v \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & v \\ 0 & b \end{pmatrix} E \in J^{\#}(T_2(R))$. Thus, $ax - xb = v$, and so $l_a - r_b : R \to R$ is surjective. Similarly, $l_b - r_a : R \to R$ is surjective. Therefore R is bleached.

(3) $\Rightarrow$ (2) Let $A = \begin{pmatrix} a & v \\ 0 & b \end{pmatrix} \in T_2(R)$. If $a, b \in U(R)$ or $a, b \in J(R)$, then $A \in T_2(R)$ is strongly J_1-clean. If $a \in U(R), b \in J(R)$, then we have some $x \in R$ such that $ax - xb = -v$. Thus, $A - \begin{pmatrix} 0 & x \\ 0 & 1 \end{pmatrix} \in U\big(T_2(R)\big)$; hence, $A \in T_2(R)$ is strongly J_1-clean. If $a \in J(R), b \in U(R)$, analogously, $A \in T_2(R)$ is strongly J_1-clean.

(2) $\Rightarrow$ (1) This is trivial. □

Let R be a commutative local ring. Then $T_2(R)$ is strongly J_π-clean. In the next, we discuss the strong J_π-cleanness for full matrices.

Lemma 6.3.4. *Let* $a \in R, u \in U(R)$. *Then the following are equivalent:*

(1) $a \in R$ *is strongly* J_π*-clean.*
(2) $uau^{-1} \in R$ *is strongly* J_π*-clean.*

Proof. (1) $\Rightarrow$ (2) Write $a = e + w, e = e^2, w \in U(R), ew = we$ and $ea^n \in J(R)$. Then $uau^{-1} = ueu^{-1} + uwu^{-1}$. Clearly, $ueu^{-1} \in R$ is an idempotent and $uwu^{-1} \in U(R)$. In addition, $\big(ueu^{-1}\big)\big(uwu^{-1}\big) = \big(uwu^{-1}\big)\big(ueu^{-1}\big)$. Further, $(ueu^{-1})(uau^{-1})^n = uea^nu^{-1} \in J(R)$. Therefore $uau^{-1} \in R$ is strongly J_π-clean.

(2) $\Rightarrow$ (1) This is proved in the same manner. □

Theorem 6.3.5. *Let* R *be a local ring. Then* $A \in M_2(R)$ *is strongly* J_π*-clean if and only if* $A \in GL_2(R)$ *or* $A^2 \in J\big(M_2(R)\big)$ *or* A *admits a diagonal reduction.*

Proof. If $A \in GL_2(R)$, then $A = 0 + A$ is a strongly J_π-clean decomposition. If $A^2 \in J\big(M_2(R)\big)$, then $A = I_2 + (A - I_2)$ is a strongly J_π-clean decomposition. Suppose A is similar to a matrix $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$. If $\alpha \in J(R), \beta \in U(R)$,

then

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \alpha - 1 & 0 \\ 0 & \beta \end{pmatrix}$$

is a strongly J_π-clean decomposition. In view of Lemma 6.3.4, A is strongly J_π-clean. If $\alpha \in U(R), \beta \in J(R)$, analogously, A is strongly J_π-clean. If $\alpha, \beta \in U(R)$, then $A \in GL_2(R)$. If $\alpha, \beta \in J(R)$, then $A \in J\big(M_2(R)\big)$, and so A is strongly J_π-clean. Thus one direction is proved.

Conversely, assume that $A \in M_2(R)$ is strongly J_π-clean. Then there exists an idempotent $E \in M_2(R)$ and a $W \in GL_2(R)$ such that $A = E + U, EU = UE$ and $EA^n \in J\big(M_2(R)\big)$ for some $n \in \mathbb{N}$. According to Lemma 1.2.12, E is similar to $diag(I_m, 0)$. If $m = 0$, then $A \in GL_2(R)$. If $m = 2$, then $E = I_2$; hence, $A^n \in J\big(M_2(R)\big)$. Then $\overline{A} \in M_2\big(R/J(R)\big)$ is nilpotent. Therefore $\chi(\overline{A}) \equiv t^2 \big(mod\ N(R/J(R))\big)$, by Lemma 4.3.3. Write $\chi(A) = t^2 + at + b$. Then there exists some $m \in \mathbb{N}$ such that $a^m \in J(R)$. As $J(R)$ is maximal, it is prime. Hence, $a \in J(R)$. Likewise, $b \in J(R)$. This shows that $\chi(A) \equiv t^2 \big(mod\ J(R)\big)$. By the Cayley-Hamilton Theorem, $A^2 \in J\big(M_2(R)\big)$. So we may assume that $m = 1$, i.e., there exists some $H \in GL_2(R)$ such that $HEH^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Hence

$$HAH^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + HUH^{-1}.$$

Set $V = (v_{ij}) := HUH^{-1}$. It follows from $EU = UE$ that $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} V = V \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and so $v_{12} = v_{21} = 0$ and $v_{11}, v_{22} \in U(R)$. We infer that $HAH^{-1} = \begin{pmatrix} 1 + v_{11} & 0 \\ 0 & v_{22} \end{pmatrix}$, and therefore A admits a diagonal reduction. □

Let $A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \in M_2\big(\mathbb{Z}_{(2)}\big)$. Clearly, $A^2 \in J\big(M_2(\mathbb{Z}_{(2)})\big)$, and so $A \in M_2\big(\mathbb{Z}_{(2)}\big)$ is strongly J_π-clean. But $A \notin GL_2\big(\mathbb{Z}_{(2)}\big)$ and $A \notin J\big(M_2(\mathbb{Z}_{(2)})\big)$. On the other hand, χ_A has no a root in $J\big(\mathbb{Z}_{(2)}\big)$. Accordingly, $A \in M_2\big(\mathbb{Z}_{(2)}\big)$ is not strongly J_1-clean, by Theorem 6.2.18.

Corollary 6.3.6. *Let R be a commutative local ring. Then $A \in M_2(R)$ is strongly J_π-clean if and only if $det(A) \in U(R)$, or $\chi(A) \equiv t^2 \big(mod\ J(R)\big)$, or $\chi(A)$ has a root in $J(R)$ and a root in $U(R)$.*

Proof. Suppose $A \in M_2(R)$ is strongly J_π-clean. In light of Theorem 6.3.5, $A \in GL_2(R)$ or $A^2 \in J\big(M_2(R)\big)$ or A admits a diagonal reduction. If $A \in GL_2(R)$, then $det(A) \in U(R)$. If $A^2 \in J\big(M_2(R)\big)$, as in the proof of Theorem 6.3.5, we prove that $\chi(A) \equiv t^2 \big(mod\ J(R)\big)$. If A admits a diagonal reduction, $UAU^{-1} = \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}$ for $U \in GL_2(R)$. We may assume that $\alpha \in J(R), \beta \in U(R)$. Hence, $\chi(A)$ has a root in $\alpha \in J(R)$ and a root in $\beta \in U(R)$, as desired.

Conversely, assume that $det(A) \in U(R)$, or $\chi(A) \equiv t^2 \big(mod\ J(R)\big)$, or $\chi(A)$ has a root $\alpha \in J(R)$ and a root $\beta \in U(R)$. If $det(A) \in U(R)$, then $A \in GL_2(R)$. If $\chi(A) \equiv t^2 \big(mod\ J(R)\big)$, then $\chi(\overline{A}) \equiv t^2 \big(mod\ R/J(R)\big)$. By the Cayley-Hamilton Theorem, $\overline{A}^2 = \overline{0}$, and therefore $A^2 \in J\big(M_2(R)\big)$. If $\chi(A)$ has a root in $J(R)$ and a root in $U(R)$, by Theorem 6.2.18, $A \in M_2(R)$ is strongly J_1-clean, as required. □

Analogously to Corollary 6.2.19, by using Corollary 6.3.6, we can derive the following.

Corollary 6.3.7. *Let R be a commutative local ring, and let $A \in M_2(R)$. Then the following are equivalent:*

(1) *$A \in M_2(R)$ is strongly J_π-clean.*
(2) *$A \in GL_2(R)$ or $A^2 \in J\big(M_2(R)\big)$, or $trA \in U(R)$ and the quadratic equation $x^2 - x = -\frac{detA}{tr^2A}$ has a root in $J(R)$.*
(3) *$A \in GL_2(R)$ or $A^2 \in J\big(M_2(R)\big)$, or $trA \in U(R), detA \in J(R)$ and the quadratic equation $x^2 - x = \frac{detA}{tr^2A - 4detA}$ is solvable.*

Corollary 6.3.8. *Let R be a commutative local ring, and let $A \in M_2(R)$. If $\frac{1}{2} \in R$, then the following are equivalent:*

(1) *$A \in M_2(R)$ is strongly J_π-clean.*
(2) *$A \in GL_2(R)$ or $A^2 \in J\big(M_2(R)\big)$, or $tr^2(A) - 4det(A)$ is an invertible square.*

Example 6.3.9. Let $R = \mathbb{Z}_{(3)}$ and $A = \begin{pmatrix} 3 & 3 \\ -2 & 2 \end{pmatrix} \in M_2(R)$. Then $A \in M_2(R)$ is strongly clean, but it is not strongly J_π-clean.

Proof. Since $I_2 - A = \begin{pmatrix} -2 & -3 \\ 2 & -1 \end{pmatrix} \in GL_2(R)$, $A \in M_2(R)$ is strongly clean. It is easy to check that $A \notin GL_2(R)$, $A^2 = \begin{pmatrix} 3 & 15 \\ -10 & -2 \end{pmatrix} \notin J\big(M_2(R)\big)$, and

that $tr^2(A) - 4det(A) = -23$ is not square. Therefore $A \in M_2(R)$ is not strongly J_π-clean, by Corollary 6.3.8. □

Theorem 6.3.10. *Let R be a commutative local ring, and let $A \in M_2(R)$. Then the following are equivalent:*

(1) *$A \in M_2(R)$ is strongly J_π-clean.*
(2) *$A \in M_2(R)$ is strongly J_2-clean.*
(3) *$A^2 \in J\big(M_2(R)\big)$ or $A \in M_2(R)$ is strongly J_1-clean.*

Proof. (3) $\Rightarrow$ (2) $\Rightarrow$ (1) These are obvious.

(1) $\Rightarrow$ (3) If $A \in M_2(R)$ is strongly J_π-clean, by virtue of Corollary 6.3.6, $A \in GL_2(R)$, or $A^2 \in J\big(M_2(R)\big)$, or $\chi(A)$ has a root in $J(R)$ and a root in $U(R)$. If $A \in GL_2(R)$, then A is strongly J_1-clean. If $\chi(A)$ has a root in $J(R)$ and a root in $U(R)$, by Theorem 6.2.18, A is strongly J_1-clean, and therefore we complete the proof. □

Corollary 6.3.11. *Let R be a commutative local ring, and let $A(x) \in M_2\big(R[[x]]\big)$. Then the following are equivalent:*

(1) *$A(x) \in M_2\big(R[[x]]\big)$ is strongly J_π-clean.*
(2) *$A(x) \in M_2\big(R[[x]]/(x^m)\big)$ is strongly J_π-clean.*
(3) *$A(0) \in M_2(R)$ is strongly J_π-clean.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) These are trivial.

(3) $\Rightarrow$ (1) Clearly, $A(x)^2 \in J\big(M_2(R[[x]])\big)$ if and only if $A(0)^2 \in J\big(M_2(R)\big)$. Then the result follows from Theorem 6.3.10 and Theorem 6.2.24. □

In what follows, we discuss the uniqueness of strongly J_π-clean rings. The double commutant of $a \in R$ is defined by $comm^2(a) = \{x \in R \mid xy = yx \text{ for all } y \in comm(a)\}$. Following Cui and Chen, we call $a \in R$ is pseudopolar provided that there exists an idempotent $p \in comm^2(a)$ such that $a + p \in U(R)$ and $pap \in J^\#(R)$. A ring R is pseudopolar in the case that every element in R is pseudopolar [219].

Theorem 6.3.12. *Let R be a ring, and let $a \in R$. Then the following are equivalent:*

(1) *$a \in R$ is pseudopolar.*
(2) *There exists an idempotent $p \in comm^2(a)$ such that $a - p \in U(R)$ and $pap \in J^\#(R)$.*

(3) *There exists some* $b \in comm^2(a)$ *such that* $b = b^2a$ *and* $a^k - a^{k+1}b \in J(R)$ *for some* $k \in \mathbb{N}$.

Proof. (2) $\Rightarrow$ (3) Write $(pap)^k \in J(R)$ for some $k \in \mathbb{N}$. Set $b = (1-p)(a-p)^{-1}$. Then $b \in comm^2(a)$ as $p \in comm^2(a)$. One easily checks that $b^2a = (1-p)(a-p)^{-2}a = (1-p)(a-p)(a-p)^{-2} = (1-p)(a-p)^{-1} = b$ and $a^k - a^{k+1}b = a^k - a^{k+1}(1-p)(a-p)^{-1} = a^k - (a-p)^{k+1}(1-p)(a-p)^{-1} = a^k - a^k(1-p) = a^kp \in J(R)$, proving (3).

(3) $\Rightarrow$ (2) Set $p = 1-ab$. Then $p = p^2 \in comm^2(a)$ as $b \in comm^2(a)$. Thus, $a - p = a - 1 + ab$ and $(a - 1 + ab)(b - 1 + ab) = 1 - ap$. As $a^k - a^{k+1}b \in J(R)$, we have $a^kp \in J(R)$, and so $(ap)^k \in J(R)$. We infer that $1 - ap \in U(R)$; hence, $a - p \in U(R)$. Furthermore, we see that $pap \in J^{\#}(R)$, proving (2).

(1) $\Rightarrow$ (3) Write $(pap)^k \in J(R)$ for some $k \in \mathbb{N}$. Set $b = (1-p)(a+p)^{-1}$. As in the proof of "(2) $\Rightarrow$ (3)", we check that $b \in comm^2(a), b^2a = b$ and $a^k - a^{k+1}b \in J(R)$, as required.

(3) $\Rightarrow$ (1) Set $p = 1-ab$. Then $p = p^2 \in comm^2(a)$ and $pap \in J^{\#}(R)$. As in the proof of "(3) $\Rightarrow$ (2)", we verify that $(a+1-ab)(b+1-ab) = 1+ap \in U(R)$. This implies that $a+p \in U(R)$, and therefore we complete the proof. □

Corollary 6.3.13. *Let* R *be a ring, and let* $a \in R$. *Then the following are equivalent:*

(1) $a \in R$ *is pseudopolar.*
(2) *There exists a unique idempotent* $e \in comm(a)$ *such that* $a - e \in U(R)$ *and* $eae \in J^{\#}(R)$.

Proof. (1) $\Rightarrow$ (2) By Theorem 6.3.12, there exists an idempotent $p \in comm^2(a)$ such that $a - p \in U(R)$ and $pap \in J^{\#}(R)$. Assume that there exists an idempotent $e \in comm(a)$ such that $a-e \in U(R)$ and $eae \in J^{\#}(R)$. Say $(pap)^m, (eae)^n \in J(R)$ for some $m, n \in \mathbb{N}$. As $p \in comm^2(a)$, we get $pe = ep$. Thus, $1 - (1-p)e = 1 - (1-p)(a-p)^n(a-p)^{-n}e = 1 - (1-p)a^n(a-p)^{-n}e = 1 - (1-p)ea^n(a-p)^{-n} \in 1 + J(R) \subseteq U(R)$. But $1 - (1-p)e \in R$ is an idempotent, and then $e = pe$. Similarly, we obtain $p = ep$. Thus $e = p$, as desired.

(2) $\Rightarrow$ (1) By hypothesis, there exists a unique idempotent $e \in comm(a)$ such that $a - e \in U(R)$ and $eae \in J^{\#}(R)$. Say $(eae)^n \in J(R)$ for some $n \in \mathbb{N}$. Let $x \in comm(a)$. Choose $b = (a-e)^{-1}(1-e)$. Then $b \in comm(a)$.

Further, $ab = a(a-e)^{-1}(1-e) = (a-e)^{-1}a(1-e) = (a-e)^{-1}(a-e)(1-e) = 1-e$, and that $(1-e)xe = (1-e)^n xe = (ab)^n x(1-e) = b^n xa^n e = b^n x(eae)^n \in J(R)$. Set $f = e + (1-e)xe$. Then $f = f^2 \in comm(a)$. One easily checks that

$$a - f = (a-e) - (1-e)xe \in U(R), faf = ea + (1-e)(xa)e \in J^{\#}(R).$$

The uniqueness forces that $e = f$, and so $xe = exe$. Likewise, $ex = exe$. Hence, $e \in comm^2(a)$. Therefore $a \in R$ is pseudopolar, by Theorem 6.3.12. □

Lemma 6.3.14. *Let $a \in R, u \in U(R)$. Then the following are equivalent:*

(1) *$a \in R$ is pseudopolar.*
(2) *$uau^{-1} \in R$ is pseudopolar.*

Proof. (1) $\Rightarrow$ (2) Write $a = p + w, p = p^2 \in comm^2(a), w \in U(R)$ and $ap \in J^{\#}(R)$. Then $uau^{-1} = upu^{-1} + uwu^{-1}$. One easily checks that $upu^{-1} \in R$ is an idempotent, $uwu^{-1} \in U(R)$ and $(uau^{-1})(upu^{-1}) = uapu^{-1} \in J^{\#}(R)$. For any $x \in comm(uau^{-1})$, we have $xuau^{-1} = uau^{-1}x$; hence, $u^{-1}xu \in comm(a)$. It follows that $pu^{-1}xu = u^{-1}xup$, and then $upu^{-1}x = xupu^{-1}$. This implies that $upu^{-1} \in comm^2(uau^{-1})$. Therefore $uau^{-1} \in R$ is pseudopolar, by Theorem 6.3.12.

(2) $\Rightarrow$ (1) This is analogous. □

A ring R is cobleached provided that for any $a \in J(R), b \in U(R)$, $l_a - r_b$ and $r_b - r_a$ are both injective. For instance, every commutative ring is cobleached.

Theorem 6.3.15. *Let R be a cobleached local ring, and let $A \in M_2(R)$. Then the following are equivalent:*

(1) *$A \in M_2(R)$ is pseudopolar.*
(2) *$A \in M_2(R)$ is strongly J_π-clean.*

Proof. (1) $\Rightarrow$ (2) This is obvious by Theorem 6.3.12.

(2) $\Rightarrow$ (1) In view of Theorem 6.3.5, $A \in GL_2(R)$ or $A^2 \in J\big(M_2(R)\big)$ or A admits a diagonal reduction. If $A \in GL_2(R)$, then A is pseudopolar. If $A^2 \in J\big(M_2(R)\big)$, then $I_2 - A \in GL_2(R)$. Hence, $A = I_2 + (A - I_2)$ with $A \in J^{\#}\big(M_2(R)\big)$, and so A is pseudopolar. If A is similar to a diagonal matrix, we may write $UAU^{-1} = \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}$, where $\alpha \in J(R), \beta \in U(R)$.

Clearly,

$$\begin{pmatrix} \alpha & \\ & \beta \end{pmatrix} - \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} = \begin{pmatrix} \alpha - 1 & \\ & \beta \end{pmatrix} \in GL_2(R).$$

For any $(x_{ij}) \in comm\left(\begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}\right)$, we have $\alpha x_{12} = x_{12}\beta, \beta x_{21} = x_{21}\alpha$. By hypothesis, we get $x_{12} = x_{21} = 0$. Hence,

$$\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}(x_{ij}) = (x_{ij})\begin{pmatrix} 1 & \\ & 0 \end{pmatrix},$$

and so $\begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \in comm^2\left(\begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}\right)$. Furthermore, $\begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}\begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \in J\big(M_2(R)\big)$. By Lemma 6.3.14, $A \in M_2(R)$ is pseudopolar, as required. □

As an immediate consequence of Theorem 6.3.15, we now derive

Corollary 6.3.16. *Let R be a commutative local ring, and let $A \in M_2(R)$. Then the following are equivalent:*

(1) *$A \in M_2(R)$ is pseudopolar.*
(2) *$A \in M_2(R)$ is strongly J_π-clean.*

Lemma 6.3.17. *Let R be pseudopolar, and let $e \in R$ be an idempotent. Then eRe is pseudopolar.*

Proof. Let $a \in eRe$. Then we can find an idempotent $f \in comm^2(a)$ such that $u := a - f \in U(R)$ and $(faf)^k \in J(R)$ for some $k \in \mathbb{N}$. Thus, $a = efe + eue$. As $e \in comm(a)$, we see that $ef = fe$; hence, $efe \in eRe$ is an idempotent. Clearly, $eu = e(a-f) = (a-f)e = ue$, and so $eue \in U(eRe)$. Further, $(efe)a(efe) = e(faf)e \in J^{\#}(eRe)$. For any $x \in comm_{eRe}(a)$, we see that $x \in comm(a)$. It follows that $fx = xf$, and then $(efe)x = x(efe)$. This implies that $efe \in comm^2_{eRe}(a)$, hence the result. □

Lemma 6.3.18. *Let R be a ring and suppose that $A \in T_n(R)$. Then for any set $\{e_{ii}\}$ of idempotents 0 and 1 in R such that $e_{ii} = e_{jj}$ whenever $l_{A_{ii}} - r_{A_{jj}}$ is not injective, there is at most one idempotent $E \in T_n(R)$ such that $AE = EA$ and $E_{ii} = e_{ii}$ for every $i \in \{1, \ldots, n\}$.*

Proof. Clearly, the result holds for $n = 1$. Assume that the result holds for $n \leq k (k \in \mathbb{N})$. Suppose that $n = k + 1$ and there are at least two

idempotents E and E' in $T_n(R)$ such that

$$A-E, A-E' \in U\big(T_n(R)\big), AE = EA \text{ and } AE' = E'A.$$

Then $E_{ii} = E'_{ii}$ for $i = 1, \ldots, n$. Further, we may assume that $E_{ij} = E'_{ij}$ for all $(i,j) \neq (1,n)$. Thus, we may write

$$E = \begin{pmatrix} E_{11} & \beta & E_{1n} \\ & F & \gamma \\ & & E_{nn} \end{pmatrix} \text{ and } E' = \begin{pmatrix} E_{11} & \beta & E'_{1n} \\ & F & \gamma \\ & & E_{nn} \end{pmatrix},$$

by the uniqueness. If $E_{11} = E_{nn}$, it follows by Lemma 3.1.2 that $E_{1n} = E'_{1n}$ as E and F are idempotents. If $E_{11} \neq E_{nn}$, it follows by Lemma 3.1.2 that $\big(\ell_{A_{11}} - r_{A_{nn}}\big)\big(E_{1n}\big) = \big(\ell_{A_{11}} - r_{A_{nn}}\big)\big(E'_{1n}\big)$. By hypothesis, $\ell_{A_{11}} - r_{A_{nn}}$ is injective, and therefore $E_{1n} = E'_{1n}$. By induction, we complete the proof.

The following result was first proved by Wang and Chen (see [536, Theorem 2.13]).

Theorem 6.3.19. *Let R be a local ring. Then the following are equivalent:*

(1) *R is uniquely bleached.*
(2) *$T_n(R)$ is pseudopolar for all $n \in \mathbb{N}$.*
(3) *$T_n(R)$ is pseudopolar for some $n \geq 2$.*

Proof. (1) $\Rightarrow$ (2) Let $A \in T_n(R)$. As R is bleached, it follows by Corollary 3.1.5 that there exists an idempotent $E \in T_n(R)$ such that

$$A - E \in U\big(T_n(R)\big) \text{ and } E \in comm(A).$$

By the proof of Theorem 3.1.4, we know that $E_{ii} = 0$, if $A_{ii} \in U(R)$ and $E_{ii} = 1$, if $A_{ii} \in J(R)$. Thus, $EAE \in J\big(T_n(R)\big)$.

Suppose there exists an idempotent $F \in T_n(R)$ such that

$$A - F \in U\big(T_n(R)\big), F \in comm(A) \text{ and } FAF \in J\big(T_n(R)\big).$$

If $A_{ii} \in U(R)$, then $F_{ii} = 0$; otherwise, $FAF \notin J\big(T_n(R)\big)$. If $A_{ii} \in J(R)$, then $F_{ii} = 1$; otherwise, $A - F \notin U\big(T_n(R)\big)$. As R is cobleached, $F_{ii} = F_{jj}$ whenever $l_{A_{ii}} - r_{A_{jj}}$ is not injective. According to Lemma 6.3.18, we get $E = F$. In light of Corollary 6.3.13, $T_n(R)$ is pseudopolar for all $n \in \mathbb{N}$.

(2) $\Rightarrow$ (3) This is obvious.

(3) $\Rightarrow$ (1) In view of Lemma 6.3.17, $T_2(R)$ is pseudopolar. Hence, $T_2(R)$ is strongly J_π-clean. It follows by Theorem 6.3.3 that R is bleached. Let $a \in J(R)$ and $b \in U(R)$ and $ax - xb = 0$ for some $x \in R$. Choose

$A = \begin{pmatrix} a & \\ & b \end{pmatrix} \in T_2(R)$. Then there exists an idempotent $E \in comm^2(A)$ such that $A - E \in U\big(T_2(R)\big)$ and $EA \in J^{\#}\big(T_2(R)\big)$, by Theorem 6.3.12. This forces that $E = \begin{pmatrix} 1 & z \\ & e \end{pmatrix}$, where $e = e^2, z \in R$. As $EA \in J^{\#}\big(T_2(R)\big)$, $eb^k \in J(R)$ for some $k \in \mathbb{N}$, and so $e \in J(R)$. Hence, $e = 0$. It follows from $ax = xb$ that $\begin{pmatrix} 0 & x \\ & 0 \end{pmatrix} \in comm(A)$, whence, $E\begin{pmatrix} 0 & x \\ & 0 \end{pmatrix} = \begin{pmatrix} 0 & x \\ & 0 \end{pmatrix} E$. This implies that $x = 0$, i.e., $l_a - r_b : R \to R$ is injective. Similarly, $l_b - r_a : R \to R$ is injective. Accordingly, R is cobleached, proving (1). □

Let R be a commutative local ring. It follows by Theorem 6.3.19 that $T_n(R)$ is pseudopolar for all $n \in \mathbb{N}$. Also we note that the proof of Theorem 6.3.19 shows that $T_n(R)$ is strongly J_1-clean for any uniquely bleached local ring R.

6.4 Quasipolar Rings

Set $R^{qnil} = \{a \in R \mid 1 + ax \in U(R) \text{ for every } x \in comm(a)\}$. We say $a \in R$ is quasi-nilpotent if $a \in R^{qnil}$. For a Banach algebra R it is well known that

$$a \in R^{qnil} \Leftrightarrow \lim_{n\to\infty} \| a^n \|^{\frac{1}{n}} = 0.$$

For any commutative ring R, we note that

$$M_2(R)^{qnil} = \{A \in M_2(R) \mid A^2 \in M_2\big(J(R)\big)\} \text{(see [536, Example 4.3])}.$$

Definition 6.4.1. An element $a \in R$ is quasipolar if there exists an idempotent $p \in R$ such that

$$a + p \in U(R), p \in comm^2(a), ap \in R^{qnil}.$$

If a is quasipolar and $ap \in N(R)$ with the nilpotency index k, we say that a is polar of order k. Any idempotent p satisfying the above conditions is called a spectral idempotent of a. A ring R is quasipolar if every element in R is quasipolar.

Proposition 6.4.2. *Let R be a ring. Then the following are equivalent:*

(1) *R is quasipolar.*

(2) *For any* $a \in R$, *there exists an idempotent* $p \in R$ *such that* $a - p \in U(R), p \in comm^2(a), ap \in R^{qnil}$.
(3) *For any* $a \in R$, *there exists a unique idempotent* $p \in R$ *such that* $a - p \in U(R), p \in comm^2(a), ap \in R^{qnil}$.

Proof. (1) $\Rightarrow$ (2) For any $a \in R$, $-a \in R$ is quasipolar. Thus, we can find an idempotent $p \in R$ such that $-a + p \in U(R), p \in comm^2(a), ap \in R^{qnil}$, as desired.

(2) $\Rightarrow$ (3) For any $a \in R$, there exist two idempotents $p, q \in R$ such that $a - p, a - q \in U(R), p, q \in comm^2(a)$ and $ap, aq \in R^{qnil}$. Then

$$\begin{aligned} 1 - (1-p)q &= 1 - (1-p)(a-p)^{-1}(a-p)q \\ &= 1 - (1-p)(a-p)^{-1}aq \\ &= 1 - baq, \end{aligned}$$

where $b = (1-p)(a-p)^{-1}$. As $p \in comm^2(a)$, we have $b \in comm(aq)$. It follows from $aq \in R^{qnil}$ that $1 - baq \in U(R)$. Therefore

$$1 - (1-p)q = 1 - (1-p)^2q^2 = \big(1 - (1-p)q\big)\big(1 + (1-p)q\big),$$

and so $1 + (1-p)q = 1$. This implies that $q = pq$. Likewise, $p = qp$. Accordingly, $p = pq = qp = q$, as required.

(3) $\Rightarrow$ (1) For any $a \in R$, there exist a unique idempotent $p \in R$ such that $(-a) - p \in U(R), p \in comm^2(a), ap \in R^{qnil}$. Accordingly, $a + p \in U(R), ap \in R^{qnil}$, as required. □

As $J^{\#}(R) \subseteq R^{qnil}$, we see that every pseudopolar element in a ring is quasipolar, by Theorem 6.3.12. Hence, every pseudopolar ring is quasipolar.

Corollary 6.4.3. *Every strongly* π*-regular ring is quasipolar.*

Proof. Let $a \in R$. By Lemma 6.1.12, we have an idempotent $p \in comm(a)$ such that $a - p \in U(R)$ and $pa \in N(R) \subseteq R^{qnil}$. Explicitly, $p = 1 - a^n c$ with $n \in \mathbb{N}$ and $c \in comm(a)$, as in the proof of Theorem 1.1.2. Write $(pa)^m = 0$. For any $x \in comm(a)$, we see that

$$px - pxp = px(1-p) = px(1-p)^m = pxa^{mn}c^m = (pa)^{mn}xc^m = 0;$$

hence, $px = pxp$. Likewise, $xp = pxp$. Therefore $p \in comm^2(a)$, as desired. □

Corollary 6.4.4. *Every uniquely clean ring is quasipolar.*

Proof. Let R be a uniquely clean ring. For any $a \in R$, it follows from Theorem 4.4.9 that $w := (a+1) - p \in J(R)$ for a central idempotent $p \in R$.

Hence, $a-p=-1+w\in U(R)$. Clearly, $pa=pw\in J(R)\subseteq R^{qnil}$ and $p\in comm^2(a)$. Therefore we complete the proof by Proposition 6.4.2. □

Proposition 6.4.5. *Let R be a quasipolar ring, and let $e\in R$ be an idempotent. Then eRe is quasipolar.*

Proof. Let $a\in eRe$. Then there exists an idempotent $p\in R$ such that $a-p\in U(R), p\in comm^2(a), ap\in R^{qnil}$. As $p\in comm^2(a)$, we see that $ep=pe$; hence, $a-epe\in U(eRe), epe\in comm^2(a)$. For any $exe\in comm\big(a(epe)\big)$, we have $exe\in comm(ap)$, and so $1-ap(exe)\in U(R)$. This shows that $e-a(epe)(exe)\in U(eRe)$. Hence, $a(epe)\in (eRe)^{qnil}$. Accordingly, eRe is quasipolar. □

Lemma 6.4.6. *Let R be a ring, and let $a\in R$. Then the following are equivalent:*

(1) *$a\in R$ is quasipolar.*
(2) *There exists an idempotent $e\in comm^2(a)$ such that $1-e\in aRa$ and $eae\in R^{qnil}$.*
(3) *There exists an idempotent $e\in comm^2(a)$ such that $eae\in (eRe)^{qnil}$ and $(1-e)a(1-e)\in U\big((1-e)R(1-e)\big)$.*

Proof. (1) $\Rightarrow$ (2) In view of Proposition 6.4.2, there exists an idempotent $e\in comm^2(a)$ such that $u:=a-e\in U(R)$ and $eae\in R^{qnil}$. Thus, $(1-e)a=(1-e)u$, and so $(1-e)au^{-1}=1-e$. Likewise, $u^{-1}a(1-e)=1-e$. Therefore, $1-e=a(1-e)u^{-2}(1-e)a\in aRa$, as desired.

(2) $\Rightarrow$ (3) As $eae\in R^{qnil}$, we see that $eae\in (eRe)^{qnil}$. Write $1-e=ara$ for some $r\in R$. Then $(1-e)a(1-e)(1-e)ra(1-e)=1-e=(1-e)ar(1-e)a(1-e)$. Hence, $(1-e)a(1-e)\in U\big((1-e)R(1-e)\big)$, as required.

(3) $\Rightarrow$ (1) Since $(1-e)a(1-e)\in U\big((1-e)R(1-e)\big)$, we have some $b\in R$ such that $(1-e)a(1-e)b(1-e)=(1-e)b(1-e)a(1-e)=1-e$. Thus,

$$(a+e)\big((1-e)b(1-e)+e\big)=a(1-e)b(1-e)+ae+e=1+ae\in U(R).$$

Likewise, $\big((1-e)b(1-e)+e\big)(a+e)=1+ea\in U(R)$. Therefore $a+e\in U(R)$ and $eae\in R^{qnil}$, hence the result. □

We note that the condition $1-e\in aRa$ in the preceding lemma can be replaced by $1-e\in aR\cap Ra$. Evidently, $a\in R$ is quasipolar if and only if there exists an idempotent $e\in comm^2(a)$ such that $\ell(1-ay)\subseteq R(1-e)\subseteq Ra$ and $\ell(a)\subseteq Re\subseteq R(1-ay)$ for any $y\in comm(a)$, if and

only if there exists an idempotent $e \in comm^2(a)$ such that $r(1-ay) \subseteq (1-e)R \subseteq aR$ and $r(a) \subseteq eR \subseteq (1-ay)R$ for any $y \in comm(a)$ (see [211, Theorem 2.1.11]).

Let P be a submodule of a left R-module M, and let $\alpha \in End(M)$. Following Cui abd Chen, P strongly α-invariant if for any $\beta \in comm_{End(M)}(\alpha)$, P is β-invariant [218]. We now derive

Proposition 6.4.7. *Let M be a left R-module, and let $\alpha \in End(M)$. Then the following are equivalent:*

(1) *$\alpha \in End(M)$ is quasipolar;*
(2) *There exists an R-module decomposition $M = P \oplus Q$, P and Q are strongly α-invariant, $\alpha|_P \in Aut(P)$ and $\alpha|_Q \in End(Q)^{qnil}$.*

Proof. (1) $\Rightarrow$ (2) Set $E = End(M)$. By Lemma 6.4.6, we have an idempotent $e \in comm_E^2(a)$ such that $eae \in (eEe)^{qnil}$ and $(1_E-e)a(1_E-e) \in U\big((1-e)E(1-e)\big)$. Let $P = Me$ and $Q = M(1_E - e)$. For any $\beta \in comm_E(\alpha)$, $e\beta = \beta e$; hence, P and Q are β-invariant. That is, P and Q are strongly α-invariant. As $End(P) \cong eEe$ and $End(Q) \cong (1_E - e)E(1_E - e)$, we conclude that $\alpha|_P \in Aut(P)$ and $\alpha|_Q \in End(Q)^{qnil}$.

(2) $\Rightarrow$ (1) Let $e : M \to P$ be the projection onto P. Then $P = Me$ and $Q = M(1_E - e)$. For any $\beta \in comm_E(\alpha)$, Me and $M(1_E - e)$ are β-invariant. Thus $e\beta = \beta e$, and so $e \in comm_E^2(\alpha)$. Obviously, $End(P) \cong eEe$ and $End(Q) \cong (1_E - e)E(1_E - e)$. This implies that $e\alpha e \in U(eEe)$ and $(1_E - e)\alpha(1_E - e) \in \big((1_E - e)E(1_E - e)\big)^{qnil}$, and therefore we obtain the result by Lemma 6.4.6. □

Theorem 6.4.8. *Let R be a ring, and let $a \in R$. Then the following are equivalent:*

(1) *a is quasipolar.*
(2) *There exists some $b \in comm^2(a)$ such that $b = b^2a$ and $a - a^2b \in R^{qnil}$.*

Proof. (1) $\Rightarrow$ (2) By hypothesis, there exists an idempotent $p \in comm^2(a)$ such that $a - p \in U(R)$ and $ap \in R^{qnil}$. Set $b = (1-p)(a-p)^{-1}$. Then $b \in comm^2(a)$ and $b^2a = b$. Further, $a - a^2b = a - a^2(1-p)(a-p)^{-1} = a - (a-p)^2(1-p)(a-p)^{-1} = a - a(1-p) = ap \in R^{qnil}$, proving (2).

(2) $\Rightarrow$ (1) Let $p = 1-ab$. Then $p = p^2 \in comm^2(a)$ and $a-p = a-1+ab$. We check that $(a-1+ab)(b-1+ab) = 1-a(1-ab) = 1-(a-a^2b) \in U(R)$, and so $a - p \in U(R)$. Additionally, $ap = a - a^2b \in R^{qnil}$, as desired. □

Corollary 6.4.9. *Let R be a ring, and let $a \in R$. Then the following are equivalent:*

(1) *a is quasipolar.*
(2) *There exists a unique $b \in comm^2(a)$ such that $b = b^2a$ and $a - a^2b \in R^{qnil}$.*

Proof. (1) $\Rightarrow$ (2) Suppose that $c = c^2a, c \in comm^2(a)$ and $a - a^2c \in R^{qnil}$. Set $p = 1 - ab$ and $q = 1 - ac$. As in the proof of Theorem 6.4.8, we have

$$a - p \in U(R), ap \in R^{qnil}, a - q \in U(R) \text{ and } aq \in R^{qnil}.$$

In light of Proposition 6.4.2, we get $p = q$, and so $ab = ac$. Therefore $b = b^2a = b(ac) = (ba)c = (ac)c = c^2a = c$, as required.

(2) $\Rightarrow$ (1) This is obvious by Theorem 6.4.8. □

Let $a \in R$ be quasipolar. We say that the preceding b in Corollary 6.4.9 the generalized Drazin inverse of a, and denote it by a^d.

Lemma 6.4.10. *Let R be a ring, and let $a, b, c, d \in R$ satisfying*

$$(ac)^2 = (db)(ac), (db)^2 = (ac)(db);$$
$$b(ac)a = b(db)a, c(ac)d = c(db)d.$$

Then $ac \in R^{qnil}$ if and only if $bd \in R^{qnil}$.

Proof. $\Longrightarrow$ Let $x \in comm(bd)$. Then we check that

$$\begin{aligned}(dbdx^5bdbac)ac &= dbdx^5bd(baca)c \\ &= dbdx^5bd(bdba)c \\ &= dbdx^5b(dbdba)c \\ &= (dbdbd)x^5bdbac \\ &= (dbdb)dx^5bdbac \\ &= (acdb)dx^5bdbac \\ &= (acdbd)x^5bdbac \\ &= ac(dbdx^5bdbac)\end{aligned}$$

Hence, $dbdx^5bdbac \in comm(ac)$, and so $1 - dbd(x^5bdbacac) = 1 - (dbdx^5bdbac)ac \in R^{-1}$. By using Jacobson's Lemma, we see that

$$\begin{aligned}1 - x^5bdbdbdbdbd &= 1 - (x^5bdbdb(dbdb)d \\ &= 1 - (x^5bdbdb(acdb)d \\ &= 1 - (x^5bdbdbac)dbd \\ &= 1 - (x^5bdbacac)dbd \\ &\in U(R).\end{aligned}$$

Then

$$\begin{aligned}&(1-xbd)(1+xbd+x^2bdbd+x^3bdbdbd+x^4bdbdbdbd)\\ =\ &(1+xbd+x^2bdbd+x^3bdbdbd+x^4bdbdbdbd)(1-xbd)\\ =\ &1-x^5bdbdbdbdbd\\ \in\ &U(R),\end{aligned}$$

and so $bd \in R^{qnil}$.

$\Longrightarrow$ Since $bd \in R^{qnil}$, we easily see that $db \in R^{qnil}$. Applying the preceding discussion, we see that $ca \in R^{qnil}$. Therefore $ac \in R^{qnil}$, as desired. □

We are now ready to prove:

Theorem 6.4.11. *Let R be a ring, and let $a, b, c, d \in R$ satisfying*

$$\begin{aligned}&(ac)^2 = (db)(ac), (db)^2 = (ac)(db);\\ &b(ac)a = b(db)a, c(ac)d = c(db)d.\end{aligned}$$

Then $ac \in R$ has generalized Drazin inverse if and only if $bd \in R$ has generalized Drazin inverse.

Proof. $\Longrightarrow$ Let $h = (ac)^d$. Let $e = bh^2d$. We shall prove that $e = (bd)^d$.

Step 1. Let $t \in comm(bd)$. Then we check that

$$\begin{aligned}ac(dtbdbdbac) &= (acdbdbd)(tbac)\\ &= (dbdbdbd)(tbac)\\ &= dtbdb(dbdba)c\\ &= (dtbdbdbac)ac.\end{aligned}$$

Thus $dtbdbdbac \in comm(ac)$, and so $(dtbdbdbac)h = h(dtbdbdbac)$. It is easy to verify that $acdbd = a(cdbd) = a(cacd) = (ac)^2d = (dbac)d = dbacd$, and so $(bacd)(bd) = (bd)(bacd)$. Then we compute that

$$\begin{aligned}et &= (bh^6(ac)^4d)t = bh^6(db)^3acdt = bh^6dbdbd(bacd)t\\ &= bh^6d(bacd)bdbdt = bh^6db(acdb)dbdt = bh^6(dbdbdbdbd)t\\ &= bh^6dtbdb(dbdb)d = bh^6dtbdba(cdbd) = bh^6dtbdb(acac)d\\ &= bh^6(dtbdbdbac)d = b(dtbdbdbac)h^6d = tbdbdbdbach^6d\\ &= tb(ac)^4h^6d = tbh^2d = te,\end{aligned}$$

and then $e \in comm^2(bd)$.

Step 2. We directly verify that

$$\begin{aligned}e(bd)e &= bh^2dbdbh^2d = bh^2acdbh^2d\\ &= bh^2acdb(ac)h^3d = bh^2(ac)^3h^3d = b(ac)^3h^5d = e.\end{aligned}$$

Step 3. Let $p = 1-(ac)h$. Then $(pa)c = ac-(ac)^2h \in R^{qnil}$. One easily checks that

$$\begin{aligned} bd-(bd)^2e &= bd - bdbdbh^2d \\ &= bd - b(dbdb)ach^3d \\ &= bd - b(acdb)ach^3d \\ &= bd - bac(dbac)h^3d \\ &= bd - b(ac)^3h^3d \\ &= b(1-ach)d \\ &= b(pd). \end{aligned}$$

We directly compute that

$$\begin{aligned} (pac)^2 &= p(ac)^2p = p(dbac)p = (pdb)(pac), \\ (pdb)^2 &= pdbpdb = pdb[1-(1-p)]db \\ &= p(db)^2 - pdbac(ac)^ddb \\ &= p(db)^2 - p(ac)^2(ac)^ddb \\ &= p(db)^2 = pacdb = (pac)(pdb); \\ b(pac)pa &= bacpa - b(ac)^d(ac)^2pa \\ &= bacpa - b(ac)^dd(bacpa) = bdbpa - b(ac)^dd(bdbpa) \\ &= bdbpa - b(ac)^d(acdb)pa = b(pdb)pa, \\ c(pac)pd &= cacd - c(ac)^d(ac)^2d = cdbd - c(ac)^d(ac)^2d \\ &\quad - [c(ac)^da(cdbd) - c(ac)^da(cacd)] = cdbd - cac(ac)^dacd \\ &\quad - [c(ac)^dacdbd - c(ac)^dacdbac(ac)^dd] \\ &= cdbd - cac(ac)^dacd - c(ac)^dacdb[1-ac(ac)^d]d \\ &= cdbd - cdb(ac)^dacd - c(ac)^dacdbpd \\ &= [cdb - c(ac)^dacdb]pd = c(pdb)pd. \end{aligned}$$

Since $(pa)c = ac-(ac)^2(ac)^d \in R^{qnil}$. In view of Lemma 6.4.10, $b(pd) \in R^{qnil}$. Therefore bd has g-Drazin inverse.

$\Longleftarrow$ Clearly, $db \in R$ has generalized Drazin inverse. Applying the preceding discussion, we have $ca \in R$ has generalized Drazin inverse. Therefore $ac \in R$ has generalized Drazin inverse. This completes the proof. □

Corollary 6.4.12. *Let R be a ring, let $k \in \mathbb{N}$, and let $a, b \in R$. If $(ab)^k$ has generalized Drazin inverse, then so does $(ba)^k$.*

Proof. Let $k = 1$, then by Theorem 6.4.11, ab has generalized Drazin inverse if and only if ba has generalized Drazin inverse. Now let $k \geq 2$, as $(ab)^k = a(b(ab)^{k-1})$, then by Theorem 6.4.11, $(ba)^k = (b(ab)^{k-1})a$ has generalized Drazin inverse. □

In the case that $k = 1$ in Corollary 6.4.12, we recover the Cline's formula for generalized Drazin inverse: $ab \in R$ has generalized Drazin inverse if and only if $ba \in R$ has generalized Drazin inverse.

Theorem 6.4.13. *Let R be a ring, and let $a, b, c, d \in R$ satisfying*

$$(ac)^2 = (db)(ac), (db)^2 = (ac)(db);$$
$$b(ac)a = b(db)a, c(ac)d = c(db)d.$$

Then $\alpha = 1 - bd \in R$ has generalized Drazin inverse if and only if $\beta = 1 - ac \in R$ has generalized Drazin inverse.

Proof. Let $p = \alpha^\pi, x = \alpha^d$. Then $1 - p\alpha(1 + bd) \in R^{-1}$. Let

$$y = \big[1 - dp(1 - p\alpha(1 + bd))^{-1}bac\big](1 + ac) + dxbac.$$

We shall prove that $\beta^d = y$.

Step 1. $y\beta y = y$. We see that

$$y\beta = 1 - (ac)^2 - dp(1 - p\alpha(1 + bd))^{-1}bac\big[1 - (ac)^2\big] + dxbac(1 - ac).$$

We compute that

$$\begin{aligned}
y\beta &= 1 - (ac)^2 - dp(1 - p\alpha(1 + bd))^{-1}bac\big[1 - (ac)^2\big] + dxbac(1 - ac)\\
&= 1 - [dbac - dxbac(1 - ac)] - dp(1 - p\alpha(1 + bd))^{-1}[bac - bac(ac)^2]\\
&= 1 - [dbac - dx(bac - bacac)] - dp(1 - p\alpha(1 + bd))^{-1}[ba - bacdba]c\\
&= 1 - [dbac - dx(1 - bd)bac] - dp(1 - p\alpha(1 + bd))^{-1}[1 - (bd)^2]bac\\
&= 1 - dpbac - dp(1 - p\alpha(1 + bd))^{-1}p\alpha(1 + bd)bac\\
&= 1 - dp(1 - p\alpha(1 + bd))^{-1}\big[(1 - p\alpha(1 + bd)) + p\alpha(1 + bd)\big]bac\\
&= 1 - dp(1 - p\alpha(1 + bd))^{-1}bac.
\end{aligned}$$

Since $acdbd = a(cdbd) = a(cacd) = (ac)^2d = (dbac)d = dbacd$, we have $(bacd)(bd) = (bd)(bacd)$, and so $(bacd)\alpha = \alpha(bacd)$. Hence, $(bacd)x = x(bacd)$, and then

$$\begin{aligned}
&dp(1 - p\alpha(1 + bd))^{-1}bacdxbac(1 - ac)\\
&= d(1 - p\alpha(1 + bd))^{-1}pxbacdbac(1 - ac)\\
&= 0.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&y\beta y\\
&= y - dp(1 - p\alpha(1 + bd))^{-1}bacy\\
&= y - dp(1 - p\alpha(1 + bd))^{-1}bac\big[1 - dp(1 - p\alpha(1 + bd))^{-1}bac\big](1 + ac)\\
&= y - dp(1 - p\alpha(1 + bd))^{-1}bac(1 + ac) + dp(1 - p\alpha(1 + bd))^{-2}(bacd)bac(1 + ac)\\
&= y - dp(1 - p\alpha(1 + bd))^{-1}(1 + bd)bac + dp(1 - p\alpha(1 + bd))^{-2}(bd)^2(1 + bd)bac\\
&= y - dp(1 - p\alpha(1 + bd))^{-2}\big[p - p\alpha(1 + bd) - p(bd)^2\big](1 + bd)bac\\
&= y.
\end{aligned}$$

Step 2. $\beta-\beta y\beta \in R^{qnil}$. Since $y=y\beta y$, we see that $(1-y\beta)^2=1-y\beta$. Hence,

$$\begin{aligned}\beta-\beta y\beta &= \beta(1-y\beta)\\ &= \beta dp(1-p\alpha(1+bd))^{-1}(bacd)p(1-p\alpha(1+bd))^{-1}bac\\ &= \beta d(bacd)p(1-p\alpha(1+bd))^{-2}bac\\ &= (1-ac)d(bacd)p(1-p\alpha(1+bd))^{-2}bac\\ &= d\alpha bacdp(1-p\alpha(1+bd))^{-2}bac.\end{aligned}$$

Let $z\in comm(\beta-\beta y\beta)$. Then

$$zd\alpha bacdp(1-p\alpha(1+bd))^{-2}bac = d\alpha bacdp(1-p\alpha(1+bd))^{-2}bacz.$$

We will suffice to prove

$$1+d\alpha bacdp(1-p\alpha(1+bd))^{-2}bacz\in R^{-1}.$$

Clearly, we have

$$p=(bd)^2p[1-p\alpha(1+bd)]^{-1}=(bd)^4p[1-p\alpha(1+bd)]^{-2}.$$

Hence, we get

$$\begin{aligned}(baczdbacd)\alpha p &= (baczd)\alpha bacdp[1-p\alpha(1+bd)]^{-2}(bd)^4\\ &= (baczd)\alpha bacdp[1-p\alpha(1+bd)]^{-2}bac(db)^2d\\ &= bac\big[zd\alpha bacdp(1-p\alpha(1+bd))^{-2}bac\big]acdbd\\ &= bac\big[d\alpha bacdp(1-p\alpha(1+bd))^{-2}bacz\big]acdbd\\ &= bdbdbacd\big[\alpha p(1-p\alpha(1+bd))^{-2}bacz\big]acdbd\\ &= bacdbdbd\big[\alpha p(1-p\alpha(1+bd))^{-2}bacz\big]a(cdbd)\\ &= bdbdbdbd\big[\alpha p(1-p\alpha(1+bd))^{-2}bacz\big]a(cacd)\\ &= bdbdbdbd\alpha p[1-p\alpha(1+bd)]^{-2}baczdbacd\\ &= bdbdbdbd\alpha p[1-p\alpha(1+bd)]^{-2}baczdbacd\\ &= \alpha p(baczdbacd).\end{aligned}$$

Step 3. $y\in comm^2(\beta)$. Let $s\in comm(\beta)$. Then $s\beta=\beta s$, and so $s(ac)=(ac)s$.

Claim 1. $s(dxbac)=(dxbac)s$. We easily check that $(bacsdbd)\alpha = bacs\beta dbd\alpha = bac\beta sdbd\alpha = \alpha(bacsdbd)$. Hence $(bacsdbd)x=x(bacsdbd)$, and then

$$\begin{aligned}s(dpbac) &= sd(bd)^4p[1-p\alpha(1+bd)]^{-2}bac\\ &= s(ac)^2dbdbdp[1-p\alpha(1+bd)]^{-2}bac\\ &= d(bacsdbd)bdp[1-p\alpha(1+bd)]^{-2}bac\\ &= dbdp[1-p\alpha(1+bd)]^{-2}(bacsdbd)bac\\ &= dbdp[1-p\alpha(1+bd)]^{-2}bacs(ac)^3\\ &= dbdp[1-p\alpha(1+bd)]^{-2}(bd)^3bacs\\ &= d(bd)^4p[1-p\alpha(1+bd)]^{-2}bacs\\ &= (dpbac)s.\end{aligned}$$

Since $sdbac = s(ac)^2 = (ac)^2 s = dbacs$, we have

$$sd\alpha xbac = d\alpha xbacs,$$

and so

$$sdxbac - sdbdxbac = dxbacs - dbdxbacs.$$

On the other hand, we have

$$\begin{aligned} s(dbdpbac) &= sd(bd)^5 p[1 - p\alpha(1 + bd)]^{-2} bac \\ &= s(ac)^4 dbdp[1 - p\alpha(1 + bd)]^{-2} bac \\ &= dbdbd(bacsdbd)p[1 - p\alpha(1 + bd)]^{-2} bac \\ &= dbdbdp[1 - p\alpha(1 + bd)]^{-2}(bacsdbd)bac \\ &= dbdbdp[1 - p\alpha(1 + bd)]^{-2} bacs(ac)^3 \\ &= dbdbdp[1 - p\alpha(1 + bd)]^{-2}(bd)^3 bacs \\ &= dbd(bd)^4 p[1 - p\alpha(1 + bd)]^{-2} bacs \\ &= (dbdpbac)s. \end{aligned}$$

Since $sdbdbac = s(ac)^3 = (ac)^3 s = dbdbacs$, we have

$$dbd\alpha xbacs = sdbd\alpha xbac.$$

Then we have

$$\begin{aligned} dbdbd\alpha xbacs &= ac(dbd\alpha xbacs) \\ &= ac(sdbd\alpha xbac) \\ &= sacdbd\alpha xbac \\ &= sdbdbd\alpha xbac, \end{aligned}$$

and so

$$dbd(1 + bd)\alpha xbacs = sdbd(1 + bd)\alpha xbac,$$

and then

$$dbdxbacs - dbd(bd)^2 xbacs = sdbdxbac - sdbd(bd)^2 xbac.$$

One easily checks that

$$\begin{aligned} d(bd)^3 xbacs &= dx(bd)^3 bacs \\ &= dxb(ac)^4 s \\ &= dx(bacsdbd)bac \\ &= d(bacsdbd)xbac \\ &= (ac)^2 sdbdxbac \\ &= sd(bd)^3 xbac. \end{aligned}$$

This implies that $dbdxbacs = sdbdxbac$, and therefore $s(dxbac) = (dxbac)s$.

Claim 2. $sdp(1-p\alpha(1+bd))^{-1}bac(1+ac) = dp(1-p\alpha(1+bd))^{-1}bac(1+ac)s$. Set $t = dp(1-p\alpha(1+bd))^{-1}bac(1+ac)$. Then we have

$$\begin{aligned} st &= sdp(1-p\alpha(1+bd))^{-1}bac(1+ac) \\ &= sd(bd)^4p[1-p\alpha(1+bd)]^{-3}bac(1+ac) \\ &= (ac)^3sdbdp[1-p\alpha(1+bd)]^{-3}bac(1+ac) \\ &= dbdp[1-p\alpha(1+bd)]^{-3}bs(ac)^4(1+ac) \end{aligned}$$

Also we have

$$\begin{aligned} ts &= dp(1-p\alpha(1+bd))^{-1}bac(1+ac)s \\ &= dp[1-p\alpha(1+bd)]^{-3}(bd)^4bsac(1+ac) \\ &= dbdp[1-p\alpha(1+bd)]^{-3}b(ac)^3sac(1+ac) \\ &= dbdp[1-p\alpha(1+bd)]^{-3}bs(ac)^4(1+ac) \end{aligned}$$

Hence $st = ts$, and then $y \in comm^2(\beta)$. This sows that $y = \beta^d$, as required.

$\Longleftarrow$ Since $1-ac \in R$ has generalized Drazin inverse, it follows that $1-ca \in R$ has generalized Drazin inverse. Applying the preceding discussion, we obtain that $1-bd \in R$ has generalized Drazin inverse, as desired. □

For any $a, b \in R$, we recover the Jacobson's Lemma for the generalized Drazin inverse: $1-ab \in R$ has generalized Drazin inverse if and only if $1-ba \in R$ has generalized Drazin inverse. Furthermore, we have

Corollary 6.4.14. *Let R be a ring, and let $A \in M_{m\times n}(R), B \in M_{n\times m}(R)$. Then $I_m + AB \in M_m(R)$ has generalized Drazin inverse if and only if $I_n + BA \in M_n(R)$ has generalized Drazin inverse*

Proof. Let $k = m + n$. Set

$$C = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in M_{k\times k}(R).$$

Then we observe that

$$I_k + CD = \begin{pmatrix} I_n & 0 \\ 0 & I_m + AB \end{pmatrix}, I_k + DC = \begin{pmatrix} I_n + BA & 0 \\ 0 & I_m \end{pmatrix}.$$

Theorem 6.4.13 applies to give the following result: $I_k + CD \in M_{k\times k}(R)$ has generalized Drazin inverse if and only if so has $I_k + DC \in M_{k\times k}(R)$. Therefore we easily obtain the result. □

Lemma 6.4.15. *The ideal-extension $I(R;S)$ is quasipolar provided that*

(1) *R is quasipolar;*
(2) *If $e = e^2 \in R$, then $es = se$ for all $s \in S$;*

(3) *If $s \in S$, then there exists some $s' \in S$ such that $ss' = s's$ and $s + s' + ss' = 0$.*

Proof. Let $(a, s) \in I(R; S)$. By (1), $a \in R$ is quasipolar. Write $u := a + p \in U(R), p = p^2 \in comm^2(a)$ and $ap \in R^{qnil}$. Then $(a, s) = (p, 0) + (u, s)$. By (2), we see that $(p, 0)^2 = (p, 0) \in comm^2(a, s)$. By (3), there exists some $t \in S$ such that $u^{-1}st = tu^{-1}s, t + u^{-1}s + u^{-1}st = 0$. Hence, $(u, s)^{-1} = (u^{-1}, tu^{-1})$. Furthermore, $(p, 0)(a, s) = (ap, sp) \in I(R, S)^{qnil}$, and therefore $I(R; S)$ is quasipolar. □

Theorem 6.4.16. *Let R be an abelian ring, and let $\alpha : R \to R$ be a ring endomorphism. If $\alpha(e) = e$ for all $e = e^2 \in R$, then the following are equivalent:*

(1) *R is quasipolar.*
(2) *$R[[x, \alpha]]$ is quasipolar.*
(3) *$R[[x, \alpha]]/(x^n)(n \in \mathbb{N})$ is quasipolar.*

Proof. (1) $\Rightarrow$ (2) Let $S = R[[x, \alpha]] \cdot x$. Then $R[[x, \alpha]] \cong I\big(R, S\big)$. If $s \in S$, then $1 + s \in U\big(R[[x, \alpha]]\big)$. So $(1 + s)(1 + s') = (1 + s')(1 + s) = 1$ for some $s' \in S$. Hence, $ss' = s's$ and $s + s' + ss' = 0$. If $e = e^2 \in R$, then $se = \alpha(e)s = es$ for all $s \in S$. Therefore, $R[[x, \alpha]]$ is quasipolar, by Lemma 6.4.15.

(2) $\Rightarrow$ (3) Let $T = R[[x, \alpha]]/(x^n)(n \in \mathbb{N})$. Then $T = \left\{ \sum_{i=0}^{n-1} a_i x^i \;\middle|\; x^n = 0, a_0, \ldots, a_{n-1} \in R \right\}$. For any $a(x) \in T$, $a(x) \in R[[x, \alpha]]$ is quasipolar; hence, there exists an idempotent $g(x) = \sum_{i=0}^{\infty} e_i x^i \in comm^2\big(a(x)\big)$ such that $a(x) - g(x) \in U\big(R[[x, \alpha]]\big)$ and $a(x)g(x) \in R[[x, \alpha]]^{qnil}$. Choose $e(x) = \sum_{i=0}^{n-1} e_i x^i \in T$. Then $e(x) \in comm^2\big(a(x)\big)$ is an idempotent. Further, $a(x) - e(x) \in U(T)$, and that $a(x)e(x) \in T^{qnil}$. Therefore, T is quasipolar.

(3) $\Rightarrow$ (1) For any $a \in R$, there exists an idempotent $e(x) = \sum_{i=0}^{n-1} e_i x^i \in comm^2(a)$ such that $a - e(x) \in U(S)$ and $ae(x) \in S^{qnil}$. As $e(x) = e(x)^2$, we get $e_0 = e_0^2; e_0e_1 + e_1e_0 = e_1; e_0e_2 + e_1^2 + e_2e_0 = e_2; \ldots, e_0e_{n-1} + \cdots + e_{n-1}e_0 = e_{n-1}$. Since R is abelian, $(2e_0 - 1)e_1 = 0$, and then $e_1 = 0$. By iteration of this process, $e_2 = \cdots = e_{n-1} = 0$. Hence, $e(x) = e_0 \in comm^2(a)$ is an idempotent. Moreover, $a - e_0 \in U(R)$ and $ae_0 \in R^{nil}$, proving (1). □

Corollary 6.4.17. *Let R be an abelian ring. Then the following are equivalent:*

(1) *R is quasipolar.*

(2) $R_n = \left\{ \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ & a_0 & a_1 & \cdots & a_{n-3} & a_{n-2} \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & a_0 & a_1 & a_2 \\ & & & & a_0 & a_1 \\ & & & & & a_0 \end{pmatrix} \middle| \; a_0, \ldots, a_{n-1} \in R \right\}$ *is quasipolar.*

Proof. Clearly,

$$R_n \cong \{a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} \mid a_0, \ldots, a_{n-1} \in R\} \cong R[[x]]/(x^n).$$

Therefore we complete the proof by Theorem 6.4.16. □

Example 6.4.18. Let R be an abelian quasipolar ring. Then the ring $T = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn}\}$ is quasipolar.

Proof. Let $S = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn} = 0\}$. Then $T \cong I(R; S)$, and the conditions in Lemma 6.4.15 are satisfied, hence the result. □

Theorem 6.4.19. *Let R be a cobleached local ring, and let $A \in M_2(R)$. Then the following are equivalent:*

(1) *A is strongly clean.*
(2) *$I_2 - A \in GL_2(R)$, or A is quasipolar.*

Proof. (1) $\Rightarrow$ (2) In view of Theorem 1.2.13, $A \in GL_2(R)$, or $I_2 - A \in GL_2(R)$, or A is similar to a diagonal reduction. If $A \in GL_2(R)$, then A is quasipolar. If A is similar to a diagonal matrix, we may write $UAU^{-1} = \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}$, where $\alpha \in J(R), \beta \in U(R)$. As in the proof of Theorem 6.3.15, $A \in M_2(R)$ is pseudopolar. Hence, $A \in M_2(R)$ is quasipolar.

(2) $\Rightarrow$ (1) This is trivial. □

Corollary 6.4.20. *Let R be a commutative ring and $R/J(R) \cong \mathbb{Z}_2$. Then $A \in M_2(R)$ is strongly clean if and only if A is quasipolar.*

Proof. $\Longrightarrow$ Clearly, R is a commutative local ring. If $I_2 - A \notin GL_2(R)$, then A is quasipolar, by Theorem 6.4.19. If $I_2 - A \in GL_2(R)$, then $det(I_2 - A) \in U(R)$. Hence, $1 - tr(A) + det(A) \in 1 + J(R)$. If $A \in GL_2(R)$, then

$A \in M_2(R)$ is quasipolar. If $A \notin GL_2(R)$, then $det(A) \in J(R)$; hence, $tr(A) \in J(R)$. By Cayley-Hamilton Theorem, $A^2 - tr(A)A + det(A)I_2 = 0$; hence, $A^2 \in J\big(M_2(R)\big)$. Thus, $A \in M_2(R)^{qnil}$; hence, it is quasipolar.

$\Longleftarrow$ This is obvious. □

The preceding condition "$R/J(R) \cong \mathbb{Z}_2$" is necessary. For instance, $\begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix} \in M_2(\mathbb{Z}_{(5)})$ is not quasipolar, but it is strongly clean.

Theorem 6.4.21. *Let R be a commutative local ring, and let $A \in M_2(R)$. Then the following are equivalent:*

(1) *$A \in M_2(R)$ is quasipolar.*
(2) *$A \in M_2(R)$ is pseudopolar.*
(3) *$A \in M_2(R)$ is strongly J_2-clean.*

Proof. (1) $\Rightarrow$ (3) Write $A = E + W, E \in comm^2(A), W \in GL_2(R), EA \in M_2(R)^{qnil}$. If $A \in GL_2(R)$, then $A \in M_2(R)$ is strongly J_2-clean. Thus, we may assume that $A \notin GL_2(R)$; hence, $det(A) \in J(R)$. In view of Lemma 1.2.12, $UEU^{-1} = \begin{pmatrix} I_r & \\ & 0 \end{pmatrix}$ for $U \in GL_2(R)$. Here, $r = 1$ or 2. If $tr(A) \in J(R)$, it follows from Cayley-Hamilton Theorem that $A^2 - tr(A)A + det(A)I_2 = 0$; hence, $A^2 \in J\big(M_2(R)\big)$. This shows that $A \in M_2(R)$ is strongly J_2-clean. Thus, we may assume that $tr(A) \in U(R)$. If $r = 1$, then

$$UAU^{-1} = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} + \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}$$

is a quasipolar decomposition. Here, $\alpha, \beta \in U(R)$. As $\big(UAU^{-1}\big)\begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \in M_2(R)^{qnil}$, we see that $1 + \alpha \in J(R)$. Thus, the equation $x^2 - tr(A)x + det(A) = 0$ has a root in $J(R)$ and a root in $U(R)$. By using Corollary 6.3.6 and Theorem 6.3.10, $A \in M_2(R)$ is strongly J_2-clean. If $r = 2$, then $E = I_2$. Clearly, $det(A) \in J(R), tr(A) \in U(R)$. Choose $X = \begin{pmatrix} -tr^{-1}(A) & \\ & -tr^{-1}(A) \end{pmatrix}$. Then $X \in comm\big(AE\big)$, and so $I_2 + AEX \in GL_2(R)$. But $det(I_2 + AEX) = tr^{-2}det\big(-tr(A)I_2 + A\big) = tr^{-2}(A)\big(tr^2(A) - tr^2(A) + det(A)\big) = tr^{-2}(A)det(A) \in J(R)$, a contradiction. Therefore, $A \in M_2(R)$ is strongly J_2-clean.

(3) $\Rightarrow$ (2) In view of Theorem 6.3.16, $A \in M_2(R)$ is pseudopolar, proving (2).

(2) $\Rightarrow$ (1) This is trivial. □

The following example is due to Cui and Chen (see [214, Theorem 3.3]).

Example 6.4.22. Let p be a prime. Then $M_2(\widehat{\mathbb{Z}}_p)$ is quasipolar.

Proof. Let $R = \widehat{\mathbb{Z}}_p$. Case I. $p = 2$. Then $R/J(R) \cong \mathbb{Z}_2$. In view of Corollary 2.2.14, $M_2(R)$ is strongly clean. It follows from Corollary 6.4.20 that $M_2(R)$ is quasipolar.

Case II. $p \geq 3$. Then $2 \in U(R)$. Let $A \in M_2(R)$. If $det(A) \in U(R)$, then $A \in GL_2(R)$ is strongly J_2-clean. If $det(A), tr(A) \in J(R)$, then $A^2 \in J\big(M_2(R)\big)$. This shows that $A \in M_2(R)$ is strongly J_2-clean. Thus, we my assume that $det(A) \in J(R), tr(A) \in U(R)$. Write $tr(A) = a_0 + a_1p + a_2p^2 + \cdots$, where $a_0 \neq 0$. As $det(A) \in J(R)$, we get $det(A) = b_1p + b_2p^2 + \cdots$, and so $tr^2(A) - 4det(A) = a_0^2\big(1 + c_1 + c_2p^2 + \cdots\big)$. We will find some $x = x_1p + x_2p^2 + \cdots \in J(R)$ such that

$$1 + c_1 + c_2p^2 + \cdots 1 + c_1 + c_2p^2 + \cdots = (1 + x)^2.$$

That is,

$$\begin{gathered} 1 \times 1 = 1 + 0 \times p; \\ x_1 + x_1 + 0 = c_1 + l_1p; \\ x_2 + x_1^2 + x_2 + l_1 = c_2 + l_2p; \\ x_3 + x_1x_2 + x_2x_1 + x_3 + l_2 = c_3 + l_3p; \\ \vdots \end{gathered}$$

As $(2, p) = 1$, we have x_1, l_1 such that $2x_1 - l_1p = c_1$. That is, $x_1 + x_1 + 0 = c_1 + l_1p$. Further, we can find x_2, l_2 such that $2x_2 - l_2p = c_2 - x_1^2 - l_1$. That is, $x_2 + x_1^2 + x_2 + l_1 = c_2 + l_2p$. By iteration, we have $x_3, l_3; x_4, l_4; \ldots$ such that the preceding equalities hold. By changing $l_1, l_2, \ldots$, we may assume that $0 \leq x_i \leq p - 1$ for each i. Therefore $tr^2(A) - 4det(A)$ is an invertible square. In view of Corollary 6.3.8, $A \in M_2(R)$ is strongly J_π-clean. It follows by Theorem 6.3.10 that A is strongly J_2-clean. Therefore $M_2(R)$ is quasipolar, by Theorem 6.4.21. □

Theorem 6.4.23. *Let R be a commutative local ring, and let $A \in M_2(R)$.*

(1) *If $det(A) \in U(R)$, then A is quasipolar.*
(2) *If $det(A), tr(A) \in J(R)$, then A is quasipolar.*
(3) *If $det(A) \in J(R), tr(A) \in U(R)$, then A is quasipolar if and only if $x^2 - tr(A)x + det(A) = 0$ is solvable.*

Proof. Case I. $det(A) \in U(R)$. Then $A \in GL_2(R)$ is quasipolar.

Case II. $det(A), tr(A) \in J(R)$. By Cayley-Hamilton Theorem, we have $A^2 - tr(A)A + det(A)I_2 = 0$; hence, $A^2 = tr(A)A - det(A)I_2 \in J(M_2(R))$. Thus, $A \in M_2(R)^{qnil}$. It follows from $A = I_2 + (A - I_2)$ that A is quasipolar.

Case III. $det(A) \in J(R), tr(A) \in U(R)$. Write $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $a + d = tr(A) \in U(R)$. Hence, a or d are invertible. Say $a \in U(R)$. Suppose $x^2 - tr(A)x + det(A) = 0$ is solvable. Say $x_1^2 - tr(A)x_1 + det(A) = 0$. Set $x_2 = tr(A) - x_1$. Then $x_2^2 - tr(A)x_2 + det(A) = 0$. As $x_1x_2 = det(A) \in J(R)$, we see that $x_1 \in J(R), x_2 \in U(R)$ or $x_1 \in U(R), x_2 \in J(R)$. Say $x_1 \in J(R), x_2 \in U(R)$. Let $U = \begin{pmatrix} b & a - x_1 \\ x_1 - a & c \end{pmatrix}$. It follows from $det(U) = atr(A) + (x_1^2 - 2ax_1 - det(A)) \in U(R)$ that $U \in GL_2(R)$. One easily checks that $U^{-1}AU = \begin{pmatrix} x_1 & \\ & x_2 \end{pmatrix} = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} + \begin{pmatrix} x_1 - 1 & \\ & x_2 \end{pmatrix}$ is quasipolar. Therefore $A \in M_2(R)$ is quasipolar.

Suppose that A is quasipolar. Then $A = E + U, E \in comm^2(A)$ and $EA \in M_2(R)^{qnil}$. Clearly, $E \neq 0$. If $E = I_2$, then $A \in M_2(R)^{qnil}$. Let $X = diag(tr^{-1}(A), tr^{-1}(A))$. Then $X \in comm(A)$. Hence, $I_2 + AX \in GL_2(R)$. But $det(I_2 + AX) = tr^{-2}(A)det(A) \in J(R)$, a contradiction. Thus, $E \neq I_2$. Clearly, A is strongly clean. In light of Corollary 1.2.6, $x^2 - tr(A)x + det(A) = 0$ is solvable, as asserted. □

Corollary 6.4.24. *Let R be a commutative local ring. If $\frac{1}{2} \in R$, then $A \in M_2(R)$ is quasipolar if and only if*

(1) $det(A) \in U(R)$; *or*
(2) $det(A), tr(A) \in J(R)$; *or*
(2) $det(A) \in J(R), tr(A) \in U(R)$ *and there exists* $u \in U(R)$ *such that* $tr^2(A) - 4det(A) = u^2$.

Proof. Suppose that $det(A) \in J(R), tr(A) \in U(R)$. If $x^2 - tr(A)x + det(A) = 0$ is solvable, then this equation has a root x_1. Set $x_2 = tr(A) - x_1$. Then $det(A) = x_1(tr(A) - x_1) = x_1x_2 \in J(R)$. Set $u = x_1 - x_2$. Then $u \in U(R)$. Therefore $tr^2(A) - 4det(A) = (x_1 + x_2)^2 - 4x_1x_2 = (x_1 - x_2)^2 = u^2$. Conversely, assume that $tr^2(A) - 4det(A) = u^2$ for a $u \in U(R)$. Then $2^{-1}(tr(A) + u)$ is a root of $x^2 - tr(A)x + det(A) = 0$. In light of Theorem 6.4.23, we complete the proof. □

An element a in a ring R is perfectly clean if there exists an idempotent $e \in comm^2(a)$ such that $a - e \in U(R)$. A ring R is perfectly

clean if every element in R is perfectly clean. Clearly, abelian exchange ring is perfectly clean. In fact, we have {quasipolar rings} $\subsetneq$ {perfectly clean rings} $\subsetneq$ {strongly clean rings}. For instance, $R\big[\mathbb{Q}, \mathbb{Z}_2\big] = \{(x_1, x_2, \ldots, x_n, y, y, \ldots) \mid \text{each } x_i \in \mathbb{Q}, y \in \mathbb{Z}_2, n \geq 1\}$ is perfect clean, while it is not quasipolar.

Theorem 6.4.25. *Let R be a ring. Then the following are equivalent:*

(1) *R is perfectly clean.*
(2) *For any $a \in R$, there exists $x \in comm^2(a)$ such that $x = xax$ and $1 - x \in (1 - a)R \cap R(1 - a)$.*

Proof. (1) $\Rightarrow$ (2) For any $a \in R$, there exists an idempotent $e \in comm^2(a)$ such that $u := a - e \in U(R)$. Set $x = u^{-1}(1 - e)$. Let $y \in comm(a)$. Then $ay = ya$. As $uy = (a - e)y = y(a - e) = yu$, we get $u^{-1}y = yu^{-1}$. Thus, $xy = u^{-1}(1 - e)y = u^{-1}y(1 - e) = yu^{-1}(1 - e) = yx$. This implies that $x \in comm^2(a)$. Further, $xax = u^{-1}(1-e)(u+e)u^{-1}(1-e) = u^{-1}(1-e) = x$. Clearly, $u = (1-e)-(1-a)$, and so $1-u^{-1}(1-e) = u^{-1}(1-a)$. This implies that $1-x \in R(1-a)$. Likewise, $1-x \in (1-a)R$ as $(1-e)u^{-1} = u^{-1}(1-e)$. Therefore $1 - x \in (1 - a)R \cap R(1 - a)$.

(2) $\Rightarrow$ (1) For any $a \in R$, there exists $x \in comm^2(a)$ such that $x = xax$ and $1 - x \in (1 - a)R \cap R(1 - a)$. Write $e = 1 - ax$. If $y \in comm(a)$, then $ay = ya$, and so $axy = ayx = yax$. This shows that $ey = ye$; hence, $e \in comm^2(a)$. In addition, $ex = xe = 0$. Write $1 - x = (1 - a)s = t(1 - a)$ for some $s, t \in R$. Then

$$\begin{aligned}(a - e)(x - es) &= ax - aes + es \\ &= ax + (1 - a)es \\ &= ax + e(1 - a)s \\ &= ax + e(1 - x) \\ &= ax + e \\ &= 1.\end{aligned}$$

Likewise, $(x - te)(a - e) = 1$. Therefore $a - e \in U(R)$, as desired. □

Corollary 6.4.26. *Let R be a ring. Then the following are equivalent:*

(1) *R is perfectly clean.*
(2) *For any $a \in R$, there exists an idempotent $e \in comm^2(a)$ such that $eae \in U(eRe)$ and $(1 - e)(1 - a)(1 - e) \in U\big((1 - e)R(1 - e)\big)$.*

Proof. (1) $\Rightarrow$ (2) For any $a \in R$, it follows from Theorem 6.4.25 that there exists $x \in comm^2(a)$ such that $x = xax$ and $1 - x \in (1 - a)R \cap R(1 - a)$. Write $1 - x = (1 - a)s = t(1 - a)$ for some $s, t \in R$. Set $e = ax$. For any

$y \in comm(a)$, we have $ay = ya$, and so $ey = (ax)y = a(yx) = (ay)x = y(ax) = ye$. Hence, $e^2 = e \in comm^2(a)$. Clearly, $(eae)(exe) = (exe)(eae) = e$; hence, $eae \in U(eRe)$. Furthermore, $1-e = (1-x)+(1-a)x = (1-a)(s+x)$. This shows that $(1-e)(1-a)(1-e)(s+x)(1-e) = 1-e$. Likewise, $(1-e)(t+x)(1-e)(1-a)(1-e) = 1-e$. Therefore $(1-e)(1-a)(1-e) \in U\big((1-e)R(1-e)\big)$.

(2) $\Rightarrow$ (1) For any $a \in R$, we have an idempotent $e \in comm^2(a)$ such that $eae \in U(eRe)$ and $(1-e)(1-a)(1-e) \in U\big((1-e)R(1-e)\big)$. Hence, $a-(1-e) = eae-(1-e)(1-a)(1-e) \in U(R)$. Set $p = 1-e$. Then $a-p \in U(R)$ with $p \in comm^2(a)$, as desired. □

Theorem 6.4.27. *Let R be a ring. Then the following are equivalent:*

(1) *R is perfectly clean.*
(2) *For any $a \in R$, there exists an idempotent $e \in comm^2(a)$ such that $a-e$ and $a+e$ are invertible.*

Proof. (1) $\Rightarrow$ (2) Let $a \in R$. Then $a^2 \in R$ is perfectly clean. Thus, we can find an idempotent $e \in comm^2(a^2)$ such that $a^2 - e \in U(R)$. As $a \in comm(a^2)$, we see that $ae = ea$. Hence, $a^2 - e = (a-e)(a+e)$, and therefore $a-e, a+e \in U(R)$.

(2) $\Rightarrow$ (1) is trivial. □

Corollary 6.4.28. *Let R be perfectly clean. Then for any $A \in M_n(R)$ there exist $U, V \in GL_n(R)$ such that $2A = U + V$.*

Proof. We prove the result by induction on n. For any $a \in R$, there exists an idempotent $e \in comm^2(a)$ such that $u := a-e, v := a+e \in U(R)$, by Lemma 6.4.27. Hence $2a = u+v$, and so the result holds for $n = 1$. Assume that the result holds for $n \le k$ ($k \in \mathbb{N}$). Let $n = k+1$, and let $A \in M_n(R)$. Write $A = \begin{pmatrix} x & \alpha \\ \beta & X \end{pmatrix}$, where $x \in R, \alpha \in M_{1\times k}(R), \beta \in M_{k\times 1}(R)$ and $X \in M_k(R)$. In view of Theorem 6.4.27, we have $u \in U(R)$ such that $2x - u = v \in U(R)$. By hypothesis, we have $U \in GL_k(R)$ such that $2\big(X - 2\beta v^{-1}\alpha\big) - U = V \in GL_k(R)$. Hence

$$2A - \begin{pmatrix} u & 0 \\ 0 & U \end{pmatrix} = \begin{pmatrix} v & 2\alpha \\ 2\beta & V + 4\beta v^{-1}\alpha \end{pmatrix}.$$

It is easy to verify that

$$\begin{pmatrix} v & 2\alpha \\ 2\beta & V + 4\beta v^{-1}\alpha \end{pmatrix} = \begin{pmatrix} 1 & \\ 2\beta v^{-1} & I_k \end{pmatrix} \begin{pmatrix} v & 2\alpha \\ 0 & V \end{pmatrix} \in GL_n(R).$$

By induction, we complete the proof. □

Let R be a quasipolar ring and $\frac{1}{2} \in R$. Then every square matrix over R is the sum of two invertible matrices.

Lemma 6.4.29. *Let R be a ring, and let $u \in U(R)$. Then the following are equivalent:*

(1) *$a \in R$ is perfectly clean.*
(2) *$uau^{-1} \in R$ is perfectly clean.*

Proof. (1) $\Rightarrow$ (2) By hypothesis, there exists an idempotent $e \in comm^2(a)$ such that $a - e \in U(R)$. Hence, $uau^{-1} - ueu^{-1} \in U(R)$. For any $x \in comm(uau^{-1})$, we see that $x(uau^{-1}) = (uau^{-1})x$, and so $(u^{-1}xu)a = a(u^{-1}xu)$. Thus, $(u^{-1}xu)e = e(u^{-1}xu)$. Hence, $x(ueu^{-1}) = (ueu^{-1})x$. We conclude that $ueu^{-1} \in comm^2(uau^{-1})$, as required.

(1) $\Rightarrow$ (2) Applying the preceding discussion to uau^{-1}, we obtain the result. □

We say that a ring is weakly cobleached provided that for any $a \in 1 + J(R), b \in J(R)$, $l_a - r_b$ is injective. Obviously, a ring R is weakly cobleached if and only if for any $a \in 1 + J(R), b \in J(R)$, $l_a - r_b$ and $l_b - r_a$ are injective. For instance, commutative local rings and local rings with nil Jacobson radicals.

Theorem 6.4.30. *Let R be a weakly cobleached local ring. Then the following are equivalent:*

(1) *$M_2(R)$ is perfectly clean.*
(2) *$M_2(R)$ is strongly clean.*
(3) *For any $A \in M_2(R)$, $A \in GL_2(R)$, or $I_2 - A \in GL_2(R)$, or A is similar to a diagonal matrix.*

Proof. (1) $\Rightarrow$ (2) This is trivial.

(2) $\Rightarrow$ (3) is obtained by Theorem 1.2.13.

(3) $\Rightarrow$ (1)If A or $I_2 - A \in GL_2(R)$, then A is perfectly clean. Assume now that A is similar to a diagonal matrix and $A, I_2 - A \notin G_2(R)$. Then A is similar to $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, where $\lambda \in U(R), \mu \in J(R)$. If $\lambda \in 1 + U(R)$, then $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} - I_2 \in GL_2(R)$; hence, it is perfectly clean. In view of Lemma 6.4.29, A is perfectly clean. Thus, we assume that $\lambda \in 1 + J(R)$. We shall suffice

to prove that $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \in GL_2(R)$ is perfectly clean. Clearly, $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \lambda & 0 \\ 0 & \mu - 1 \end{pmatrix}$, where $\begin{pmatrix} \lambda & 0 \\ 0 & \mu - 1 \end{pmatrix} \in GL_2(R)$. For any $\begin{pmatrix} x & s \\ t & y \end{pmatrix} \in comm\left(\begin{pmatrix} \lambda & \\ 0 & \mu \end{pmatrix}\right)$, we see that

$$\lambda s = s\mu \text{ and } \mu t = t\lambda;$$

hence, $s = t = 0$. This implies that

$$\begin{pmatrix} x & s \\ t & y \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & s \\ t & y \end{pmatrix}.$$

Therefore $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in comm^2\left(\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}\right)$, hence the result. □

Theorem 6.4.31. *Let R and S be local rings, let V be an R-S-bimodule, and let $T = \begin{pmatrix} R & V \\ 0 & S \end{pmatrix}$. Then the following are equivalent:*

(1) *T is perfectly clean.*
(2) *T is quasipolar.*
(3) *For any $a \in 1 + J(R), b \in J(R), v \in V$, there exists a unique $x \in V$ such that $ax - xb = v$.*

Proof. (1) ⇒ (3) Let $a \in 1 + J(R), b \in J(R)$. Set $A = \begin{pmatrix} a & -v \\ 0 & b \end{pmatrix}$. By hypothesis, we can find an idempotent $E \in comm^2(A)$ such that $A - E \in T$ is invertible. Clearly, $E = \begin{pmatrix} 0 & x \\ 0 & 1 \end{pmatrix}$ for some $x \in V$. Thus, $ax - xb = v$.

Suppose that $ay - yb = v$ for $y \in R$. Then

$$A \begin{pmatrix} 0 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & y \\ 0 & 1 \end{pmatrix} A,$$

and so $\begin{pmatrix} 0 & y \\ 0 & 1 \end{pmatrix} \in comm(A)$. This implies that

$$E \begin{pmatrix} 0 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & y \\ 0 & 1 \end{pmatrix} E;$$

hence, $x = y$. Therefore there exists a unique $x \in V$ such that $ax - xb = v$, as desired.

(3) $\Rightarrow$ (2) Let $A = \begin{pmatrix} a & v \\ 0 & b \end{pmatrix} \in T$. If $a, b \in J(R)$, then $A - I_2 \in U(T)$ and $A \in J(T) \subseteq T^{qnil}$; hence, A is quasipolar. If $a, b \in U(R)$, then $A - 0 \in U(T)$; hence, A is quasipolar.

If $a \in 1 + J(R), b \in J(R)$, then we can find $t \in V$ such that $at - tb = -v$. Let $\begin{pmatrix} x & s \\ 0 & y \end{pmatrix} \in comm(A)$. Then

$$A \begin{pmatrix} x & s \\ 0 & y \end{pmatrix} = \begin{pmatrix} x & s \\ 0 & y \end{pmatrix} A,$$

and so

$$ax = xa, by = yb, as - sb = xv - vy.$$

Hence, we check that

$$\begin{aligned} a(xt - ty + s) - (xt - ty + s)b &= x(at - bt) - (at - tb)y + (as - sb) \\ &= -xv + vy + (as - sb) \\ &= 0. \end{aligned}$$

By hypothesis, $xt - ty = -s$, and so

$$\begin{pmatrix} x & s \\ 0 & y \end{pmatrix} \begin{pmatrix} 0 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & ty \\ 0 & y \end{pmatrix} = \begin{pmatrix} 0 & xt + s \\ 0 & y \end{pmatrix} = \begin{pmatrix} x & s \\ 0 & y \end{pmatrix} \begin{pmatrix} 0 & t \\ 0 & 1 \end{pmatrix}.$$

We infer that

$$\begin{pmatrix} 0 & t \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 0 & t \\ 0 & 1 \end{pmatrix} \in comm^2(A).$$

Furthermore, $A - \begin{pmatrix} 0 & t \\ 0 & 1 \end{pmatrix} \in U(T)$ and $A \begin{pmatrix} 0 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & at + v \\ 0 & b \end{pmatrix} \in J(T) \subseteq T^{qnil}$. Therefore A is quasipolar.

If $a \in J(R), b \in 1 + J(R)$, then A is perfectly clean, as in the preceding discussion.

(2) $\Rightarrow$ (1) This is obvious. □

For instance, if R is a commutative local ring or a local ring with nil Jacobson radical, then $T_2(R)$ is quasipolar. From this observation, we see that $T_2(\mathbb{Z}_{(2)})$ is quasipolar, but it is not strongly π-regular.

Notes. Theorem 6.1.7 is due to Burgess and Menal [86, Theorem 1.3]. Theorem 6.1.10 was proved by Badawi [43, Theorem 3] and Vasconcelos [523, Theorem]. Theorem 6.1.13 is due to Nicholson [432]. Theorem 6.1.19

is due to Borooah et al. [78, Proposition 38]. Theorem 6.1.36 was proved by Ara [25, Theorem 4]. Theorem 6.3.12 is due to Wang and Chen [536, Theorem 3.2]. Theorem 6.4.8 was proved in [369, Theorem 4.2]. Theorem 6.4.16 is due to [217, Theorem 2]. Theorem 6.4.23 were first observed by Cui and Chen [214].

This page intentionally left blank

Chapter 7

Clean Properties

An element of a ring is called clean if it is the sum of an idempotent and a unit. A ring R is called clean if every element of R is clean. Clearly, every strongly clean ring is clean. This concept was first introduced by Nicholson in 1977 (see [430]). Over the last ten to fifteen years there has been an explosion of interest in this class of rings as well as the many generalizations and variations. This chapter will outline various properties of cleanness.

7.1 Clean Rings

The goal of this section is to investigate some elementary properties of clean rings. We begin with

Proposition 7.1.1. *Let R_i be a ring for each $i \in I$. Then $\bigoplus_{i\in I} R_i$ is clean if and only if each R_i is clean.*

Proof. This is an easy computation. □

Call an element $v \in R$ is a square root of 1 provided that $v^2 = 1$.

Proposition 7.1.2. *Let R be a ring in which 2 is invertible. Then R is clean if and only if every element in R is the sum of a unit and a square root of 1.*

Proof. $\Longrightarrow$ Let $a \in R$. Then there exist an idempotent $e \in R$ and a unit $u \in R$ such that $\frac{a+1}{2} = e + u$. Hence, $a = 2u + (2e - 1)$. As $(2e-1)^2 = 1$, we see that a is the sum of a unit and a square root of 1.

$\Longleftarrow$ Let $a \in R$. Then we have $2a - 1 = u + v$ where $u \in U(R), v^2 = 1$. Hence, $a = \frac{u}{2} + \frac{v+1}{2}$. Clearly, $\left(\frac{v+1}{2}\right)^2 = \frac{v+1}{2}$, as desired. □

The following is a generalization of [487, Proposition 3.1].

Proposition 7.1.3. *Let I be an ideal of a clean ring R. If R/I is connected, then units in R/I lift to units in R.*

Proof. Take $\overline{a} \in U(R/I)$. Then we have $\overline{b} \in R/I$ such that $ab \equiv 1, ba \equiv 1 \pmod{I}$. Since R is clean, we can find idempotents $e, f \in R$ and $u, v \in U(R)$ such that $a = e + u$ and $b = f + v$. By hypothesis, $\{\overline{e}, \overline{f}\} \subseteq \{\overline{0}, \overline{1}\}$. If $e \equiv 0 \pmod{I}$, then $a \equiv u \pmod{I}$. If $f \equiv 0 \pmod{I}$, then $b \equiv v \pmod{I}$. Hence, $av \equiv 1, va \equiv 1 \pmod{I}$. This shows that $a - v^{-1} = (av-1)v^{-1} \in I$, i.e., $a \equiv v^{-1} \pmod{I}$. If $e \equiv f \equiv 1 \pmod{I}$, then we check that

$$\begin{aligned} 1 &\equiv ab \\ &\equiv a(1+v) \\ &\equiv (1+u) + av \pmod{I}; \end{aligned}$$

hence, $a(-vu^{-1}) \equiv 1 \pmod{I}$. Thus, $a + uv^{-1} = \big(a(-vu^{-1}) - 1\big)(-uv^{-1}) \in I$. Therefore $a \equiv -uv^{-1} \pmod{I}$, as required. □

Let I be an ideal of a clean ring R. Analogously, for any $\overline{a} \in \overline{1} + J(R/I)$, there exists $u \in U(R/I)$ such that $a \equiv u \pmod{I}$.

Clearly, every homomorphic image of a clean ring is clean. We now consider when a ring is clean in terms of various homomorphic images.

Lemma 7.1.4. *Let P be an ideal generated by central idempotents of a ring R. If P is not a Pierce ideal, then there exists a central idempotent $e \in R$ such that $P + eR$ and $P + (1-e)R$ are ideals properly containing the ideal P.*

Proof. Since P is generated by central idempotents and P is not a Pierce ideal of R, there is a central idempotent $e \in R - P$ such that $P + eR$ is an ideal of R. This shows that $P \subsetneq P + eR$. If $P + (1-e)R = R$, then $p + (1-e)r = 1$ for some $p \in P, r \in R$. Hence, $e = ep \in P$, an absurd. We infer that $P + (1-e)R$ is an ideal of R. If $P = P + (1-e)R$, then

$1-e \in P$. This implies that $1 = e + (1-e) \in P + eR$, a contradiction. So the lemma is true. □

Lemma 7.1.5. *Let $a_1, \ldots, a_m \in R$, and let $f_1, \ldots, f_k \in \mathbb{Z}[x_1, \ldots, x_m, y_1, \ldots, y_n]$. Then the following are equivalent*:

(1) *There exist $b_1, \ldots, b_n \in R$ such that $f_i(a_1, \ldots, a_m, b_1, \ldots, b_n) = 0$ for $i = 1, \ldots, k$.*
(2) *For any $P \in Pier(R)$, there exist $\overline{b_1}, \ldots, \overline{b_n} \in R/P$ such that $f_i\big(\varphi_P(a_1), \ldots, \varphi_P(a_m), \overline{b_1}, \ldots, \overline{b_n}\big) = 0$ for $i = 1, \ldots, k$.*

Proof. (1) ⇒ (2) This is trivial.

(2) ⇒ (1) We say that R/P is special if there exist $\overline{b_1}, \ldots, \overline{b_n} \in R/P$ such that $f_i\big(\varphi_P(a_1), \ldots, \varphi_P(a_m), \overline{b_1}, \ldots, \overline{b_n}\big) = 0$ in R/P for $i = 1, \ldots, k$. Here, $\varphi_P : R \to R/P$ is the natural epimorphism. Assume the contrary. Let Ω be the set of all ideals A generated by central idempotents of R such that R/A is not special. Then $0 \in \Omega$. One easily checks that the union of every ascending chain of ideals from Ω is contained in Ω. Thus, Ω is an inductive set. By using Zorn's Lemma, Ω contains a maximal element Q. It will suffice to prove that Q is a Pierce ideal of R. Assume the contrary. By Lemma 7.1.4, there exists a central idempotent $e \in R$ such that $P \subsetneq Q + eR, Q + (1-e)R$, and that

$$\begin{gathered} R/Q \cong R/\big(Q + eR\big) \oplus R/\big(Q + (1-e)R\big); \\ r + Q \mapsto \big(r + (Q + eR), r + (Q + (1-e)R)\big). \end{gathered}$$

Clearly, $Q + eR, Q + (1-e)R$ are not in Ω, and so $R/\big(Q + eR\big)$ and $R/\big(Q + (1-e)R\big)$ are both special. This implies that R/Q is special, a contradiction. □

Theorem 7.1.6. *Let R be a ring. Then the following are equivalent*:

(1) *R is clean.*
(2) *R/P is clean for all $P \in Pier(R)$.*

Proof. (1) ⇒ (2) This is obvious.

(2) ⇒ (1) Let $P \in Pier(R)$, and let $\varphi_P : R \to R/P$ be the natural epimorphism. We consider the following polynomials in noncommutative indeterminates X, Y_1, Y_2, Y_3: $f_1(X, Y_1, Y_2, Y_3) = X - Y_1 - Y_2$, $f_2(X, Y_1, Y_2, Y_3) = Y_1 - Y_1^2$, $f_3(X, Y_1, Y_2, Y_3) = 1 - Y_2Y_3$, $f_4(X, Y_1, Y_2, Y_3) = 1 - Y_3Y_2$. Let $a \in R$. Since R/P is clean, we can find $\overline{e}, \overline{s}, \overline{t} \in R/P$ such that $f_i\big(\varphi_P(a), \overline{e}, \overline{s}, \overline{t}\big) = 0$ for each i. In view of Lemma 7.1.5, there are $e, u, v \in R$ such that each

$f_i(a,e,u,v)=0$. We infer that $e=e^2$ and $u=v^{-1}\in U(R)$, and that $a=e+u$, as required. □

Corollary 7.1.7. *Let R be abelian. Then the following are equivalent*:

(1) *R is clean.*
(2) *R/P is local for all $P\in Pier(R)$.*

Proof. (1) $\Rightarrow$ (2) Let $P\in Pier(R)$, and let $\overline{a}\in R/P$. Then there exists an idempotent $e\in R$ such that $a-e\in U(R)$. As R is abelian, we see that either $eR+P=R$ or $(1-e)R+P=R$. This shows that $\overline{e}\big(R/P\big)=R/P$ or $\overline{1-e}\big(R/P\big)=R/P$. It follows that $\overline{e}=\overline{0}$ or $\overline{1}$, and so $\overline{a}$ or $\overline{1-a}$ is invertible in R/P. Thus, R/P is a local ring.

(2) $\Rightarrow$ (1) This is obvious by Theorem 7.1.6, as every local ring is clean. □

Corollary 7.1.8. *Let R be abelian. Then the following are equivalent*:

(1) *R is clean.*
(2) *$aR+bR=R$ with $a,b\in R$ implies that there exists an idempotent $e\in R$ such that $a+be\in U(R)$.*

Proof. (1) $\Rightarrow$ (2) Assume that $aR+bR=R$ with $a,b\in R$. Let $f(X_1,X_2,Y_1,Y_2)=1-(X_1+X_2Y_1)Y_2, g(X_1,X_2,Y_1,Y_2)=1-Y_2(X_1+X_2Y_1)$ and $h(Y_1)=Y_1-Y_1^2$ be the polynomials in noncommutative indeterminate X_1,X_2,Y_1 and Y_2. Let $P\in Pier(R)$. Then R/P is a local ring by Corollary 7.1.7. If $\overline{a}\notin J(R/P)$, then $\overline{a}+\overline{b}\times\overline{0}\in U(R/P)$. If $\overline{a}\in J(R/P)$, then $\overline{a}+\overline{b}\times\overline{1}\in U(R/P)$. In any case, we can find an idempotent $\overline{t}\in R/P$ such that $\overline{a}+\overline{bt}\in U(R/P)$. Thus, we have $\overline{w}\in R/P$ such that $(\overline{a}+\overline{bt})\overline{w}=\overline{1}=\overline{w}(\overline{a}+\overline{bt})$. Hence, $f(\overline{a},\overline{b},\overline{t},\overline{w})=\overline{0}, g(\overline{a},\overline{b},\overline{t},\overline{w})=\overline{0}$ and $h(\overline{t})=\overline{0}$. In view of Lemma 7.1.5, there exist some $e,d\in R$ such that $f(a,b,e,d)=0, g(a,b,e,d)=0$ and $h(e)=0$. Therefore $a+be\in U(R)$ for a central idempotent $e\in R$.

(2) $\Rightarrow$ (1) Let $a\in R$. Then $aR+(-1)R=R$. By hypothesis, there exists an idempotent $e\in R$ such that $a-e\in U(R)$, as asserted. □

Analogously to Theorem 7.1.6, a ring R is clean if and only if so is its every indecomposable homomorphic image.

Lemma 7.1.9. *Every clean ring is an exchange ring.*

Proof. Let R be clean, and let $a \in R$. Then there exists an idempotent $e \in R$ such that $u := a - e \in U(R)$. It is easy to verify that $u\big(a - u^{-1}(1 - e)u\big) = ue + u^2 - u + eu = a^2 - a$. Set $f = u^{-1}(1 - e)u$. Then $f = f^2 \in R$. Further, $a - f = u^{-1}(a^2 - a)$, and so $f \in Ra$ and $1 - f \in R(1 - a)$. Therefore R is an exchange ring, in terms of Proposition 1.1.7. □

Theorem 7.1.10. *Let R be a ring, and let $I \subseteq J(R)$. Then R is clean if and only if*

(1) *R/I is clean;*
(2) *Every idempotent lifts modulo I.*

Proof. (1) $\Rightarrow$ (2) Clearly, R/I is clean. By virtue of Lemma 7.1.9, R is an exchange ring, and then proving (2) by Lemma 4.4.5.

(2) $\Rightarrow$ (1) Let $a \in R$. Then we can find an idempotent $\overline{e} \in R/I$ such that $\overline{a - e} \in U\big(R/I\big)$. By (2), we may assume that $e^2 = e \in R$. As $I \subseteq J(R)$, we see that units lift modulo I, and then we have $u \in U(R)$ such that $\overline{a - e} = \overline{u}$. Write $a - e = u + r$ with $r \in I \subseteq J(R)$. Therefore $a - e \in U(R)$, as asserted. □

We note that there exists a ring R and an ideal I of R such that R/I is clean and idempotents lift modulo I, but R is not clean (see [380]).

Corollary 7.1.11. *A ring R is clean if and only if so is $R/P(R)$.*

Proof. By Lemma 4.3.1, $P(R)$ is nil, and so every idempotent lifts modulo $P(R)$. Clearly, $P(R) \subseteq J(R)$. This completes the proof, by Theorem 7.1.10. □

The converse of Lemma 7.1.9 is not true. Here, we include a counterexample of Bergman [300, Example 1].

Example 7.1.12. Let F be a field, and let $A = F[[x]]$. Let K be the field of fractions of A. Define a ring

$$R = \{r \in End_F(A) \mid \exists q \in K, n \in \mathbb{N} \text{ with } r(a) = qa \text{ for all } a \in (x^n)\}.$$

Then R is an exchange ring which is not clean.

An element a in a commutative ring R is adequate if for any $b \in R$ there exist some $r, s \in R$ such that (1) $a = rs$; (2) $(r, b) = 1$; (3) $(s', b) \neq 1$ for each noninvertible divisor s' of s.

Lemma 7.1.13. *Let R be a commutative Bézout ring. If $a \in R$ is adequate, then $\overline{0} \in R/aR$ is adequate.*

Proof. Let $\overline{b} \in R/aR$. Then there exist $r, s \in R$ such that $a = rs, (r, b) = 1$ and $(s', b) \neq 1$ for any noninvertible divisor s' of s. Hence, $\overline{a} = \overline{rs}$ and $\big(\overline{r}, \overline{b}\big) = \overline{1}$. Let $\overline{t} \in R/aR$ be a noninvertible divisor of $\overline{s}$. Then $t \mid (s + ak)$ for some $k \in R$. Write $s + ak = t\beta$ for some $\beta \in R$. Then $s + rsk = t\beta$, and so $s(1 + rk) = t\beta$. If $(s, t) = 1$, then $t \mid (1 + rk)$. This implies that $(r, t) = 1$; hence, $(rs, t) = 1$. It follows that $(a, t) = 1$, and then $\overline{t} \in R/aR$ is invertible, a contradiction. Therefore $(s, t) \neq 1$. Since R is a commutative Bézout ring, we have a noninvertible $u \in R$ such that $sR + tR = uR$. We infer that u is a noninvertible divisor of s. Hence, $(u, b) \neq 1$. This proves that $\big(\overline{u}, \overline{b}\big) \neq \overline{1}$; otherwise, there exist $x, y, z \in R$ such that $ux + by = 1 + az$. Hence, $ux + by = 1 + rsz = 1 + ucrz$ for a $c \in R$, and so $u(x - crz) + by = 1$, a contradiction. Therefore $\big(\overline{t}, \overline{b}\big) \neq \overline{1}$, as desired. □

Theorem 7.1.14. *Let R be a commutative Bézout ring. If $a \in R$ is adequate, then R/aR is clean.*

Proof. Set $S = R/aR$. Then S is a commutative Bézout ring. By virtue of Lemma 7.1.13, $\overline{0} \in S$ is adequate. Let $x \in S$. Then we have some $r, s \in S$ such that $\overline{0} = rs$, where $(r, x) = \overline{1}$ and $(s', x) \neq \overline{1}$ for any noninvertible divisor s' of s. We claim that $(r, s) = \overline{1}$. If not, $rS + sS = hS \neq S$. Thus, $h \in S$ is a noninvertible divisor of s; hence that $(h, x) \neq \overline{1}$. But h is a divisor of r, we get $(h, x) = \overline{1}$, which is impossible. Write $rc + sd = \overline{1}$ in S. Then $(rc)^2 = rc\big(rc + sd\big) = rc$. Set $e = rc$. Then $e \in S$ is an idempotent.

Claim I. $(x - e)S + rS = S$. If not, $(x - e)S + rS = tS \neq S$. Then $rS \subseteq tS$, and so $xS \subseteq tS$. This implies that $rS + xS \subseteq tS \neq S$, a contradiction.

Claim II. $(x - e)S + sS = S$. If not, $(x - e)S + sS = tS \neq S$. Then t is a noninvertible divisor of s, and so $(t, x) \neq \overline{1}$. But $eS + sS = S$, whence, $eS + tS = S$. Write $x - e = tw$ with $w \in S$. Then $e = x - tw$, and so $eS + tS \subseteq tS + xS \neq S$, which is impossible.

Finally, $(x - e)S + rsS = S$, and therefore $x - e \in U(S)$, hence the result. □

A commutative Bézout ring in which every element is adequate is called zero adequate. For instances, regular rings and valuation rings. As an immediate consequence of Theorem 7.1.14, we now derive

Corollary 7.1.15. *Every zero adequate ring is clean.*

A ring R has artinian primitive factors provided that R/P is right artinian for all right primitive ideals P of R.

Theorem 7.1.16. *Every exchange ring with artinian primitive factors is clean.*

Proof. Let R be an exchange ring with artinian primitive factors. Assume that R is not clean. Then we can find some $a \in R$ such that for any $e^2 = e \in R$ and $u \in U(R)$, we have $a \neq e + u$. Let Ω be the set of all ideals A of R such that for any $\overline{e}^2 = \overline{e} \in R/A$ and $\overline{u} \in U(R/A)$, we have $\overline{a} \neq \overline{e+u}$. Obviously, $\Omega \neq \emptyset$.

Given any ascending chain $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_k \subseteq \cdots$ in Ω, we set $M = \bigcup_{n=1}^{\infty} A_n$. Then M is an ideal of R. If M is not in Ω, then there exist $\overline{g} = \overline{g^2} \in R/M, \overline{w} \in U(R/M)$ such that $\overline{a} = \overline{g+w}$. Hence $a - (g+w), g - g^2, ws - 1, sw - 1 \in M$ for some $s \in R$, and so we can find positive integers $n_i (1 \leq i \leq 4)$ such that $a - (g+w) \in A_{n_1}, g - g^2 \in A_{n_2}, ws - 1 \in A_{n_3}, sw - 1 \in A_{n_4}$. Let $p = \max\{n_1, n_2, n_3, n_4\}$. Then $a - (g+w), g - g^2, ws - 1, sw - 1 \in A_p$. This contradicts the choice of A_p, so $M \in \Omega$. Thus, Ω is inductive. By using Zorn's Lemma, we have an ideal Q of R such that Q is maximal in Ω. Let $S = R/Q$. The maximality of $Q \in \Omega$ implies that S is indecomposable as a ring.

If $J(S) \neq 0$, we may assume that $J(S) = N/Q$ with $Q \subsetneqq N$. By the maximality of Q, $a \in R$ is the sum of an idempotent and a unit that commute modulo N. From $S/J(S) \cong R/N$, we have $\overline{a} + J(S) = \big(\overline{f} + J(S)\big) + \big(\overline{v} + J(S)\big)$, where $\overline{f} + J(S) = \big(\overline{f} + J(S)\big)^2 \in S/J(S), \overline{v} + J(S) \in U\big(S/J(S)\big)$. Since $S/J(S)$ is an exchange ring, we may assume that $\overline{f} = \overline{f^2} \in S$. On the other hand, units lift modulo the Jacobson radical of S. We may assume that $\overline{v} \in U(S)$. Thus $a + Q = (f + Q) + \big((v + Q) + (r + Q)\big)$ for some $r + Q \in J(S)$. Clearly, $(v + Q) + (r + Q) \in U(S)$, and it yields a contradiction. This implies that $J(S) = 0$, so S is an indecomposable ring with $J(S) = 0$.

Since R is an exchange ring with artinian primitive factors, by virtue of [563, Lemma 3.7], S is simple artinian. That is, $S \cong M_m(D)$ for some $m \in \mathbb{N}$. In view of Theorem 1.1.2, S is strongly clean, and so it is clean. Thus, $a \in R$ can be seen as the sum of an idempotent and a unit modulo Q. This gives a contradiction, whence R is a clean ring. □

Corollary 7.1.17. *Every exchange ring of bounded index is clean.*

Proof. Let R be an exchange ring of bounded index. Let P be a primitive ideal of R, let $S = R/P$ and let n be the bounded index of R. Then S is an exchange ring as well. Suppose that $e_1, e_2, \ldots, e_{n+1}$ are orthogonal idempotents in S. In view of Lemma 3.1.10, there exist orthogonal idempotents $f_1, f_2, \ldots, f_{n+1}$ in R such that each $e_i = f_i + P$. Without loss of generality we may assume that each $f_i \notin P$. Since P is primitive, it is prime, so there exist $x_1, x_2, \ldots, x_n \in R$ such that $f_1x_1f_2, f_1x_1f_2x_2f_3, \ldots, f_1x_1f_2x_2f_3 \cdots f_nx_nf_{n+1} \notin P$. Put $x = f_1x_1f_2 + f_2x_2f_3 + \cdots + f_nx_nf_{n+1}$. Then $x^{n+1} = 0$; hence, $x^n = 0$. But $x^n = f_1x_1f_2x_2f_3 \cdots f_nx_nf_{n+1} \notin P$, a contradiction. Thus S is an exchange ring which contains no infinite set of orthogonal idempotents. According to [107, Corollary 12], S is semiperfect; whence, $S/J(S)$ is artinian. As P is primitive, $J(S) = 0$, and so S is artinian. Accordingly, R is an exchange ring with artinian primitive factors, to which we can apply Theorem 7.1.16. □

We come now to consider clean property of related rings. We have

Proposition 7.1.18. *The ring $R[[x]]$ is clean if and only if R is clean.*

Proof. Suppose that $R[[x]]$ is clean. Let $\varepsilon : R[[x]] \to R, f(x) \mapsto f(0)$. Then $R \cong R[[x]]/Ker(\varepsilon)$. Hence, R is clean.

Conversely, assume that R is clean. For any $f(x) \in R[[x]]$, we can find an idempotent $e \in R$ such that $f(0) - e \in U(R)$. Hence, $f(x) - e = \big(f(0) - e\big) + xg(x) \in U\big(R[[x]]\big)$ for some $g(x) \in R[[x]]$, which completes the proof. □

But we note that $R[x]$ is not clean for any commutative ring R. If $x \in R[x]$ is clean, we can find an idempotent $e \in R[x]$ such that $x - e \in U\big(R[x]\big)$. As R is commutative, we see that $e \in R$. This is imposable, and we are done.

Theorem 7.1.19. *Let T be the ring of a Morita context $(A, B, M, N, \psi, \varphi)$. If A and B are clean, then so is T.*

Proof. Let $\begin{pmatrix} a & n \\ m & b \end{pmatrix} \in T$. Then we have idempotents $e \in A$ and $f \in B$ such that $a - e = u \in U(A)$ and $\big(b - \varphi(mu^{-1} \otimes n)\big) - f = v \in U(B)$. Hence

$$\begin{pmatrix} a & n \\ m & b \end{pmatrix} - \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} = \begin{pmatrix} u & n \\ m & b - f \end{pmatrix}.$$

One easily checks that

$$\begin{pmatrix} u & n \\ m & b-f \end{pmatrix} = \begin{pmatrix} 1_A & 0 \\ mu^{-1} & 1_B \end{pmatrix} \begin{pmatrix} u & n \\ 0 & v \end{pmatrix} \in U(T).$$

This completes the proof. □

Corollary 7.1.20. *Let R be a clean ring. Then $M_n(R)$ is clean for all $n \in \mathbb{N}$.*

Proof. The result holds for $n = 1$. Assume that the result holds for $n \leq k$ $(k \geq 1)$. Let $n = k + 1$. Then $M_n(R) \cong \begin{pmatrix} R & M_{12} \\ M_{21} & M_k(R) \end{pmatrix}$, where $M_{12} = \{(a_2, \ldots, a_{k+1}) \mid a_2, \ldots, a_{k+1} \in R\}$ and $M_{21} = \left\{ \begin{pmatrix} b_2 \\ \vdots \\ b_{k+1} \end{pmatrix} \mid b_2, \ldots, b_{k+1} \in R \right\}$. By virtue of Theorem 7.1.19, $M_{k+1}(R)$ is clean, and therefore the result follows by induction. □

Corollary 7.1.21. *Let R be a clean ring. Then $T_n(R)$ is clean for all $n \in \mathbb{N}$.*

Proof. If $n = 1$, then the result follows. Suppose the result holds for $n \geq 1$. Then $T_n(R)$ is clean. Set $M = \begin{pmatrix} R \\ \vdots \\ R \end{pmatrix}_{1 \times n}$. In view of Theorem 7.1.19, $\begin{pmatrix} R & 0 \\ M & T_n(R) \end{pmatrix}_{(n+1)\times(n+1)}$ is clean. That is, $T_{n+1}(R)$ is clean. By induction, we obtain the result. □

Corollary 7.1.22. *Let $e_1, \ldots, e_n$ be orthogonal idempotents of R. If $1 = e_1 + \cdots + e_n = 1$ and each $e_i R e_i$ is clean, then R is clean.*

Proof. It is easy to verify that $R \cong \begin{pmatrix} e_1Re_1 & \cdots & e_1Re_n \\ \vdots & \ddots & \vdots \\ e_nRe_1 & \cdots & e_nRe_n \end{pmatrix}$. By Theorem 7.1.19 and induction, we are through. □

Let $M_1, \ldots, M_n$ be right R-modules. If $End_R(M_1), \ldots, End_R(M_n)$ are clean, then so is $End_R(M_1 \oplus \cdots \oplus M_n)$. Let $M = M_1 \oplus \cdots \oplus M_n$, and let e_i be the element of $End_R(M)$ that projects onto the ith coordinate.

Then $End_R(M)$ has a set $\{e_1, \ldots, e_n\}$ of orthogonal idempotents such that $e_1 + \cdots + e_n = 1_M$. Since $e_i End_R(M) e_i \cong End_R(e_i M) \cong End_R(M_i)$ for all i, each $e_i End_R(M) e_i$ are clean. Therefore we are through, in terms of Corollary 7.1.22.

The classes of the following rings are to be clean: division rings; local rings; semisimple rings; left (right) artinian rings; finite rings; semiprimary rings (i.e., semilocal rings with nilpotent Jacobson radicals), left (right) perfect rings (i.e., semilocal rings with left (right) T-nilpotent Jacobson radicals), semiperfect rings, Boolean rings, etc. In fact, clean rings are enriched. Burgess and Raphael proved that every ring can be embedded in a clean ring as an essential ring extension. We refer the reader to [89] for details. Though the matrix ring of a clean ring is clean, it is hard to determine a single matrix over a ring is clean. We now consider a criterion to a kind of matrices over commutative rings. $A \in M_n(R)$ is e-clean if $A - E \in GL_n(R)$ and $det(E) = e$ for some $E^2 = E \in M_n(R)$. We have

Lemma 7.1.23. *Let R be commutative, $E = (e_{ij}) \in M_n(R)$ be an idempotent, and let $e = det(E)$. If $(e_{i1}, \ldots, e_{in}) = 1$, then $ee_{ii} = e, ee_{ij} = 0\,(j \neq i)$.*

Proof. Suppose that there are some x_i such that $e_{n1}x_1 + \cdots + e_{nn}x_n = 1$. Let $f = 1 - e$. Over the factor ring R/fR, $\overline{E}$ has determinant $\overline{1}$. As $\overline{E} = \overline{E}^2$, we get $\overline{E} = \overline{I_n}$. Hence, $e_{ni} \in fR$ for any $i(\neq n)$, and so $ee_{nj} = 0\,(j \neq n)$.

Obviously, $e_{nn}x_n \equiv 1 \left(mod\, e_{n1}R + \cdots + e_{n(n-1)}R\right)$. Thus,

$$\overline{E} = \begin{pmatrix} \overline{e_{11}} & \overline{e_{12}} & \cdots & \overline{e_{1n}} \\ \overline{e_{21}} & \overline{e_{22}} & \cdots & \overline{e_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{0} & \overline{0} & \cdots & \overline{e_{nn}} \end{pmatrix} \left(mod\, e_{n1}R + \cdots + e_{n(n-1)}R\right).$$

Since $E = E^2$, we see that $\overline{e_{nn}} = \overline{e_{nn}}^2$. This implies that $\overline{e_{nn}} = \overline{1}$, i.e., $1 - e_{nn} \in e_{n1}R + \cdots + e_{n(n-1)}R$. Hence, we get $e = ee_{nn}$. The general case is proved in a similar fashion. □

Theorem 7.1.24. *Let R be commutative, let $a_1, \ldots, a_n \in R$, and let $e \in R$ be an idempotent. Then* $\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in M_n(R)$ *is e-clean if and only if the following hold:*

(1) *There exist* $x_1, \ldots, x_n \in R$ *such that* $a_1x_1 + \cdots + a_nx_n \in R$ *is e-clean;*
(2) $ex_2 = \cdots = ex_n = 0$*;*
(3) $x_1 \equiv 1 \left(mod\, x_2R + \cdots + x_nR\right)$.

Proof. Let A be the $n \times n$ matrix whose first row is $(a_1, a_2, \ldots, a_n)$, and zeros elsewhere. Suppose that A is e-clean. Then we have an idempotent matrix $E = (e_{ij}) \in M_n(R)$ and $U = (u_{ij}) \in GL_n(R)$ such that $A = E + U \text{and} det(E) = e$. Thus,

$$\begin{pmatrix} a_1 - e_{11} & a_2 - e_{12} & \cdots & a_n - e_{1n} \\ -e_{21} & -e_{22} & \cdots & -e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -e_{n1} & -e_{n2} & \cdots & -e_{nn} \end{pmatrix} = U.$$

This implies that

$$\begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ -e_{21} & -e_{22} & \cdots & -e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -e_{n1} & -e_{n2} & \cdots & -e_{nn} \end{vmatrix} + (-1)^n det(E) = det(U).$$

Thus, $a_1A_{11} + a_2A_{12} + \cdots + a_nA_{1n} = e + (-1)^{n+1}u$, where $u = detU \in U(R)$ and each A_{1i} is the algebraic complement of E corresponding to $e_{1i} (1 \leq i \leq n)$. Let $x_1 = A_{11}, x_2 = A_{12}, \ldots, x_n = A_{1n}$. Then $a_1x_1 + a_2x_2 + \cdots + a_nx_n = e + (-1)^{n+1}u$ is e-clean. Since $E^2 = E \in M_n(R)$ and $det(E) = e$, it follows by Lemma 7.1.23 that $ee_{ii} = e, ee_{ij} = 0 \, (1 \leq i \neq j \leq n)$. This implies that $eA_{12} = \cdots = eA_{1n} = 0$; hence, $ex_2 = \cdots = ex_n = 0$.

Clearly, $e_{21}A_{11} + e_{22}A_{12} + \cdots + e_{2n}A_{1n} = 0$, and so

$$e_{21}A_{11} \equiv 0 \left(mod\, x_2R + \cdots + x_nR\right).$$

On the other hand, $u_{11}A_{11} + u_{12}A_{12} + \cdots + u_{1n}A_{1n} = (-1)^{n+1}u$, and then

$$u_{11}A_{11} \equiv (-1)^{n+1}u \left(mod\, x_2R + \cdots + x_nR\right).$$

As $u \in U(R)$, we have

$$e_{21} \equiv 0 \left(mod\, x_2R + \cdots + x_nR\right).$$

Likewise,

$$e_{31}, \ldots, e_{n1} \equiv 0 \left(mod\, x_2R + \cdots + x_nR\right).$$

Since $E^2 = E$, we see that

$$\begin{pmatrix} e_{11} & e_{12} & \cdots & e_{1n} \\ 0 & e_{22} & \cdots & e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & e_{n2} & \cdots & e_{nn} \end{pmatrix} \begin{pmatrix} e_{11} & e_{12} & \cdots & e_{1n} \\ 0 & e_{22} & \cdots & e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & e_{n2} & \cdots & e_{nn} \end{pmatrix}$$
$$\equiv \begin{pmatrix} e_{11} & e_{12} & \cdots & e_{1n} \\ 0 & e_{22} & \cdots & e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & e_{n2} & \cdots & e_{nn} \end{pmatrix} \big(mod\, x_2R + \cdots + x_nR\big).$$

This implies that

$$\begin{pmatrix} e_{22} & \cdots & e_{2n} \\ \vdots & \ddots & \vdots \\ e_{n2} & \cdots & e_{nn} \end{pmatrix} \begin{pmatrix} e_{22} & \cdots & e_{2n} \\ \vdots & \ddots & \vdots \\ e_{n2} & \cdots & e_{nn} \end{pmatrix}$$
$$\equiv \begin{pmatrix} e_{22} & \cdots & e_{2n} \\ \vdots & \ddots & \vdots \\ e_{n2} & \cdots & e_{nn} \end{pmatrix} \big(mod\, x_2R + \cdots + x_nR\big).$$

Accordingly,

$$A_{11}^2 = A_{11}\,\big(mod\, x_2R + \cdots + x_nR\big).$$

Hence,

$$u_{11}A_{11}^2 \equiv u_{11}A_{11}\,\big(mod\, x_2R + \cdots + x_nR\big).$$

Therefore

$$A_{11} \equiv 1\,\big(mod\, x_2R + \cdots + x_nR\big),$$

i.e.,

$$x_1 \equiv 1\,\big(mod\, x_2R + \cdots + x_nR\big).$$

Conversely, assume that (1)−(3) hold. By (1), we have some $x_1, \ldots, x_n \in R$ such that $a_1x_1 + \cdots + a_ix_i + \cdots + a_nx_n$ is e-clean. Let $c_1 = x_1$ and $c_i = -x_i\,(2 \le i \le n)$. Then $a_1c_1 - \cdots - a_ic_i - \cdots - a_nc_n$ is e-clean. By (3), we have $k_2, \ldots, k_n \in R$ such that $c_1 = 1 + k_2c_2 + \cdots + k_nc_n$. Let

$$E = (e_{ij}) = \begin{pmatrix} 1 & -k_2 & \cdots & -k_n \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \begin{pmatrix} e & & & \\ c_2 & 1 & & \\ \vdots & & \ddots & \\ c_n & & & 1 \end{pmatrix} \begin{pmatrix} 1 & k_2 & \cdots & k_n \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

By (2), one easily checks that $E = E^2 \in M_n(R)$ and $detE = e$. Let

$$U = (u_{ij}) = \begin{pmatrix} a_1 & \cdots & a_n \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} - E.$$

Then

$$\begin{aligned} det(U) &= \begin{vmatrix} a_1 - e_{11} & a_2 - e_{12} & \cdots & a_n - e_{1n} \\ -e_{21} & -e_{22} & \cdots & -e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -e_{n1} & -e_{n2} & \cdots & -e_{nn} \end{vmatrix} \\ &= \begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ -e_{21} & -e_{22} & \cdots & -e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -e_{n1} & -e_{n2} & \cdots & -e_{nn} \end{vmatrix} + (-1)^n det(E) \\ &= (-1)^{n-1}\big(a_1A_{11} + \cdots + a_nA_{1n}\big) + (-1)^n e, \end{aligned}$$

where $A_{11}, \ldots, A_{1n}$ are algebraic complements of E corresponding to $e_{11}, \ldots, e_{1n}$ respectively. Clearly,

$$E = \begin{pmatrix} 1 + (e - c_1) & (e - c_1)k_2 & (e - c_1)k_3 & \cdots & (e - c_1)k_n \\ c_2 & 1 + k_2c_2 & k_3c_2 & \cdots & k_nc_2 \\ c_3 & k_2c_3 & 1 + k_3c_3 & \cdots & k_nc_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_n & k_2c_n & k_3c_n & \cdots & 1 + k_nc_n \end{pmatrix}.$$

We see that $A_{11} = 1 + k_2c_2 + \cdots + k_nc_n = c_1$. Further, each $A_{1i} = -c_i$ $(2 \leq i \leq n)$. Obviously, there is some $u \in U(R)$ such that $a_1A_{11} + \cdots + a_nA_{1n} = a_1c_1 - \cdots - a_ic_i - \cdots - a_nc_n = e + u$. Hence, $det(U) = (-1)^{n-1}(e + u) + (-1)^n e = (-1)^{n-1}u \in U(R)$, and so $U \in GL_n(R)$. Therefore A is e-clean, as desired. □

As immediate consequences, $\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in M_n(R)$ is 1-clean if and only if $a_1 \in 1 + U(R)$, and $\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in M_n(R)$ is 0-clean if and only if the following hold:

(1) There exist $x_1, \ldots, x_n \in R$ such that $a_1x_1 + \cdots + a_nx_n \in U(R)$;
(2) $x_1 \equiv 1 \left(mod\, x_2R + \cdots + x_nR\right)$.

Example 7.1.25. $\begin{pmatrix} 12 & 5 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in M_3(\mathbb{Z})$ is clean, while $\begin{pmatrix} 12 & 5 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{Z})$ is not clean. Since $12 \times (-2) + 5 \times 2 + 3 \times 5 = 1$ and $-2 \equiv 1\,(mod\, 2R + 5R)$, it follows by Theorem 7.1.24 that $\begin{pmatrix} 12 & 5 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in M_3(\mathbb{Z})$ is 0-clean.

As $12 \notin 1 + U(\mathbb{Z})$, $\begin{pmatrix} 12 & 5 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{Z})$ is not 1-clean. If it is 0-clean, then there exist some $x_1, x_2 \in \mathbb{N}$ such that $12x_1 + 5x_2 = \pm 1$ and $x_1 \equiv 1\,(mod\, x_2\mathbb{Z})$. Write $x_1 + x_2y = 1$. Then $(12y - 5)x_2 = 13$ or 11. Hence, $12y - 5 = \pm 1, \pm 11$ or ± 13. So $12y = -8, -6, 4, 6, 16$ or 18, a contradiction. Therefore $\begin{pmatrix} 12 & 5 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{Z})$ is not 0-clean, and we are done. □

Let R be a commutative ring, and let $a \in R$. In view of Theorem 7.1.24, $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in M_2(R)$ is clean if and only if there exists an idempotent $e \in R$ and $x, y \in R$ such that $ax - e \in U(R), ey = 0$ and $x \equiv 1 \left(mod\, yR\right)$. Furthermore, we record a related result which is due to [486, Proposition 2.2].

Proposition 7.1.26. *Let R be a ring, and let $a \in R$. Then the following are equivalent:*

(1) $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in M_2(R)$ *is clean.*
(2) *There exists an idempotent $e \in R$ and a unit $u \in R$ such that $a - e - u \in (1 - e)Ra$.*
(3) *There exists an idempotent $e \in Ra$ and a unit $u \in R$ such that $1 - e = (1 - e)u(1 - a)$.*

Proof. (1) $\Rightarrow$ (2) Let $a \in R$. Then there exists an idempotent $E \in M_2(R)$ and a nilpotent $U \in GL_2(R)$ such that $\begin{pmatrix} a & \\ & 0 \end{pmatrix} = E + U$. Hence,

$$U^{-1}\begin{pmatrix} a & \\ & 0 \end{pmatrix} = U^{-1}E + I_2 = \left(U^{-1}EU\right)U^{-1} + I_2.$$

Set $F = U^{-1}EU$. Then $F = F^2 \in M_2(R)$, and that

$$(I_2 - F)U^{-1}\begin{pmatrix} a & \\ & 0 \end{pmatrix} = I_2 - F.$$

Write $I_2 - F = \begin{pmatrix} e & \\ x & 0 \end{pmatrix}$, where $e = e^2 \in R$ and $x \in Re$. Write $f = 1 - e$ and $U^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. It follows from

$$U^{-1}\begin{pmatrix} a & \\ & 0 \end{pmatrix} = FU^{-1} + I_2$$

that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} a & \\ & 0 \end{pmatrix} = \begin{pmatrix} f & \\ -x & 1 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} + I_2.$$

Thus, we get

$$\alpha a = f\alpha + 1, 0 = f\beta,$$
$$\gamma a = -x\alpha + \gamma, 0 = -x\beta + \delta + 1.$$

One easily checks that

$$\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix}U^{-1}\begin{pmatrix} 1 & 0 \\ \gamma a & 1 \end{pmatrix} = \begin{pmatrix} \alpha + \beta\gamma a & 0 \\ 0 & -1 \end{pmatrix}.$$

This implies that $u := \alpha + \beta\gamma a \in U(R)$. Hence, $\alpha = u - \beta\gamma a$. It follows from $\alpha a = f\alpha + 1, 0 = f\beta$ that $(u - \beta\gamma a)a = f(u - \beta\gamma a) + 1$, and then $ua - \beta\gamma a^2 = fu + 1$. Thus,

$$a = u^{-1}\beta\gamma a^2 + u^{-1}fu + u^{-1}.$$

Set $g = u^{-1}fu$. Then $g = g^2 \in R$. As $\beta = e\beta$, we see that

$$\begin{aligned} a - g - u^{-1} &= u^{-1}\beta\gamma a^2 = u^{-1}e\beta\gamma a^2 \\ &= u^{-1}(1 - f)\beta\gamma a^2 = (u^{-1} - u^{-1}f)\beta\gamma a^2 \\ &= (1 - u^{-1}fu)u^{-1}\beta\gamma a^2 \in (1 - u^{-1}fu)Ra, \end{aligned}$$

as required.

(2) $\Rightarrow$ (1) Suppose that there exists an idempotent $e \in R$ and a unit $u \in R$ such that $a - e - u \in (1 - e)Ra$. Write $f = 1 - e$ and $a - e - u = fxa$. Choose

$$E = \begin{pmatrix} u^{-1}eu & 0 \\ u^{-1}x(a-1)u^{-1}fu & 1 \end{pmatrix}.$$

Then $E = E^2 \in M_2(R)$. Choose

$$U = \begin{pmatrix} 1 & 0 \\ u^{-1}x(1-a) & 1 \end{pmatrix} \begin{pmatrix} 1 & -u^{-1}fu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \\ u^{-1}x & -1 \end{pmatrix}.$$

Then $U \in GL_2(R)$. One easily checks that $U \begin{pmatrix} a \\ 0 \end{pmatrix} = EU + I_2$, and therefore $\begin{pmatrix} a \\ 0 \end{pmatrix} - U^{-1}EU = U^{-1} \in GL_2(R)$, as required.

(2) $\Rightarrow$ (3) By hypothesis, we have an idempotent $e \in R$ and a unit $u \in R$ such that $a - e - u \in (1-e)Ra$. Write $a - e - u = (1-e)xa$ for some $x \in R$. Then $g := 1 - u^{-1}eu \in R$ is an idempotent, and that $g = 1 - u^{-1}(ea - e) = 1 + u^{-1}e - u^{-1}ea = u^{-1}(e+u) - u^{-1}ea = u^{-1}\big(1 - (1-e)x\big)a - u^{-1}ea \in Ra$. Further, $1 - g = u^{-1}e(a-1) = (1-g)(-u^{-1})(1-a)$, as needed.

(3) $\Rightarrow$ (2) By assumption, there exists an idempotent $e \in Ra$ and a unit $u \in R$ such that $1 - e = (1-e)u(1-a)$. Write $e = xa$ for some $x \in R$. Then $a - \big(1 - u^{-1}eu\big) + u^{-1} = u^{-1}e(u+x)a = u^{-1}eu\big(1 + u^{-1}x\big)a \in u^{-1}euRa$, as desired. □

Corollary 7.1.27. *Let R be an abelian ring, and let $a \in R$. Then the following are equivalent:*

(1) $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in M_2(R)$ *is clean.*

(2) $a \in R$ *is clean.*

Proof. (1) $\Rightarrow$ (2) In view of Proposition 7.1.26, there exists an idempotent $e \in Ra$ and a unit $u \in R$ such that $1 - e = (1-e)u(1-a) \in R(1-a)$. Write $e = xa, ex = x, 1 - e = y(1-a)$ and $(1-e)y = y$. Then $ax = (ax)(ax) = (xa)(ax) = x(ax)a = xa$. Likewise, $(1-a)y = y(1-a)$. Clearly, $\big(a - (1-e)\big)^{-1} = x - y \in U(R)$; hence, $a \in R$ is clean.

(2) $\Rightarrow$ (1) Write $a = e + u$ for an idempotent $e \in R$ and a unit $u \in R$. Then

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} u & 0 \\ 0 & -1 \end{pmatrix}$$

is clean, as asserted. □

As in the proof of Proposition 7.1.26, Khurana et al. observed that $a \in R$ is clean if and only if there exist an idempotent $e \in R$ and a unit $x \in R$ such that $xa - ex = 1$. Furthermore, Šter constructed a clean ring R while eRe is not clean for an idempotent $e \in R$. We refer the reader to [486, Example 3.4] for details.

7.2 Exchange Rings

The purpose of this section is to investigate clean elements in exchange rings. For further use, we record more elementary properties of exchange rings.

Lemma 7.2.1. *Let R be an exchange ring, and let $e \in R$ be an idempotent. Then eRe is an exchange ring.*

Proof. For any $a \in eRe$, there exists an idempotent $f \in R$ such that $f = as, 1-f = (1-a)t, s, t \in R$. Clearly, $ef = f$, and so $(efe)^2 = efefe = efe \in eRe$. Further, $efe = a(ese), e - efe = (e-a)(ete), ese, ete \in R$. This implies that $efe \in a(eRe)$ and $e - efe \in (e-a)(eRe)$; hence the result follows. □

Lemma 7.2.2. *Let R be an exchange ring, and let $a_1, \ldots, a_n \in R (n \geq 2)$. If $a_1 + \cdots + a_n = 1$, then there exist orthogonal idempotents $e_1, \ldots, e_n \in R$ such that $e_1 \in Ra_1, \ldots, e_n \in Ra_n$ and $e_1 + \cdots + e_n = 1$.*

Proof. It is obvious for $n = 2$ by Proposition 1.1.7. Assume that the result holds for $n = k (k \geq 2)$. Given $a_1 + \cdots + a_k + a_{k+1} = 1$, then there exists an idempotent $e \in R$ such that $e = s(a_1 + \cdots + a_k), 1 - e = ta_{k+1}, s, t \in R$; hence, $esa_1e + \cdots + esa_ke = e$. As R is an exchange ring, so is eRe by Lemma 7.2.1. By hypothesis, we have orthogonal idempotents $f_1, \ldots, f_k \in eRe$ such that $f_1 \in eResa_1e, \ldots, f_k \in eResa_ke$ and $f_1 + \cdots + f_k = e$. Write $f_i = er_iesa_ie$ and $e_i = f_ier_iesa_i$. Then $e_ie_j = f_ier_iesa_if_jer_jesa_j = f_if_jer_jesa_j$. Thus, if $i \neq j$, then $e_ie_j = 0$. If $i = j$, then $e_i^2 = f_ier_iesa_i = e_i$. Set $e_{k+1} = 1 - e_1 - \cdots - e_k$. If $i \neq k+1$, then $e_ie_{k+1} = e_{k+1}e_i = 0$ and $e_{k+1}^2 = e_{k+1}$. This shows that $e_1, \ldots, e_{k+1} \in R$ are orthogonal idempotents. Obviously, $e_1 \in Ra_1, \ldots, e_k \in Ra_k$. As $e_{k+1}e = e - f_1 - \cdots - f_k = 0$, we get $e_{k+1} = e_{k+1}(1-e) \in Ra_{k+1}$. By induction, we complete the proof. □

Lemma 7.2.3. *If $M = A \oplus B$, $p : M \to A, (a, b) \to a$ and $C \subseteq M$, then the following are equivalent:*

(1) *$p|_C : C \to A, c \mapsto p(c)$ is an isomorphism.*
(2) *$M = C \oplus B$.*

Proof. (1) $\Rightarrow$ (2) Since $p|_C : C \to A$ is an isomorphism, we can find $\varphi : A \to C$ such that $\varphi p|_C = 1_C$ and $p|_C \cdot \varphi = 1_A$. If $m \in C \cap B$, then $p|_C(m) = 0$, and so $m = \varphi p|_C(m) = \varphi(0) = 0$. This implies that

$C \cap B = 0$. For any $m \in M$, we have $a \in A, b \in B$ such that $m = a + b$. Clearly, $\varphi(a) \in C$. Thus, $p\big(a - \varphi(a)\big) = p(a) - p|_C \cdot \varphi(a) = 0$; whence, $a - \varphi(a) \in Ker(p) = B$. Let $d = b + a - \varphi(a)$. Then $m = \varphi(a) + d$ with $\varphi(a) \in C, d \in B$. Hence $M = C + B$. Therefore $M = C \oplus B$.

(2) $\Rightarrow$ (1) For any $a \in A$, have $a = c + b, c \in C, b \in B$. Hence, $c = a - b$, and so $p|_C(c) = p(a - b) = a$. Thus, $p|_C : C \to A$ is surjective. If $p|_C(m) = 0, m \in C$, then we have some $a \in A, b \in B$ such that $m = a + b$. We infer that $a = 0$, and then $m = b \in C \cap B = 0$. Therefore $p|_C$ is an isomorphism. □

A right R-module A is said to have the finite exchange property if for every right R-module M and any two decompositions $M = A' \oplus N = \bigoplus_{i\in I} A_i$, where $A'_R \cong A_R$ and the index set I is finite, then there exist submodules $A'_i \subseteq A_i$ such that $M = A' \oplus \big(\bigoplus_{i\in I} A'_i\big)$. We now have at our disposal all the information necessary to prove the following.

Theorem 7.2.4. *Let P be a right R-module. Then the following are equivalent:*

(1) *$End_R(P)$ is an exchange ring.*
(2) *Given any right R-module decompositions $M = C \oplus D = A_1 \oplus A_2$ with $C \cong P$, there exist $Q_i \subseteq A_i$ such that $M = C \oplus Q_1 \oplus Q_2$.*
(3) *Given any right R-module decompositions $M = C \oplus D = \bigoplus_{i=1}^{n} A_i$ with $C \cong P$, there exist $Q_i \subseteq^{\oplus} A_i$ such that $M = C \oplus \left(\bigoplus_{i=1}^{n} Q_i\right)$.*
(4) *P has the finite exchange property.*

Proof. (1) $\Rightarrow$ (3) Given any right R-module decompositions $M = C \oplus D = \bigoplus_{i=1}^{n} A_i$ with $C \cong P$. Let $E = End_R(M)$, and let $\pi : M \to C, \pi(c, d) = c$. Then $\pi = \pi^2 \in E$ and $C = \pi(M)$. Let $p_i : M \to A_i, \pi(a_1, \ldots, a_n) = a_i$. Then $p_i = p_i^2 \in E$ and $A_i = p_i(M)$. Obviously, $p_1 + \cdots + p_n = 1_M$; hence, $\pi p_1 \pi + \cdots + \pi p_n \pi = \pi$. By hypothesis, $\pi E \pi \cong End_R\big(\pi(M)\big) \cong End_R(P)$ is an exchange ring. In light of Lemma 7.2.2, there exist orthogonal idempotents $e_1, \ldots, e_n \in \pi E \pi$ such that $e_1 \in \pi E \pi p_1 \pi, \ldots, e_n \in \pi E \pi p_n \pi$ and $e_1 + \cdots + e_n = \pi$. Write $e_i = \pi \alpha_i \pi p_i \pi, \alpha_i \in E$. Choose $\sigma_i = e_i \pi \alpha_i \pi$, Clearly, $e_i = \sigma_i p_i \pi$ and $e_i \sigma_i = \sigma_i$. Choose $\eta_i = p_i \sigma_i p_i$. Then $\eta_i \eta_j = p_i \sigma_i p_i p_j \sigma_j p_j$, and so $\eta_1, \ldots, \eta_n \in E$ are orthogonal idempotents. In addition, $\eta_i(M) \subseteq A_i$. Further,

$$A_i = A_i \cap \big(\eta_i(M) \oplus (1_M - \eta_i)(M)\big) = \eta_i(M) \oplus \big(A_i \cap (1_M - \eta_i)(M)\big).$$

Let $Q_i = A_i \cap (1_M - \eta_i)(M)$, Then Q_i is a direct summand of A_i and

$$M = \left(\bigoplus_{i=1}^{n} \eta_i(M)\right) \bigoplus \left(\bigoplus_{i=1}^{n} Q_i\right).$$

Construct a map $p : M = \left(\bigoplus_{i=1}^{n} \eta_i(M)\right) \bigoplus \left(\bigoplus_{i=1}^{n} Q_i\right) \to \bigoplus_{i=1}^{n} \eta_i(M), x \to \big(\eta_1(x), \ldots, \eta_n(x)\big)$. Then $p \mid_C : C \to \bigoplus_{i=1}^{n} \eta_i(M)$. If $p \mid_C(x) = 0$ and $x \in C$, then $\eta_1(x) = \cdots = \eta_n(x) = 0$; whence, $x = \pi(x) = \sum_{i=1}^{n} e_i(x) = \sum_{i=1}^{n} e_i\sigma_i p_i\pi(x) = \sum_{i=1}^{n} e_i\pi\alpha_i\pi p_i\pi\sigma_i p_i(x) = \sum_{i=1}^{n} \sigma_i\eta_i(x) = 0$. This implies that $p \mid_C$ is injective.

One easily checks that $\eta_i\sigma_j\eta_j = \eta_i\pi\sigma_j p_j\eta_j = \eta_i\pi e_j\sigma_j p_j\eta_j = p_i\sigma_i p_i e_j\sigma_i p_j\eta_j = p_i e_i e_j\sigma_j p_j\eta_j$, and so $\eta_i\sigma_j\eta_j = 0 (i \neq j), \eta_i\sigma_i\eta_i = p_i e_i\sigma_i p_i\eta_i = p_i\sigma_i p_i\eta_i = \eta_i$. For any $(\eta_1(x_1), \ldots, \eta_n(x_n)) \in \bigoplus_{i=1}^{n} \eta_i(M)$, choosing $\sum_{i=1}^{n} \sigma_i\eta_i(x_i) \in C$, then $p \mid_C \left(\sum_{i=1}^{n} \sigma_i\eta_i(x_i)\right) = (\eta_1(x_1), \ldots, \eta_n(x_n))$. This means that $p \mid_C$ is surjective, and so it is an isomorphism. According to Lemma 7.2.3, $M = C \oplus \left(\bigoplus_{i=1}^{n} Q_i\right)$.

(3) $\Rightarrow$ (4) $\Rightarrow$ (2) There are obvious.

(2) $\Rightarrow$ (1) Set $M = \{(x, y) \mid x, y \in P\}$. Choose $A_1 = \{(x, 0) \mid x \in P\}, A_2 = \{(0, x) \mid x \in P\}$ and $D = \{(x, x) \mid x \in P\}$. Assume that $\alpha + \beta = 1_P, \alpha, \beta \in End(P)$. Choose $C = \{(\alpha(x), -\beta(x)) \mid x \in P\}$. Obviously, $M = A_1 \oplus A_2$. Moreover, $C + D \subseteq M$. For any $(x, y) \in M$, we have

$$(x, y) = \big(\alpha(x - y), -\beta(x - y)\big) + \big(x - \alpha(x - y), x - \alpha(x - y)\big) \in C \oplus D,$$

and so $M = C + D$. If $\big(\alpha(z), -\beta(z)\big) = (s, s)$, then $\alpha(z) = -\beta(z)$, and so $z = 0$. This implies that $\big(\alpha(z), -\beta(z)\big) = (0, 0)$. Thus, $C \cap D = 0$. Hence, $M = C \oplus D = A_1 \oplus A_2$, where $C \cong P$. By hypothesis, there exist $Q_i \subseteq A_i$ such that $M = C \oplus Q_1 \oplus Q_2$.

For any $x \in P$, there exist unique $z \in P, (x_1, 0) \in Q_1, (0, x_2) \in Q_2$ such that

$$(x, x) = \big(\alpha(z), -\beta(z)\big) + (x_1, 0) + (0, x_2).$$

If $\alpha'(x) = x_1$ and $\alpha'(z) = z_1$, then define $\alpha' : P \to P, x \to x_2$, $\beta' : P \to P, x \to x_1$. Obviously, α' is well defined. For any $x, y \in P$, if

$$(x, x) = \big(\alpha(z_1), -\beta(z_1)\big) + (x_1, 0) + (0, x_2) \text{ and}$$
$$(y, y) = \big(\alpha(z_2), -\beta(z_2)\big) + (y_1, 0) + (0, y_2),$$

then

$$(x+y, x+y) = \big(\alpha(z_1+z_2), -\beta(z_1+z_2)\big) + (x_1+y_1, 0) + (0, x_2+y_2).$$

It follows that $\alpha'(x+y) = x_2+y_2 = \alpha'(x)+\alpha'(y)$. Similarly, $\alpha'(xr) = \alpha'(x)r$ for any $x \in P, r \in R$. We infer that $\alpha' \in End_R(P)$. Likewise, $\beta' \in End_R(P)$.

For any $x \in P$, if

$$(x, x) = \big(\alpha(z), -\beta(z)\big) + (x_1, 0) + (0, x_2),$$

then $\alpha'(x) = x_2$, and so $\alpha\alpha'(x) = \alpha(x_2)$. It is easy to verify that

$$\big(\alpha(x_2), \alpha(x_2)\big) = \big(\alpha(x_2), -\beta(x_2)\big) + (0, x_2);$$

hence, $\alpha'\alpha\alpha'(x) = \alpha'\alpha(x_2) = x_2 = \alpha'(x)$. Therefore $\alpha'\alpha\alpha' = \alpha'$. Likewise, $\beta'\beta\beta' = \beta'$. Let $e = \alpha\alpha', f = \beta\beta'$. Then $e \in \alpha End_R(P), f \in \beta End_R(P)$ are idempotents. Since $\alpha(z)+x_1 = x = -\beta(z)+x_2$, $\alpha(z)+\beta'(x) = -\beta(z)+\alpha'(x)$, and then $z = (\alpha+\beta)(z) = \alpha'(x)-\beta'(x)$. As a result,

$$\begin{aligned} x &= \alpha(z)+x_1 \\ &= \alpha(z)+\beta'(x) \\ &= \alpha\alpha'(x)-\alpha\beta'(x)+\beta'(x) \\ &= \alpha\alpha'(x)+\beta\beta'(x). \end{aligned}$$

Accordingly, $e+f = 1_P$. This completes the proof. □

Lemma 7.2.5. *Let A, B and C be right R-modules. If $C \oplus A = C \oplus B$, then $\varphi : A \cong B, a \mapsto b$, where $a = c+b, c \in C$.*

Proof. For any $a \in A$, there exist $c \in C, b \in B$ such that $a = c+b$. Construct a map $\varphi(a) = b$. Clearly, φ is a right R-morphism. If $\varphi(a) = 0$ and $a = c+b$, then $b = 0$; hence, $a = c \in A \cap C = 0$. This implies that φ is injective. For any $b \in B$, there exist $c \in C, a \in A$ such that $b = c+a$, i.e., $a = -c+b$. Thus, $\varphi(a) = b$, and so φ is surjective. Therefore φ is an isomorphism. □

Lemma 7.2.6. *Let P be a right R-module. Then the following are equivalent:*

(1) *$End_R(P)$ is an exchange ring.*

(2) *Given any right R-module decompositions $M = C \oplus D \oplus E = \bigoplus_{i=1}^{n} A_i \oplus E$ with $C \cong P$, there exist $Q_i \subseteq^{\oplus} A_i$ such that $M = C \oplus \left(\bigoplus_{i=1}^{n} Q_i\right) \oplus E$.*

Proof. (1) $\Rightarrow$ (2) Given any right R-module decomposition $M = C \oplus D \oplus E = \bigoplus_{i=1}^{n} A_i \oplus E$ with $C \cong P$. By virtue of Lemma 7.2.5, there exists $\varphi : C \oplus D \cong \bigoplus_{i=1}^{n} A_i, c+d \mapsto a$, where $a \in \bigoplus_{i=1}^{n} A_i, e \in E$ such that $c+d = a+e$. Hence $N := \varphi(C) \oplus \varphi(D) = \bigoplus_{i=1}^{n} A_i$. In light of Theorem 7.2.4, there exist $Q_i \subseteq^{\oplus} A_i$ such that $N = \varphi(C) \oplus \left(\bigoplus_{i=1}^{n} Q_i \right)$. Let $p : M = C \oplus D \oplus E \to D, c+d+e \to d$ be the projection onto D. Consider $p \mid_{\bigoplus_{i=1}^{n} Q_i} : \bigoplus_{i=1}^{n} Q_i \to D$. If $p(a) = 0$ with $a \in \bigoplus_{i=1}^{n} Q_i$, writing $a = c + d + e$ with $c \in C, d \in D$ and $e \in E$, then $d = 0$. Hence, $a = c + e,$, i.e., $c = a - e$. This implies that $\varphi(c) = a \in \varphi(C) \cap \left(\bigoplus_{i=1}^{n} Q_i \right) = 0$; whence, $a = 0$. Therefore $p \mid_{\bigoplus_{i=1}^{n} Q_i}$ is injective.

For any $d \in D$, $a := \varphi(d) \in \bigoplus_{i=1}^{n} Q_i$. Thus, $d = a+e, e \in E$. This implies that $a = d-e$, and so $p(a) = d$. Therefore $p \mid_{\bigoplus_{i=1}^{n} Q_i}$ is surjective. We conclude that $p \mid_{\bigoplus_{i=1}^{n} Q_i}$ is an isomorphism. Accordingly, $M = C \oplus \left(\bigoplus_{i=1}^{n} Q_i \right) \oplus E$ by Lemma 7.2.3.

(2) $\Rightarrow$ (1) This is trivial by Theorem 7.2.4. □

Theorem 7.2.7. *Let R be a ring, and let $n \in \mathbb{N}$. Then R is an exchange ring if and only if so is $M_n(R)$.*

Proof. Suppose that R is an exchange ring. If $M = R_1 \oplus R_2 \oplus Q = A_1 \oplus A_2$ with $R_1 \cong R \cong R_2$, then $M = R_1 \oplus (R_2 \oplus Q) = A_1 \oplus A_2$. According to Theorem 7.2.4, there exist $Q_1 \subseteq^{\oplus} A_1$, $Q_2 \subseteq^{\oplus} A_2$ such that $M = R_1 \oplus Q_1 \oplus Q_2$. Thus, $M = R_2 \oplus Q \oplus R_1 = Q_1 \oplus Q_2 \oplus R_1$. In virtue of Lemma 7.2.6, $M = R_2 \oplus (C_1 \oplus C_2) \oplus R_1 = (R_1 \oplus R_2) \oplus C_1 \oplus C_2$, where $C_1 \subseteq^{\oplus} A_1$, $C_2 \subseteq^{\oplus} A_2$. In light of Theorem 7.2.4, $End_R(R_1 \oplus R_2)$ is an exchange ring. By induction, $M_{2^n}(R)$ is an exchange ring. Since $M_n(R) \cong EM_{2^n}(R)E$ with $E = E^2 \in M_{2^n}(R)$, by Lemma 7.2.1, $M_n(R)$ is an exchange ring. Conversely, assume that $M_n(R)$ is an exchange ring. Choose $e = diag(1, 0, \ldots, 0) \in M_n(R)$. Then $R \cong eM_n(R)e$. Therefore the result follows from Lemma 7.2.1. □

As a consequence of Lemma 7.2.1 and Theorem 7.2.7, the exchange property is Morita invariant. Thus, a ring R is an exchange ring if and only if so is $End_R(A)$ for all finitely generated projective right R-modules A.

Corollary 7.2.8. *Let R be a π-regular ring. Then $M_n(R)$ is an exchange ring for all $n \in \mathbb{N}$.*

Proof. For any $a \in R$, we have $n \in \mathbb{N}$ and $b \in R$ such that $a^n = a^n b a^n$. Set $g = a^n b$ and $e = g + ga^n(1-g)$. Then $e \in aR$ is an idempotent. One easily checks that $1 - e = \big(1 - ga^n\big)(1-g) = (1-a)(1+a+\cdots+a^{n-1})(1-g) \in (1-a)R$. Thus R is an exchange ring, and we complete the proof by Theorem 7.2.7. □

An element $a \in R$ is special clean if there exists an idempotent $e \in R$ and a unit $u \in R$ such that $a = e + u$ and $aR \cap eR = 0$.

Proposition 7.2.9. *Every special clean element in a ring is unit-regular.*

Proof. Let $a \in R$ be special clean. Then there exists an idempotent $e \in R$ and a unit $u \in R$ such that $a = e + u$ and $aR \cap eR = 0$. Hence, $au^{-1} = eu^{-1} + 1$. Thus, $au^{-1}e = eu^{-1}e + e \in aR \cap eR = 0$. This yields $au^{-1}(a-u) = 0$, and so $au^{-1}a = a$. Therefore $a \in R$ is unit-regular. □

The following simple fact is due to Nicholson (see [430, Lemma 2.8]).

Lemma 7.2.10. *Let A be a quasi-projective right R-module. If $A_1 \subseteq^{\oplus} A$ and $A = A_1 + B$, then there exists some $A_2 \subseteq B$ such that $A = A_1 \oplus A_2$.*

Proof. Write $A = A_1 \oplus C$. Then there exists a projection $p : A \to A_1$ such that $A_1 = pA$. Let $\varphi : A \to A/B$ be the natural map. Then there exists an R-morphism $\alpha : A \to A$ such that the following diagram

$$\begin{array}{ccc} A & \overset{\alpha}{\leftarrow} & A \\ p\downarrow & & \downarrow\varphi \\ A_1 & \overset{\varphi\,|_{A_1}}{\twoheadrightarrow} & A/B \end{array}$$

commutes. That is, $\varphi p \alpha = \varphi$. Clearly, $p = p^2 \in End_R(A)$. Let $\gamma = p + p\alpha(1-p)$. Then $\gamma = \gamma^2 \in End_R(A)$. Set $A_2 = Ker(\gamma)$. Then $A_2 = (1-\gamma)(A) = (1-p\alpha)(1-p)(A) \subseteq Ker(\varphi) = B$. For any $a \in A_1 \cap A_2$, we have $a = p(a)$ and $\gamma(a) = 0$. Hence, $a = (1-\gamma)(a) + \gamma(a) = (1 - pa)(1-p)p(a) = 0$. This shows that $A_1 \cap A_2 = 0$. For any $a \in A$, $a = \gamma(a) + (1-\gamma)(a) \in pA + (1-\gamma)(A) = A_1 + A_2$. This implies that $A = A_1 + A_2$. Therefore $A = A_1 \oplus A_2$, as asserted. □

Theorem 7.2.11. *Let R be a ring, and let $a \in R$ be regular. If*

(1) *$aR/ar(a^2)$ is projective;*
(2) *$ar(a^2) \cong R/\big(r(a)+aR\big)$;*

then $a \in R$ is special clean.

Proof. Since $aR/ar(a^2)$ is projective, the exact sequence

$$0 \to ar(a^2) \hookrightarrow aR \to aR/ar(a^2) \to 0$$

splits, and so we have a right R-module Z such that $aR = ar(a^2) \oplus Z$. Since $a \in R$ is regular, aR is projective, and then so is $ar(a^2)$. By hypothesis, $R/\big(aR + r(a)\big)$ is projective, and so

$$0 \to aR + r(a) \hookrightarrow R \to R/\big(aR + r(a)\big) \to 0$$

splits. Thus, there exists a right R-module Y such that $R = \big(aR+r(a)\big)\oplus Y$; hence that $aR + r(a)$ is projective. Write $R = aR \oplus E$. Then $aR + r(a) = \big(aR + r(a)\big) \cap \big(aR \oplus E\big) = aR \oplus \big(aR + r(a)\big) \cap E$, i.e., $aR \subseteq^{\oplus} aR + r(a)$. In view of Lemma 7.2.10, we have a right R-module $X \subseteq r(a)$ such that $aR + r(a) = aR \oplus X$. Hence, $r(a) = r(a) \cap \big(aR \oplus X\big) = K \oplus X$, where $K := r(a) \cap aR = ar(a^2)$. By assumption, we have a right R-isomorphism $\phi : Y \cong R/\big(r(a) + aR\big) \cong ar(a^2) = K$, and then $\alpha : X \oplus Y \to X \oplus K$, where $\alpha(x + y) = x + \phi(y)$ for any $x \in X$ and $y \in Y$. Let

$$h : R = K \oplus X \oplus Y \oplus Z \to K \oplus X \oplus Y \oplus Z = R,$$
$$h(k + x + y + z) = \alpha^{-1}(x + k) + y, k \in K, x \in X, y \in Y, z \in Z;$$
$$v : R = K \oplus X \oplus Y \oplus Z \to K \oplus X \oplus Y \oplus Z = R,$$
$$v(k + x + y + z) = \alpha(x + y) + \phi^{-1}(k), k \in K, x \in X, y \in Y, z \in Z.$$

For any $k \in K, x \in X, y \in Y, z \in Z$, we have $hvhv(k + x + y + z) = hvh\big(\alpha(x+y) + \phi^{-1}(k)\big) = hv\big(x+y+\phi^{-1}(k)\big) = h\big(\alpha(x+y) + \alpha\phi^{-1}(k)\big) = x + y + \phi^{-1}(k) = hv(k + x + y + z)$. Let $e = hv$. Then $e \in End_R(R)$ is an idempotent. Assume that $(a - hv)(k + x + y + z) = 0$ for some $k \in K, x \in X, y \in Y, z \in Z$. Then $a(y + z) = hv(k + x + y + z) = x+y+\phi^{-1}(k) \in aR\cap(X\oplus Y) = 0$, and then $x = -y-\phi^{-1}(k) \in X\cap Y = 0$. It follows from $a(y+z) = 0$ that $y+z \in r(a)\cap(Y\oplus Z) = 0$; hence $y+z = 0$. This implies that $y = -z \in Y \cap Z = 0$, and so $y = z = 0$. Further, $\phi^{-1}(k) = -y = 0$; hence $k = 0$. This shows that $a - hv \in End_R(R)$ is a monomorphism.

For any $k \in K, x \in X, y \in Y, z \in Z$, we have $k + z \in K \oplus Z = aR = a\big(r(a) \oplus Y \oplus Z\big) = a(Y \oplus Z)$. Thus, we can find some $y' \in Y$ and $z' \in Z$

such that $k+z=a(y'+z')$. Choose $x'=-x$ and $k'=-\phi(y+y')$. Then $(a-hv)(k'+x'+y'+z')=a(y'+z')-\big(x'+y'+\phi^{-1}(k')\big)=k+x+y+z$. That is, $a-hv\in End_R(R)$ is an epimorphism. Therefore $a-e\in Aut_R(R)$. Furthermore, $aR\cap eR=aR\cap hvR\subseteq\big(K\oplus Z\big)\cap\big(X\oplus Y\big)=0$. This yields that $aR\cap eR=0$, hence the result. □

A separative ring is one whose finitely generated projective modules satisfy the property $A\oplus A\cong A\oplus B\cong B\oplus B\Longrightarrow A\cong B$. For instances, every weakly stable exchange ring (including every exchange ring having stable range 1) and exchange ring satisfying generalized s-comparability (see [445]). A long standing problem asks that whether every regular ring is separative (see [274]). We have

Lemma 7.2.12. *Let R be an exchange ring. Then the following are equivalent:*

(1) *R is separative.*
(2) *For all finitely generated projective right R-modules A,B and C, $C\oplus A\cong C\oplus B$ with $C\lesssim^{\oplus}A,B\Longrightarrow A\cong B$.*
(3) *For all finitely generated projective right R-modules A,B and C, $C\oplus A\cong C\oplus B$ with $C\propto A,B\Longrightarrow A\cong B$.*

Proof. (1) ⇒ (2) Suppose that $C\oplus A\cong C\oplus B$ and $C\lesssim^{\oplus}A,B$ for finitely generated projective right R-modules A and B. Write $A\cong C\oplus D$. Then $A\oplus A\cong A\oplus C\oplus D\cong B\oplus C\oplus D\cong A\oplus B$. Likewise, $B\oplus B\cong A\oplus B$. By hypothesis, $A\cong B$, as desired.

(2) ⇒ (3) Suppose that $C\oplus A\cong C\oplus B$ and $C\propto A,B$ for finitely generated projective right R-modules A and B. Write $C\lesssim^{\oplus}mA,mB$ for some $m\in\mathbb{N}$. Obviously, $End_R(C)$ is an exchange ring. In light of Theorem 7.2.4, $C\cong C_1\oplus\cdots\oplus C_m$ where each $C_i\lesssim^{\oplus}A$. Clearly, $C_i\lesssim^{\oplus}mB$, and then we have $C_i\cong C_{i1}\oplus\cdots\oplus C_{im}$ with each $C_{ij}\lesssim^{\oplus}B$. Therefore

$$\bigoplus_{i=1}^{m}\bigoplus_{j=1}^{m}C_{ij}\oplus A\cong\bigoplus_{i=1}^{m}\bigoplus_{j=1}^{m}C_{ij}\oplus B$$

with each $C_{ij}\lesssim^{\oplus}A,B$. By hypothesis, we get

$$\bigoplus_{i=2}^{m}\bigoplus_{j=1}^{m}C_{ij}\oplus A\cong\bigoplus_{i=2}^{m}\bigoplus_{j=1}^{m}C_{ij}\oplus B.$$

By iteration of this process, we obtain $A\cong B$, as desired.

(3) $\Rightarrow$ (1) Suppose that $A \oplus A \cong A \oplus B \cong B \oplus B$ for finitely generated projective right R-modules A and B. Then $A \oplus A \cong A \oplus B$ with $A \propto A, B$. Consequently, $A \cong B$, as required. □

We come now to explore special clean elements in separative exchange rings.

Theorem 7.2.13. *Let R be a separative exchange ring, and let $a - a^2 \in R$ be regular. If*

(1) *$aR/ar(a^2)$ and $R/\big(aR + r(a)\big)$ are projective;*
(2) *$Rar(a^2) = \ell(a^2)aR = R(a - a^2)R$;*

then $a \in R$ is special clean.

Proof. Since $aR/ar(a^2)$ is projective, there exists a right R-module Z such that $aR = ar(a^2) \oplus Z$. Since $R/\big(aR + r(a)\big)$ is projective,

$$0 \to aR + r(a) \hookrightarrow R \to R/\big(aR + r(a)\big) \to 0$$

splits. Thus, we have a right R-module Y such that $R = \big(aR + r(a)\big) \oplus Y$. Set $K = ar(a^2)$. In light of Lemma 7.2.10, we have a right R-module $X \subseteq r(a)$ such that $aR + r(a) = aR \oplus X$. Thus, $R = aR \oplus X \oplus Y, aR = K \oplus Z, r(a) = K \oplus X$ and $aR + r(a) = aR \oplus X$.

Since $a - a^2 \in R$ is regular, we can find some $z \in R$ such that $a(1-a) = a(1-a)za(1-a)$. Let $x = 1 + (1-a)z(1-a)$ and $y = 1 + aza$. Then $a = axa$ and $1 - a = (1-a)y(1-a)$, i.e., $a, 1-a \in R$ are both regular. Let $b = 1 - a$. Then $R = r(a) \oplus xaR = r(b) \oplus ybR$. Since R is an exchange ring, so is $End_R\big(r(a)\big)$ from Lemma 7.2.1. Thus, we have $C \subseteq r(b), D \subseteq ybR$ such that $R = r(a) \oplus C \oplus D$, and thus, $aR = aC + aD = aC \oplus aD$. Given any $x \in C$, we see that $x \in r(b)$, and so $x = ax \in aC$. Further, $ax = x \in C$, and so $C \subseteq aC \subseteq C$. Hence, $aC = C$. Thus, $aR = C \oplus aD$. Obviously, $D \cong aD$. Construct a map $\varphi : D \to a(1-a)D$ given by $\varphi(d) = a(1-a)d$ for any $d \in D$. Then φ is an R-epimorphism. If $a(1-a)d = 0$ for some $d \in D$, then $(1-a)d \in r(a)$. Since $ad \in C \oplus D$, $(1-a)d = d - ad \in r(a) \cap \big(C \oplus D\big) = 0$, whence, $(1-a)d = 0$. As $d \in D \subseteq ybR$, we can find some $z \in R$ such that $d = ybz$, and so $bd = bybz = (1-a)d = 0$. Thus, $bz = 0$, and then $d = ybz = 0$. This shows that $D \cong a(1-a)D$. Clearly, $a(1-a)D \cong a(1-a)R$, and so $D \cong aD \cong a(1-a)R$. Then $R = C \oplus aD \oplus X \oplus Y = r(a) \oplus C \oplus D = C \oplus D \oplus X \oplus K$. This implies that

$$a(1-a)R \oplus K \cong a(1-a)R \oplus Y.$$

From $Rar(a^2) = R(a-a^2)R$, we get $a(1-a)R \subseteq Rar(a^2)$. Thus we can find some $x_1, \ldots, x_n \in R; y_1, \ldots, y_n \in ar(a^2)$ such that $a(1-a) = \sum_{i=1}^{n} x_i y_i$. As $a(1-a) \in R$ is regular, we may assume that each $x_i \in a(1-a)R$. Construct a map $\varphi : n\big(ar(a^2)\big) \to a(1-a)R$ given by $\varphi(z_1, \ldots, z_n) = \sum_{i=1}^{n} x_i z_i$ for any $(z_1, \ldots, z_n) \in n\big(ar(a^2)\big)$. Clearly, φ is a right R-morphism. For any $a(1-a)r \in a(1-a)R$, we can find $(y_1 r, \ldots, y_n r) \in n\big(ar(a^2)\big)$ such that $\varphi(y_1 r, \ldots, y_n r) = a(1-a)r$; hence, φ is a right R-epimorphism. Since $a(1-a)R$ is projective, φ splits. Therefore $a(1-a)R \propto ar(a^2)$.

Write $aR + r(a) = gR$ for an idempotent $g \in R$. Then $R/\big(aR + r(a)\big) \cong (1-g)R$. As $a - a^2 \in R$ is regular, there exists an idempotent $e \in R$ such that $(a-a^2)R = eR$. This implies that $e \in ReR = \ell(a^2)aR$. Thus, we have some $x_1, \ldots, x_n \in \ell(a^2)a$ and $r_1, \ldots, r_n \in R$ such that $e = \sum_{i=1}^{n} x_i r_i$. Write $x_i = s_i a$ for some $s_i \in \ell(a^2)$. Then $s_i a^2 = 0$. Write $g = ar + b$, where $r \in R, b \in r(a)$. Then $x_i g = s_i a(ar + b) = s_i a^2 r = 0$. This implies that $x_i = x_i(1-g)$. Let $f = diag(1-g, \ldots, 1-g) \in M_n(R)$, and let $c = e(x_1, \ldots, x_n)f, d = f \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} e$. Then $e = (x_1, \ldots, x_n) \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} = cd$, $c = ecf, d = fde$. Construct an R-morphism $\psi : f(nR) \to eR, fx \mapsto cx$ for any $x \in nR$. Then ψ is epimorphism. As eR is projective, $eR \lesssim^{\oplus} f(nR)$. Hence, $a(1-a)R \lesssim^{\oplus} n\big((1-g)R\big)$, i.e., $a(1-a)R \propto R/\big(aR + r(a)\big)$. According to Lemma 7.2.12, $K \cong Y$, and then, $ar(a^2) \cong R/\big(r(a) + aR\big)$. Therefore $a \in R$ is special clean, by Theorem 7.2.11. □

Corollary 7.2.14. *Let R be a separative regular ring, and let $a \in R$. Then each $a \in R$ satisfying $Rar(a^2) = \ell(a^2)aR = R(a-a^2)R$ is special clean.*

Proof. Since R is regular, $aR, r(a), aR + r(a)$ and $K = ar(a^2) = aR \cap r(a)$ are direct summands of R_R. In view of Lemma 7.2.10, there exists $X \subseteq r(a)$ such that $aR + r(a) = aR \oplus X$. Additionally, we have right R-modules Y and Z such that $R = aR \oplus X \oplus Y$ and $aR = K \oplus Z$. Obviously, $aR/ar(a^2) \cong Z$ and $R/\big(aR + r(a)\big) \cong Y$. Hence, $aR/ar(a^2)$ and $R/\big(aR + r(a)\big)$ are projective. This establishes the result by Theorem 7.2.13. □

Example 7.2.15. Let V be an infinite-dimensional vector space over a field F, and let $R = M_3\big(End_F(V)\big)$. Then R is a separative regular

ring. Choose $a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in R$. Clearly, $Rar(a^2) \subseteq Ra(1-a^2)R = Ra(1-a)(1+a)R \subseteq R(a-a^2)R$. As $a^2 = a^3$, it follows that $R(a-a^2)R = Ra(1-a)R \subseteq Rar(a^2)$. Hence $Rar(a^2) = R(a-a^2)R$. Likewise, we prove that $\ell(a^2)aR = R(a-a^2)R$. Therefore $a \in R$ satisfies $Rar(a^2) = \ell(a^2)aR = R(a-a^2)R$. According to Corollary 7.2.6, $a \in R$ is special clean.

Proposition 7.2.16. *Let R be a regular ring. Then the following are equivalent:*

(1) *R is unit-regular.*
(2) *For each $a \in R$, $ar(a^2) \cong R/\big(aR + r(a)\big)$.*
(3) *For each $a \in R$, $End_R\big((a-a^2)R\big)$ is unit-regular.*

Proof. (1) $\Rightarrow$ (3) This is obvious.

(3) $\Rightarrow$ (2) As in the proof of Theorem 7.2.13, we get $a(1-a)R \oplus ar(a^2) \cong a(1-a)R \oplus R/\big(aR \oplus r(a)\big)$. Since $End_R\big(a(1-a)R\big)$ is unit-regular, $ar(a^2) \cong R/\big(aR + r(a)\big)$, by Theorem 6.1.32.

(2) $\Rightarrow$ (1) Let $a \in R$. Construct X as in Theorem 7.2.13, we see that $r(a) = X \oplus ar(a^2)$ and $R/aR \cong X \oplus R/\big(aR + r(a)\big)$. As $ar(a^2) \cong R/\big(aR + r(a)\big)$, $r(a) \cong R/aR$, and then $a \in R$ is unit-regular. Therefore R is unit-regular. □

Theorem 7.2.17. *Let R be a separative exchange ring in which 2 is invertible, and let $a - a^3 \in R$ be regular. If*

(1) *$aR/ar(a^2)$ and $R/\big(aR + r(a)\big)$ are projective;*
(2) *$Rar(a^2) = \ell(a^2)aR = R(a-a^3)R$;*

then $a \in R$ is special clean.

Proof. Write $a - a^3 = (a-a^3)s(a-a^3)$. Then $a = axa$, where $x = a + (1-a^2)s(1-a^2)$. Set $b = 1-a$. Then $a = 1-b$. Hence, $(1-b)b(2-b) = (1-b)b(2-b)s(1-b)b(2-b)$. As $2 \in R$ is invertible, we get

$$(1-b)b\left(1-\frac{b}{2}\right) = (1-b)b\left(1-\frac{b}{2}\right)(2s)(1-b)b\left(1-\frac{b}{2}\right),$$

and so $b = byb$, where $y = \frac{3}{2} - \frac{b}{2} + (1-b)(1-\frac{b}{2})(2s)(1-b)(1-\frac{b}{2})$. Set $c = 1 + a$. Then

$$c(1-c)\left(1-\frac{c}{2}\right) = c(1-c)\left(1-\frac{c}{2}\right)(-2s)c(1-c)\left(1-\frac{c}{2}\right).$$

Thus, $c = czc$, where $z = \frac{3}{2} - \frac{c}{2} + (1-c)(1-\frac{c}{2})(-2s)(1-c)(1-\frac{c}{2})$. Clearly, $r(a) = (1-xa)R$ and $r(b) = (1-yb)R$. Set $e = 1-xa$ and $f = (1-yb)$.

Then $eR + fR = eR + (1-e)fR$. We easily check that

$$\begin{aligned}(1-e)fa(1-e)f &= xa(1-yb)axa(1-yb)\\ &= xa(1-yb)a(1-yb)\\ &= xa\big(1-y(1-a)\big)a\big(1-y(1-a)\big)\\ &= x\big(1-y(1-a)\big)\\ &= (1-e)f.\end{aligned}$$

Set $g = (1-e)fa(1-e)$. Then $g = g^2 \in R$ and $eg = ge = 0$. It follows that $eR + fR = (e+g)R$. Set $h = e+g$. Then $r(a) + r(b) = eR + fR = hR$ with $h = h^2$. Further,

$$R = \big(r(a) + r(b)\big) \oplus (1-h)R = r(c) \oplus zcR.$$

Since R is an exchange ring, so is $End_R\big(hR\big)$ by Lemma 7.2.1. Thus, we can find some $C \subseteq r(c), D \subseteq zcR$ such that $R = \big(r(a) + r(b)\big) \oplus C \oplus D$.

Construct a map $\varphi : D \to a(1-a)(1+a)D$ given by $\varphi(d) = a(1-a)(1+a)d$ for any $d \in D$. If $a(1-a)(1+a)d = 0$, then $(1-a)(1+a)d \in r(a), a(1+a)d \in r(b)$ and $a(1-a)d \in C \oplus D$. It is easy to verify that

$$\begin{aligned}d &= (1-a^2)d + a^2d\\ &= (1-a)(1+a)d + \tfrac{1}{2}a(1+a)d - \tfrac{1}{2}a(1-a)d\end{aligned}$$

and so $d + \frac{1}{2}a(1-a)d \in (C \oplus D) \cap (r(a) \oplus r(b)) = 0$. This implies that $d = -\frac{1}{2}a(1-a)d \in r(c) \cap D \subseteq r(c) \cap zcR = 0$. Therefore $\varphi : D \to (a-a^3)D$ is a right R-isomorphism. Consequently, $D \cong aD \cong (a-a^3)D = (a-a^3)R$.

Construct K, Z, X and Y as in the proof of Theorem 7.2.13, we get

$$R = aR \oplus X \oplus Y, aR = K \oplus Z, r(a) = K \oplus X \text{ and } aR + r(a) = aR \oplus X.$$

Thus, we have

$$\begin{aligned}R &= r(a) \oplus r(b) \oplus C \oplus D\\ &= K \oplus X \oplus r(b) \oplus C \oplus D\\ &= aR \oplus X \oplus Y\\ &= r(b) \oplus aC \oplus aD \oplus X \oplus Y.\end{aligned}$$

Clearly, $C = aC$, and then $D \oplus K \cong aD \oplus Y$. This implies that

$$(a-a^3)R \oplus K \cong (a-a^3)R \oplus Y.$$

Similarly to Theorem 7.2.13, it follows from $Rar(a^2) = R(a-a^3)R$ that $(a-a^3)R \propto ar(a^2)$. Write $aR + r(a) = gR$ for an idempotent $g \in R$. Then $R/\big(aR + r(a)\big) \cong (1-g)R$. Write $(a-a^3)R = eR$, where $e = e^2 \in R$. This implies that $e \in ReR = \ell(a^2)aR$. Analogously to Theorem 7.2.13, we see that $(a-a^3)R \propto R/\big(aR + r(a)\big)$. In light of Lemma 7.2.12, $ar(a^2) \cong R/\big(r(a) + aR\big)$. This completes the proof, by Theorem 7.2.11. □

Corollary 7.2.18. *Let R be a separative regular ring in which 2 is invertible, and let $a \in R$. Then each $a \in R$ satisfying $Rar(a^2) = \ell(a^2)aR = R(a - a^3)R$ is special clean.*

Proof. As in the proof of Corollary 7.2.14, $aR/ar(a^2)$ and $R/\big(aR + r(a)\big)$ are projective. Therefore the result follows by Theorem 7.2.17. □

We say that a ring R is IC (i.e., internal cancellation) if, $R = A_1 \oplus B = A_2 \oplus C$ with $A_1 \cong A_2 \Longrightarrow B \cong C$. We now derive

Lemma 7.2.19. *Let R be a ring. Then the following are equivalent:*

(1) *R is IC;*
(2) *Every regular element in R is unit-regular.*
(3) *For any idempotents $e, f \in R$, $eR \cong fR \Longrightarrow (1-e)R \cong (1-f)R$.*
(4) *For any idempotents $e, f \in R$, $eR \cong fR \Longrightarrow$ there exists $u \in U(R)$ such that $e = ufu^{-1}$.*

Proof. (1) $\Leftrightarrow$ (3) This is obvious.

(3) $\Rightarrow$ (2) Let $a \in R$ be regular. Write $a = axa$ for some $x \in R$. As $\varphi : axR \cong xaR$, we get $\phi : (1-ax)R \cong (1-xa)R$. Construct a morphism $\psi : R = xaR \oplus (1-xa)R \to axR \oplus (1-ax)R, s + t \mapsto \varphi(s) + \phi(t)$ for any $s \in xaR, t \in (1-xa)R$. Then $\psi \in Aut(R)$, and so $\psi(1) \in U(R)$. Clearly, $a = a\psi(1)a$. Therefore $a \in R$ is unit-regular, as desired.

(2) $\Rightarrow$ (4) Suppose that $\varphi : eR \cong fR$ with idempotents $e, f \in R$. Then $e = xy$ and $f = yx$, where $x \in eRf, y \in fRe$. Clearly, $x = xyx$. Then we have $x = xzx, z = zxz$, where $z = yxy$. By hypothesis, there exists $v \in U(R)$ such that $z = zvz$. Let $u = (1 - xz - vz)v(1 - zx - zv)$. One easily checks that $(1 - xz - vz)^2 = 1 = (1 - zx - zv)^2$, and so $u \in U(R)$. Obviously,

$$\begin{aligned} xzu &= -xzv(1 - zx - zv) \\ &= -xzv + xzx + xzv \\ &= xzx \\ &= x \end{aligned}$$

and

$$\begin{aligned} uzx &= (1 - xz - vz)v(-zvzx) \\ &= -(1 - xz - vz)vzx \\ &= -vzx + xzx + vzx \\ &= xzx \\ &= x. \end{aligned}$$

Thus, $x = xzu = x(yxy)u = xyu$ and $x = uzx = u(yxy)x = uyx$. Hence, $x = xyu = uyx$, and therefore $e = xy = u(yx)u^{-1} = ufu^{-1}$.

(4) $\Rightarrow$ (3) Suppose that $eR \cong fR$ with idempotents $e, f \in R$. Then there exists $u \in U(R)$ such that $e = ufu^{-1}$. Hence, $u^{-1}(1-e) = (1-f)u^{-1}$. Construct an R-morphism $\varphi : (1-e)R \to (1-f)R$ given by $\varphi\big((1-e)r\big) = u^{-1}(1-e)r$ for any $r \in R$. Then φ is an isomorphism, as required. □

Recall that two direct summands K and L of a right R-module M are perspective if there exists some $D \subseteq M$ such that $M = K \oplus D = L \oplus D$.

Lemma 7.2.20. *Let M a right S-module, and let $R = End_S(M)$. Then two submodules A and B of M are perspective if and only if there exists an idempotent $e \in R$ such that $eM = A$ and $e\mid_B : B \to A$ is an isomorphism.*

Proof. Suppose that A and B are perspective. Write $M = A \oplus D = B \oplus D$. Then the projection $e : M \to A$ is an idempotent morphism and $A = eM$. Hence, $e\mid_B : B \to A$. For any $a \in A$, we can find $b \in B, d \in D$ such that $a = b + d$. Then $e\mid_B(b) = e\mid_B(a - d) = a$; hence, $e\mid_B$ is epimorphic. For any $b \in B$, $e\mid_B(b) = 0$ implies that $b \in B \cap D = 0$, and so $e\mid_B$ is monomorphic. Therefore $e\mid_B : B \to A$ is an isomorphism.

Conversely, choose $D = (1-e)M$. Then $M = A \oplus D$. For any $x \in M$, we can find $a \in A, d \in D$ such that $x = a + d$. As $e\mid_B : B \to A$ is an isomorphism, there is some $b \in B$ such that $e\mid_B(b) = a$. Write $b = a' + d'$ with $a' \in A, d' \in D$. Then $a = a'$, and so $x = a' + d = b + (d - d') \in B + D$. This implies that $M = B + D$. If $b \in B \cap D$, then $e\mid_B(b) = 0$, and so $b = 0$. Thus, $B \cap D = 0$. As a result, we conclude that $M = B \oplus D$, as required. □

Following Garg, Grover and Khurana, a right module is called perspective provided that any two isomorphic summands have a common complement (see [276]). A ring R is perspective if, R_R is perspective as a right R-module.

Lemma 7.2.21. *Every perspective ring is IC.*

Proof. Let R be a perspective ring. Suppose that $eR \cong fR$ with $e, f \in R$. Then $R = eR \oplus (1-e)R = fR \oplus (1-f)R$. Hence, we can find an R-module D such that $R = eR \oplus D = fR \oplus D$. This implies that $(1-e)R \cong D \cong (1-f)R$. Therefore R is IC, by Lemma 7.2.19. □

Theorem 7.2.22. *Let R be a ring. Then the following are equivalent:*

(1) *R is perspective.*
(2) *If $Ra + Rb = R$ and $aR \oplus X = R$ with $a, b \in R, X \subseteq R_R$, then $br(a)$ and X are perspective.*
(3) *If $Ra + Rb = R$ and $aR \oplus X = R$ with $a, b \in R, X \subseteq R_R$, then there exists an idempotent $e \in R$ such that $a + eb \in U(R)$ and $eR = X$.*
(4) *If $aR \oplus X = R$ with $X \subseteq R_R$, then $r(a)$ and X are perspective.*

Proof. (1) $\Rightarrow$ (2) Suppose $Ra+Rb = R$ and $aR\oplus X = R$ with $a, b \in R, X \subseteq R_R$. Then $a \in R$ is regular. In view of Lemma 7.2.21, R is IC. Write $a = axa$ for some $x \in R$. Since $axR \cong xaR$, we see that $(1-ax)R \cong C \cong (1-xa)R$, by Lemma 7.2.19. Thus, $r(a) \cong R/aR$. As $r(a) \cap r(b) = 0$, we see that $br(a) \cong r(a)$. By hypothesis, $R/aR \cong X$. Therefore $br(a) \cong X$. Since $Ra + Rb = R$, we get $Rb(1 - xa) = R(1 - xa)$, and so $b(1 - xa) \in R$ is regular. This implies that $br(a) = b(1 - xa)R \subseteq^{\oplus} R$. Accordingly, $br(a)$ and X are perspective.

(2) $\Rightarrow$ (3) Suppose $Ra+Rb = R$ and $aR\oplus X = R$ with $a, b \in R, X \subseteq R_R$. Then $br(a)$ and X are perspective. In view of Lemma 7.2.20, there exists an idempotent $e \in R$ such that $eR = X$ and $e\,|_{br(a)} : br(a) \to X$ is an isomorphism. We shall prove that $a + eb \in R$ is invertible. Clearly, $(a + eb)r(a) = X$. As $ebr(a) = X = eR$, we have $k \in r(a)$ such that $eb = ebk$, and then $(a + eb)(1 - k) = a + eb - ak - ebk = a$. Thus, $R = aR \oplus X \subseteq (a + eb)R$. This implies that $(a + eb)R = R$. Write $(a + eb)c = 1$ for a $c \in R$. As $R(a + eb) + R = R$, by hypothesis, we see that $r(a + eb)$ and 0 are perspective. We infer that $r(a + eb) = 0$, and so $1 - c(a + eb) = 0$. Therefore, $c(a + eb) = 1$, and so $a + eb \in U(R)$.

(3) $\Rightarrow$ (4) Taking $b = -1$. Then we have an idempotent $e \in R$ such that $eR = X$ and $a - e \in U(R)$. Hence, $r(a) \cap r(e) = 0$, and so $e : r(a) \to er(a)$ is an isomorphism. We shall show that $er(a) = eR$. Since $a - e \in U(R)$, we can find an $r \in R$ such that $e = (a - e)r$. Hence, $ar \in aR \cap eR = 0$, and so $r \in r(a)$. Thus, $(a - e)r = -er \in er(a)$. This implies that $er(a) = eR$. Therefore $r(a)$ and X are perspective, by Lemma 7.2.20.

(4) $\Rightarrow$ (1) Suppose that $R = A_1 \oplus B = A_2 \oplus C$ with $A_1 \cong A_2$. Then we have idempotents $e, f \in R$ such that $A_1 = eR$ and $A_2 = fR$. Since $eR \cong fR$, it follows by Lemma 7.2.19 that $f = ueu^{-1}$ for $u \in U(R)$. Take $a = u(1 - e)$ and $X = ueR$. Then $aR \oplus X = R$. By hypothesis, $r(a)$ and X are perspective. Therefore eR and fR are perspective. Therefore we have some $D \subseteq R$ such that $R = A_1 \oplus D = A_2 \oplus D$, hence the result. □

Corollary 7.2.23. *Let R be a ring. If R has stable range 1, then every regular element in R is special clean.*

Proof. Suppose $R = A_1 \oplus B = A_2 \oplus C$ with $A_1 \cong A_2$. Then there exists an idempotent $e \in R$ such that $A_1 \cong eR$. Hence, $End(A_1) \cong eRe$. In view of Lemma 6.1.31, $End(A_1)$ has stable range 1. Thus, we can find some $D \subseteq R$ such that $R = A_1 \oplus D = A_2 \oplus D$, by Theorem 6.1.32. Hence, R is perspective. Let $a \in R$ be regular. Set $b = -1$ and $X = (1 - ax)R$. Then $Ra + Rb = R$ and $aR \oplus X = R$. In light of Theorem 7.2.22, there exists an idempotent $e \in R$ such that $a + eb \in U(R)$ and $eR = X$. Hence, $a - e \in U(R)$ and $aR \cap eR = 0$. Therefore $a \in R$ is special clean. □

Lemma 7.2.24. *Let R be an exchange ring. Then the following are equivalent:*

(1) *R has stable range 1.*
(2) *Every regular element in R is unit-regular.*

Proof. (1) ⇒ (2) Let $a \in R$ be regular. Write $a = axa$. Since $xa + (1 - xa) = 1$, we have a $y \in R$ such that $u := x + (1 - xa)y \in U(R)$. Thus, $a = axa = aua$, i.e., $a \in R$ is unit-regular.

(2) ⇒ (1) Given $ax + b = 1$ in R, we have an idempotent $e \in R$ such that $e = bs$ and $1 - e = (1 - b)t$ for some $s, t \in R$. Then $axt + e = 1$; hence, $(1 - e)axt + e = 1$. We infer that $(1 - e)a \in R$ is regular. By hypothesis, there exists $u \in U(R)$ such that $(1 - e)a = (1 - e)au(1 - e)a$. Let $f = u(1 - e)a$. Then $fxtu^{-1} + ueu^{-1} = 1$, and so

$$fxtu^{-1}(1 - f) + ueu^{-1}(1 - f) = 1 - f.$$

Hence,

$$\begin{aligned} & u\big(a + bs(u^{-1}(1 - f) - a)\big) \\ = {} & u\big(a + e\big(u^{-1}(1 - f) - a\big)\big) \\ = {} & u\big((1 - e)a + eu^{-1}(1 - f)\big) \\ = {} & f + ueu^{-1}(1 - f) \\ = {} & 1 - fxtu^{-1}(1 - f) \\ \in {} & U(R). \end{aligned}$$

Therefore $a + bs\big(u^{-1}(1 - f) - a\big) \in U(R)$, as desired. □

Theorem 7.2.25. *Let R be an exchange ring. Then the following are equivalent:*

(1) *R has stable range 1.*
(2) *Every regular $a \in R$ is special clean.*

Proof. (1) $\Rightarrow$ (2) This is obvious by Corollary 7.2.23.

(2) $\Rightarrow$ (1) In view of Proposition 7.2.9, every regular element in R is unit-regular. This completes the proof, by Lemma 7.2.24. □

Corollary 7.2.26. *A ring R is unit-regular if and only if every element in R is special clean.*

Proof. Since every regular ring is an exchange ring, the proof is true by Theorem 7.2.25. □

7.3 Quasi-morphic Rings

For any element a in a ring R, $R/\ell(a) \cong Ra$. Considering the dual of this isomorphism, Nicholson and Campose introduced morphic elements. An element a in a ring R is left morphic if $R/Ra \cong \ell(a)$. As is well known, $a \in R$ is left morphic if and only if there exists some $b \in R$ such that $Ra = \ell(b)$ and $\ell(a) = Rb$. A ring R is left morphic in case every element in R is left morphic. In [104], Camillo and Nicholson extended morphic rings to quasi-morphic rings so as to combine morphic property and regularity. A ring R is called a left quasi-morphic ring provided that for any $a \in R$, there exist $b, c \in R$ such that $Ra = \ell(b)$ and $\ell(a) = Rc$. Clearly, every left morphic element is quasi-morphic. Right (quasi) morphic rings are defined analogously. A right and left (quasi) morphic ring is simply called a (quasi) morphic ring. The purpose of this section is concerned with special clean elements in quasi-morphic rings.

Lemma 7.3.1. *A ring R is quasi-morphic ring if and only if for each $a \in R$ there exist $b, c \in R$ such that*

(1) *$Ra = \ell(b)$ and $\ell(a) = Rc$;*
(2) *$aR = r(c)$ and $r(a) = bR$.*

Proof. $\Longrightarrow$ Let $a \in R$. Since R is left quasi-morphic, there exist $b, c \in R$ such that $Ra = \ell(b)$ and $\ell(a) = Rc$. Always $bR \subseteq r\ell(b)$. Let $x \in r\ell(b)$. As R is right quasi-morphic, there exists $d \in R$ such that $bR = r(d)$. Hence, $db = 0$, and so $d \in \ell(b)$. It follows that $dx = 0$, and then $x \in r(d) = bR$. This implies that $r\ell(b) \subseteq bR$; hence, $r\ell(b) = bR$. It follows that $r(a) = r(Ra) = r\ell(b) = bR$. Furthermore, $aR = r\ell(a) = r(Rc) = r(c)$, as desired.

$\Longleftarrow$ This is obvious. □

Lemma 7.3.2. *Let R be a quasi-morphic ring, and let $a, b \in R$. Then the following are equivalent:*

(1) $aR \oplus bR = (a+b)R$.
(2) $Ra \oplus Rb = R(a+b)$.
(3) $aR \cap bR = 0 = Ra \cap Rb$.

Proof. (1) $\Rightarrow$ (3) Obviously, $aR \cap bR = 0$. By Lemma 7.3.1, there exist some $x, y, s, t \in R$ such that

$$Ra = \ell(x), \ell(a) = Ry; aR = r(y), r(a) = xR;$$
$$Rb = \ell(s), \ell(b) = Rt; bR = r(t), r(b) = sR.$$

Since $aR + bR = (a+b)R$, we have $m, n \in R$ such that $a = (a+b)m$. Thus, $a(1-m) = bm \in aR \cap bR = 0$; hence,$1 - m \in r(a)$ and $m \in r(b)$. Write $1 - m = xk$ and $m = sl$,where $k, l \in R$. Thus, $1 = xk + sl$. For any $q \in Ra \cap Rb$, we get $q \in \ell(x) \cap \ell(s)$. This implies that $qx = qs = 0$; hence, $q = q(xk + sl) = 0$. Therefore $Ra \cap Rb = 0$.

(3) $\Rightarrow$ (1) In view of Lemma 7.3.1, there exist some $x, y \in R$ such that

$$R(a+b) = \ell(x), \ell(a+b) = Ry; (a+b)R = r(y), r(a+b) = xR.$$

Hence, $(a+b)x = 0$, and then $ax = -bx \in aR \cap bR = 0$. This implies that $a, b \in \ell(x)$. It follows that $Ra + Rb \subseteq \ell(x) = R(a+b)$. Clearly, $R(a+b) \subseteq Ra + Rb$. Accordingly, $Ra + Rb = R(a+b)$, as needed.

(2) $\Leftrightarrow$ (3) This is proved in the same manner. □

Let R be a quasi-morphic ring, and let $a, b \in R$. We use $a \leq^{\oplus} b$ to stand for $bR = aR \oplus (b-a)R$. In view of Lemma 7.3.2, it is straightforward to verify that $\leq^{\oplus}$ is a partial order over R. Set $\{a^-\} = \{x \in R \mid a = axa\}$ and $\{a^+\} = \{x \in R \mid a = axa, x = xax\}$. We write x as $a^-(a^+)$ in case $x \in \{a^-\}(\{a^+\})$. In the next, we shall investigate the behavior of regular elements in a quasi-morphic ring by means of such partial order. Our proofs are based on the techniques developed by Blackwood, Jain, Prasad and Srivastava in [73].

Theorem 7.3.3. *Let R be a quasi-morphic ring, and let $a \in R$ be regular. Then the following are equivalent:*

(1) $a \leq^{\oplus} b$.
(2) *There exists some $x \in \{a^+\}$ such that $ax = bx, xa = xb$.*
(3) *There exists some $x \in \{a^-\}$ such that $ax = bx, xa = xb$.*

Proof. (1) $\Rightarrow$ (2) Since $a \leq^{\oplus} b$, $aR \subseteq bR$. Write $a = bs$. Then $a(1-s) = (b-a)s \in aR \cap (b-a)R$, and so $a = as = bs$. Likewise, we can find $t \in R$ such that $a = ta = tb$. As $a \in R$ is regular, there exists $z \in \{a^+\}$ such that $a = aza$. Set $x = szt \in R$. Then $a = aza = (as)z(ta) = axa$. Further, $ax = aszt = bszt = bx$ and $xa = szta = sztb = xb$. It is easy to verify that $xax = sztaszt = szazt = szt = x$, and therefore $x \in \{a^+\}$, as desired.

(2) $\Rightarrow$ (3) This is trivial.

(3) $\Rightarrow$ (1) Suppose that $a = axa, ax = bx$ and $xa = xb$. For any $r \in aR \cap (b-a)R$, then $r = as = (b-a)t$ for some $s, t \in R$. Hence, $xas = x(b-a)t = 0$, and so $as = axas = 0$. This implies that $aR \cap (b-a)R = 0$. Likewise, $Ra \cap R(b-a) = 0$. By virtue of Lemma 7.3.2, $bR = aR \oplus (b-a)R$, i.e., $a \leq^{\oplus} b$. □

Corollary 7.3.4. *Let R be a quasi-morphic ring, and let $a \in R$ be regular. Then the following are equivalent:*

(1) $a \leq^{\oplus} b$.
(2) $b = a + (1 - aa^+)s(1 - a^+a)$ *for some* $s \in R$.

Proof. (1) $\Rightarrow$ (2) In view of Theorem 7.3.3, $a = aa^+a, aa^+ = ba^+$ and $a^+a = a^+b$. Thus, $aa^+(b-a) = 0 = (b-a)a^+a$. Therefore $b - a = (1 - aa^+)(b-a)(1 - a^+a)$, as needed.

(2) $\Rightarrow$ (1) Assume that $b = a + (1 - aa^+)s(1 - a^+a)$ for some $s \in R$. Then $aa^+ = ba^+, a^+a = a^+b$. As $a = aa^+a$, we complete the proof by Theorem 7.3.3. □

Corollary 7.3.5. *Let R be a quasi-morphic ring, and let $a \in R$ be regular. If $b = a + c$, then the following are equivalent:*

(1) $a \leq^{\oplus} b$.
(2) *There exists some* $x \in \{a^-\}$ *such that* $cx = xc = 0$.

Proof. According to Theorem 7.3.3, $a \leq^{\oplus} b$ if and only if there exists some $x \in \{a^-\}$ such that $ax = bx, xa = xb$, i.e., $cx = xc = 0$. This completes the proof. □

Hartwig asked a question: If R is regular and $aR \cap cR = 0 = Ra \cap Rc$, does there exist some a^- such that $ca^- = a^-c = 0$? In [73, Proposition 6], Blackwood et al. gave an affirmative answer. Now we generalize this fact to quasi-morphic rings.

Corollary 7.3.6. *Let R be a quasi-morphic ring, and let $a \in R$ be regular. Then the following are equivalent:*

(1) $aR \cap cR = 0 = Ra \cap Rc$.
(2) *There exists some* a^- *such that* $ca^- = a^-c = 0$.

Proof. (1) $\Rightarrow$ (2) Set $b = a + c$. Then $aR \cap (b-a)R = 0 = Ra \cap R(b-a)$. In view of Lemma 7.3.2, we get $bR = aR \oplus (b-a)R$. This implies that $a \leq^{\oplus} b$. In light of Corollary 7.3.5, there exists some a^- such that $ca^- = a^-c = 0$.

(2) $\Rightarrow$ (1) Write $a = aa^-a$ and $ca^- = a^-c = 0$. For any $r \in aR \cap cR$, say $r = ax = cy$ for some $x, y \in R$. Then $a^-ax = a^-cy = 0$; hence, $ax = aa^-ax = 0$. It follows that $aR \cap cR = 0$. Likewise, $Ra \cap Rc = 0$, as asserted. □

Theorem 7.3.7. *Let R be a quasi-morphic ring. Then the following are equivalent:*

(1) *R is IC.*
(2) *For any regular element $a \in R$, there exists $u \in U(R)$ such that $a \leq^{\oplus} u$.*

Proof. (1) $\Rightarrow$ (2) For any regular element $a \in R$, it follows by Lemma 7.2.19 that $a \in R$ is unit-regular. Hence, we have an idempotent $e \in R$ and a unit $u \in R$ such that $a = eu$. Thus, $uR = R = eR \oplus (1-e)R = euR \oplus (1-e)uR = aR \oplus (u-a)R$. Therefore $a \leq^{\oplus} u$.

(2) $\Rightarrow$ (1) Let $a \in R$ be regular. Then there exists $u \in U(R)$ such that $a \leq^{\oplus} u$. By virtue of Theorem 7.3.3, we have an element $x \in R$ such that $a = axa, ax = ux$ and $xa = xu$. Hence, $a = axa = axu$. This implies that $au^{-1}a = a$, i.e., $a \in R$ is unit-regular. According to Lemma 7.2.19, R is IC. □

Immediately, we prove that a regular ring R is unit-regular if and only if for any $a \in R$, there exists $u \in U(R)$ such that $a \leq^{\oplus} u$.

Lemma 7.3.8. *Let R be a quasi-morphic ring, and let $a, b \in R$ be regular. Then the following are equivalent:*

(1) $a \leq^{\oplus} b$.
(2) $aR \subseteq bR, Ra \subseteq Rb$ *and* $\{a^-\} \cap \{b^-\} \neq \emptyset$.
(3) $aR \subseteq bR, Ra \subseteq Rb$ *and* $\{b^-\} \subseteq \{a^-\}$.

Proof. (1) $\Rightarrow$ (3) By hypothesis, $bR = aR \oplus (b-a)R$. In view of Lemma 7.3.2, $Rb = Ra \oplus R(b-a)$. Thus, $aR \subseteq bR, Ra \subseteq Rb$. Let $g \in \{b^-\}$. Then $b = bgb$. According to Theorem 7.3.3, there exists some $x \in R$ such that $a = axa, ax = bx$ and $xa = xb$. Therefore

$aga = axagaxa = axbgbxa = axbxa = axaxa = a$, and so $g \in \{a^-\}$. Therefore $\{b^-\} \subseteq \{a^-\}$.

(3) $\Rightarrow$ (2) This is obvious.

(2) $\Rightarrow$ (1) Choose $g \in \{a^-\} \cap \{b^-\}$. Then $a = aga$ and $b = bgb$. Since $aR \subseteq bR, Ra \subseteq Rb$, we can find some $s, t \in R$ such that $a = bs$ and $a = tb$. Set $x = gagbgag$. Then $axa = agagbgaga = agbga = tbgbgbs = tbgbs = aga = a$. Further, $ax = agagbgag = agbgag = tbgbgag = tbgag = agag = ag$ and $bx = bgagbgag = bgbsgbgag = bsgbgbsg = agag = ag$. Hence, $ax = bx$. Likewise, $xa = ga = xb$. In light of Theorem 7.3.3, $a \leq^{\oplus} b$. □

Theorem 7.3.9. *Let R be a quasi-morphic ring, and let $a, b \in R$ be regular. Then the following are equivalent:*

(1) $a = aba$.
(2) *There exists $c \in R$ such that $b = bcb, c = cbc$ and $a \leq^{\oplus} c$.*
(3) *There exists $c \in R$ such that $c = cbc$ and $a \leq^{\oplus} c$.*

Proof. (1) $\Rightarrow$ (2) Set $c = a + b^-(b - bab)b^- \in R$. We easily check that $b = bcb, c = cbc$. For any $r \in aR \cap (c-a)R$, say $r = as = b^-(b-bab)b^-t$ for some $s, t \in R$. Then $r = abas = abb^-(b - bab)b^-t = 0$; hence, $aR \cap (c - a)R = 0$. Likewise, $Ra \cap R(c-a) = 0$. According to Lemma 7.3.2, $a \leq^{\oplus} c$, as desired.

(2) $\Rightarrow$ (3) This is trivial.

(3) $\Rightarrow$ (1) Suppose that $c = cbc$ and $a \leq^{\oplus} c$. In view of Lemma 7.3.8, $\{c^-\} \subseteq \{a^-\}$. As $b \in \{c^-\}$, we get $b \in \{a^-\}$. Therefore $a = aba$, as asserted. □

Lemma 7.3.10. *If R is IC, then R is Dedekind-finite.*

Proof. Let R be IC and $xy = 1$. Then $x \in R$ is regular. In view of Lemma 7.2.19, $x \in R$ is unit-regular. Write $x = eu$, where $e = e^2 \in R$ and $u \in U(R)$. Then $euy = 1$, and so $e = 1 - euy(1 - e) \in U(R)$. This implies that $e = 1$; hence, $x = u \in U(R)$. As $xyx = x$, we conclude that $yx = 1$, as asserted. □

Lemma 7.3.11. *Let R be a quasi-morphic ring, and let $a_1, \ldots, a_n \in R$. Then*

(1) $a_1R \cap \cdots \cap a_nR = bR$ *for some* $b \in R$.
(2) $r(a_1) \cap r(a_2) \cap \cdots \cap r(a_n) = r(c)$ *for some* $c \in R$.

Proof. (1) It suffices to prove the result for $n = 2$. Write $a_2R = r(x)$ and $r(xa_1) = yR$. We claim that $a_1R \cap a_2R = (a_1y)R$. As $xa_1y = 0$, we see that

$a_1y \in r(x) = a_2R$. Hence, $(a_1y)R \subseteq a_1R \cap a_2R$. Let $d \in a_1R \cap a_2R$. Write $d = a_1s = a_2t$. Then $a_1s \in r(x)$, i.e., $xa_1s = 0$. Thus $s \in r(xa_1) = yR$; whence, $d \in (a_1y)R$. This implies that $a_1R \cap a_2R \subseteq (a_1y)R$. Set $b = a_1y$. Then $a_1R \cap a_2R = bR$, as required.

(2) By hypothesis, there exist $b_1, \ldots, b_n \in R$ such that each $r(a_i) = b_iR$. It follows from (1) that $b_1R \cap \cdots \cap b_nR = bR$ for some $b \in R$. Further, $bR = r(c)$ for some $c \in R$. Therefore $r(a_1) \cap r(a_2) \cap \cdots \cap r(a_n) = r(c)$, as desired. □

Lemma 7.3.12. *Let R be a quasi-morphic ring, and let $a_1, \ldots, a_n \in R$. Then*

(1) $\ell[r(a_1) \cap r(a_2) \cap \cdots \cap r(a_n)] = Ra_1 + Ra_2 + \cdots + Ra_n$.
(2) $r[\ell(a_1) \cap \ell(a_2) \cap \cdots \cap \ell(a_n)] = a_1R + a_2R + \cdots + a_nR$.

Proof. We prove (1) by induction on n; then (2) follows by similarity. Since R is quasi-morphic, there exist $b, c \in R$ such that $Ra_1 = \ell(b)$ and $\ell(a_1) = Rc$. As $a_1b = 0$, we see that $b \in r(a_1)$; hence, $Ra_1 \subseteq \ell[r(a_1)]$. If $y \in \ell[r(a_1)]$, then $yr(a_1) = 0$, and so $yb = 0$, i.e., $y \in \ell(b)$. Hence $\ell[r(a_1)] \subseteq Ra_1$, and the result holds for $n = 1$.

Assume that the result holds for $n \leq k (k \in \mathbb{N})$. Then

$$\ell[r(a_2) \cap r(a_2) \cap \cdots \cap r(a_n)] = Ra_2 + Ra_2 + \cdots + Ra_n.$$

We now consider $n = k + 1$.

In light of Lemma 7.3.11, $r(a_2) \cap r(a_2) \cap \cdots \cap r(a_n) = r(b)$ for $b \in R$. But $r(b) = cR$ for some $c \in R$. Thus,

$$\begin{aligned} \ell(c) &= \ell[r(a_2) \cap r(a_2) \cap \cdots \cap r(a_n)] \\ &= Ra_2 + \cdots + Ra_n \end{aligned}.$$

Hence,

$$\begin{aligned} \ell[r(a_1) \cap r(a_2) \cap \cdots \cap r(a_{k+1})] &= \ell[r(a_1) \cap cR] \\ &= Ra_1 + \ell(c) \\ &= Ra_1 + Ra_2 + \cdots + Ra_{k+1}. \end{aligned}$$

This completes by induction. □

Theorem 7.3.13. *Let R be a quasi-morphic ring, and let $a, a^2 \in R$ be regular. If R is IC, then $a \in R$ is special clean.*

Proof. Let $a \in R$ be regular. Then we have some $b \in R$ such that $a = aba$. Set $e = ab$. Then $aR = eR$ is a direct summand of R. Clearly, $r(a) = (1 - ba)R$ is a direct summand of R. Set $K = aR \cap r(a)$. As

$a^2 \in R$ is regular, there exists $c \in R$ such that $a^2 = a^2ca^2$. Thus, $r(a^2) = (1 - ca^2)R$. We claim that $K = a(1 - ca^2)R$. Clearly, $a(1 - ca^2)R \subseteq K$. Let $w \in K$, say $w = ar = (1 - ba)s, r, s \in R$. It suffices to show that $r \in (1 - ca^2)R$. That is, $r \in r(a^2)$. But $a^2(r) = a(w) = a(1 - ba)s = 0$, and we are through. It is easy to verify that $a(1 - ca^2)ba(1 - ca^2) = a(ba - ca^2)(1 - ca^2) = a(1 - ca^2)$. Set $f = a(1 - ca^2)b$. Then $K = fR$ is a direct summand of R. Since R is quasi-morphic, it follows by Lemma 7.3.11 that $aR + r(a) = aR + (1 - ba)R = r\big(\ell(a) \cap Ra\big)$. Set $L = \ell(a) \cap Ra$. We claim that $L = R(1 - a^2c)a$. Clearly, $R(1 - a^2c)a \subseteq L$. Let $v \in L$, say $v = ma = n(1-ab), m, n \in R$. It suffices to show that $m \in R(1-a^2c)$. That is, $m \in \ell(a^2)$. But $ma^2 = va = n(1 - ab)a = 0$, and we are through. One easily checks that $(1-a^2c)ab(1-a^2c)a = (1-a^2c)(ab-a^2c)a = (1-a^2c)a$. Set $g = (1 - a^2c)a$. Then $L = Rg$, where $g \in R$ is an idempotent. Hence, $r(L) = r(Rg) = (1 - g)R$. We infer that $aR + r(a)$ is a direct summand of R. Write $R = aR \oplus M$. Then $aR + r(a) = \big(aR + r(a)\big) \cap \big(aR \oplus M\big) = aR \oplus \big(aR + r(a)\big) \cap M$. Clearly, $aR + r(a)$ is projective. By virtue of Lemma 7.2.10, there exists $X \subseteq r(a)$ such that $aR + r(a) = aR \oplus X$. Let $R = aR \oplus X \oplus Y$. Write $R = K \oplus N$. Then $aR = aR \cap \big(K \oplus N\big) = K \oplus Z$, where $Z = aR \cap N$. Thus, $R = K \oplus X \oplus Y \oplus Z$.

As R is IC, $a \in R$ is unit-regular from Lemma 7.2.19. Thus, we see that $r(a) \cong R/aR \cong X \oplus Y$. Furthermore, $r(a) = r(a) \cap \big(aR + r(a)\big) = r(a) \cap \big(aR \oplus X\big) = X \oplus r(a) \cap aR = K \oplus X$. Thus, $\varphi : r(a) = X \oplus K \cong X \oplus Y$. Since $R = baR \oplus X \oplus K = aR \oplus X \oplus Y$ with $baR \cong aR$, we get $\phi : K \cong Y$. Let

$$\begin{aligned} h : R = K \oplus X \oplus Y \oplus Z &\to K \oplus X \oplus Y \oplus Z = R; \\ k + x + y + z &\mapsto \varphi(k + x) + y, \forall\ k \in K, x \in X, y \in Y, z \in Z; \\ v : R = K \oplus X \oplus Y \oplus Z &\to K \oplus X \oplus Y \oplus Z = R; \\ k + x + y + z &\mapsto \varphi^{-1}(x + y) + \phi(k), \forall\ k \in K, x \in X, y \in Y, z \in Z. \end{aligned}$$

For any $k \in K, x \in X, y \in Y, z \in Z$, it is easy to verify that

$$\begin{aligned} & hvhv(k + x + y + z) \\ &= hvh\big(\varphi^{-1}(x + y) + \phi(k)\big) \\ &= hv\big(x + y + \phi(k)\big) \\ &= h\varphi^{-1}\big(x + y + \phi(k)\big) \\ &= x + y + \phi(k) \\ &= hv(k + x + y + z). \end{aligned}$$

Hence $(hv)^2 = hv$. Let $e = hv$. Then $e = e^2 \in End_R(R)$.

Given any $t \in aR, x \in X, y \in Y$, we have $t \in aR = a\big(Y \oplus Z\big)$. So we can find some $y' \in Y$ and $z' \in Z$ such that $t = a(y' + z')$. Choose $x' = -x \in X$

and $k' = -\phi^{-1}(y + y') \in K$. Then

$$\begin{aligned}&(a - hv)(k' + x' + y' + z')\\ &= a(y' + z') - x' - y' - \phi(k')\\ &= t + x - y' - \phi(k')\\ &= t + x + y.\end{aligned}$$

This means that $a - hv \in End_R(R)$ is a right R-epimorphism. As R is a projective right R-module, $a - hv \in End_R(R)$ is right invertible. In view of Lemma 7.3.10, R is Dedekind-finite. Therefore $a - hv$ is invertible. Moreover, we check that $aR \cap eR \subseteq (K + Z) \cap (X + Y)$. Therefore $a \in End_R(R)$ is special clean. □

Corollary 7.3.14. *Let R be a quasi-morphic ring, and let $a, a^2 \in R$ be regular. If R has stable range 1, then $a \in R$ is special clean.*

Proof. Obviously, R is IC. Thus we obtain the result by Theorem 7.3.13. □

Lemma 7.3.15. *Every morphic ring is IC.*

Proof. Let R be a morphic ring, and let $a \in R$ be regular. Then $a = axa$ and $x = xax$ for some $x \in R$. By hypothesis, $R/aR \cong r(a)$, and so $\varphi : (1 - ax)R \cong (1 - xa)R$. Hence, we can find $c \in (1 - ax)R(1 - xa), d \in (1 - xa)R(1 - ax)$ such that $1 - ax = cd$ and $1 - xa = dc$. It is easy to verify that $(a + c)(x + d) = (x + d)(a + c) = 1$, and so $(a + c)^{-1} = x + d$. Moreover, $a = ax(a + c)$, and then $a(a + c)^{-1}a = axa = a$, i.e., $a \in R$ is unit-regular. Therefore R is IC. □

Theorem 7.3.16. *Let R be a morphic ring, and let $a, a^2 \in R$ be regular. Then $a \in R$ is special clean.*

Proof. Clearly, R is quasi-morphic. In view of Lemma 7.3.15, R is IC. Therefore the proof is true from Theorem 7.3.13. □

Immediately, we deduce that every unit-regular ring is special clean. Let $R = T(\mathbb{Z}_6, \mathbb{Z}_6)$. Then R is a morphic ring, while it is not unit-regular. Choose $a = (\overline{2}, \overline{4}) \in R$. Then $a, a^2 \in R$ are regular. In view of Theorem 7.3.16, $a \in R$ is special clean.

Proposition 7.3.17. *Let R be a quasi-morphic ring, and let $a, a^2 \in R$ be regular. If R is IC, then $2a \in R$ is the sum of two invertible elements.*

Proof. Let $a \in R$. Construct h and v as in Theorem 7.3.13. Set $e = hv$. Then $e = e^2 \in R$ and $u_1 := a - e \in U(R)$. Given any $t \in aR, x \in X, y \in Y$,

we have $t \in aR = a\big(Y \oplus Z\big)$. So we can find some $y'' \in Y$ and $z'' \in Z$ such that $t = a(y'' + z'')$. Choose $x'' = x \in X$ and $k'' = \phi^{-1}(y - y'') \in K$. Then

$$\begin{aligned}&(a+e)(k''+x''+y''+z'')\\&= a(y''+z'')+x''+y''+\phi(k'')\\&= t+x+y.\end{aligned}$$

We infer that $a + e \in End_R(R)$ is a right R-epimorphism. Since R is a projective right R-module, $a + e \in End_R(R)$ is right invertible. By Lemma 7.3.10, R is Dedekind-finite. Hence, $u_2 := a + e$ is invertible. We conclude that $2a = (a + hv) + (a - hv) = u_2 + u_1$, as desired. □

Corollary 7.3.18. *Let R be a morphic ring, and let $a, a^2 \in R$ be regular. If $\frac{1}{2} \in R$, then the element a is the sum of two invertible elements.*

Proof. Since R is morphic, it is quasi-morphic and IC. As $\frac{1}{2} \in R$, it follows by Proposition 7.3.17 that $a \in R$ is the sum of two units. □

The following example shows that $\frac{1}{2} \in R$ in Corollary 7.3.18 is not necessary.

Example 7.3.19. Let $GF(4)$ be the Galois field with 4 elements which is isomorphic to $\mathbb{Z}[t]/\big(2, t^2 + t + 1\big)$. Then $GF(4)$ is morphic. For any $\overline{1} \neq \overline{f(t)} \in \mathbb{Z}[t]/\big(2, t^2+t+1\big)$, we see that $\overline{f(t)} = \overline{1}+\big(\overline{f(t)}-\overline{1}\big)$. Additionally, $\overline{1} = \overline{t+1} - \overline{t}$. Thus, every element in $GF(4)$ is the sum of two invertible elements. In this case, $\frac{1}{2} \notin GF(4)$.

7.4 Countably Linear Transformations

The main purpose of this section is to prove that every linear transformation over a countably generated right vector space over a division ring is the sum of an idempotent and an automorphism. That is, the endomorphism ring of all linear transformations over a countably generated right vector space over a division ring is clean.

Let V be a countably generated infinite-dimensional right vector space over a division ring D, and let $\{x_1, x_2, x_3, \ldots\}$ be a basis of V. We say that $\sigma : V \to V$ is a shift operator provided that $\sigma(x_i) = x_{i+1}$ for all $i \in \mathbb{N}$.

Lemma 7.4.1. *Let V be a countably generated infinite-dimensional right vector space over a division ring D, and let $\sigma \in End_D(V)$ be a shift operator. Then there exists $\alpha = \alpha^2 \in End_D(V)$ such that $\sigma - \alpha \in Aut_D(V)$.*

Proof. Let

$$E = \begin{pmatrix} 0 & -1_D \\ 0 & 1_D \end{pmatrix}, L = \begin{pmatrix} 0 & 1_D \\ 0 & 0 \end{pmatrix},$$

and let $A = \begin{pmatrix} E & 0 & 0 & \cdots \\ L & E & 0 & \cdots \\ 0 & L & E & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ which is a column-finite matrix over D. Then $E^2 = E, LE + EL = L$ and $L^2 = 0$. We check that

$$A^2 = \begin{pmatrix} E^2 & 0 & 0 & \cdots \\ LE + EL & E^2 & 0 & \cdots \\ L^2 & LE + EL & E^2 & \cdots \\ 0 & L^2 & LE + EL & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} E & 0 & 0 & \cdots \\ L & E & 0 & \cdots \\ 0 & L & E & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = A.$$

Let

$$K = \begin{pmatrix} 0 & 1_D \\ 1_D & -1_D \end{pmatrix} \text{ and } B = \begin{pmatrix} K & 0 & 0 & \cdots \\ 0 & K & 0 & \cdots \\ 0 & 0 & K & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then B is a column-finite matrix over D and

$$B^{-1} = \begin{pmatrix} K^{-1} & 0 & 0 & \cdots \\ 0 & K^{-1} & 0 & \cdots \\ 0 & 0 & K^{-1} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let $Q = \begin{pmatrix} 0 & 0 \\ 1_D & 0 \end{pmatrix}$. Then

$$A + B = \begin{pmatrix} Q & 0 & 0 & \cdots \\ L & Q & 0 & \cdots \\ 0 & L & Q & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let $\{x_1, x_2, x_3, \ldots\}$ be a basis of V. Define $\alpha, \beta \in End_D(V)$ given by

$$\alpha(x_1, x_2, x_3, \ldots) = (x_1, x_2, x_3, \ldots)A,$$
$$\beta(x_1, x_2, x_3, \ldots) = (x_1, x_2, x_3, \ldots)B.$$

Then $\alpha = \alpha^2 \in End_D(V), \beta \in Aut_D(V)$. Moreover,

$$\begin{aligned}
&(\alpha+\beta)(x_1, x_2, x_3, \ldots)\\
&= (x_1, x_2, x_3, \ldots)(A+B)\\
&= (x_1, x_2, x_3, \ldots)\begin{pmatrix} Q & 0 & 0 & \cdots \\ L & Q & 0 & \cdots \\ 0 & L & Q & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}\\
&= \sigma(x_1, x_2, x_3, \ldots),
\end{aligned}$$

and therefore $\sigma = \alpha + \beta$, as desired. □

Lemma 7.4.2. *Let V be a countably generated right vector space over a division ring D, and let $\gamma \in End_D(V)$ be such that V is spanned by $\{y, \gamma(y), \gamma^2(y), \ldots\}$. Then there exists $\alpha = \alpha^2 \in End_D(V)$ such that $\gamma - \alpha \in Aut_D(V)$.*

Proof. We may assume that $V \neq 0$. If $\gamma^n(y) \notin yD + \gamma(y)D + \gamma^2(y)D + \cdots + \gamma^{n-1}(y)D$ for all $n \in \mathbb{N}$, then $\{y, \gamma(y), \gamma^2(y), \ldots\}$ is a basis of V. Since γ is a shift operator with respect to this basis, by virtue of Lemma 7.4.1, we have $\alpha = \alpha^2 \in End_D(V)$ such that $\gamma - \alpha \in Aut_D(V)$.

Assume that there exists some $n \in \mathbb{N}$ such that $\gamma^n(y) \in yD + \gamma(y)D + \gamma^2(y)D + \cdots + \gamma^{n-1}(y)D$. If n is minimal with this property, then $\{y, \gamma(y), \gamma^2(y), \ldots, \gamma^{n-1}(y)\}$ is a basis of V; hence, $V \cong nD$, and so $End_D(V) \cong M_n(D)$. In light of Corollary 7.1.21, we have $\alpha = \alpha^2 \in End_D(V)$ such that $\gamma - \alpha \in Aut_D(V)$, and we are done. □

Lemma 7.4.3. *Let V be a countably generated right vector space over a division ring D, let $\gamma \in End_D(V)$, and let U be a γ-invariant vector subspace of V. Assume that $V = U + K$ where $K = yD + \gamma(y)D + \gamma^2(y)D + \cdots$ for some $y \in V$. If there exists $\pi = \pi^2 \in End_D(U)$ such that $\gamma|_U - \pi \in Aut_D(U)$, then there exists $\alpha = \alpha^2 \in End_D(V)$ such that $\gamma - \alpha \in Aut_D(V)$ and $\alpha|_U = \pi$.*

Proof. Clearly, there exists a vector subspace M such that $V = M \oplus U$. As U is γ-invariant, we get $\overline{\gamma} : V/U \to V/U$ defined by $\overline{\gamma}(\overline{v}) = \overline{\gamma(v)}$. Let $\theta : V \to V$ be given by $\theta(m+u) = m$ for any $m \in M, u \in U$. Then $\theta(V) = M$ and $\theta^2 = \theta$. This induces a D-isomorphism $\theta_0 : V/U \to M$. Let $\zeta = \theta_0 \overline{\gamma} \theta_0^{-1} \in End_D(M)$. Then $\zeta\theta_0 = \theta_0\overline{\gamma}$, and so $\zeta\theta_0(\overline{v}) = \theta_0\overline{\gamma}(\overline{v})$ for any $v \in V$. This implies that $\zeta\theta(v) = \theta\gamma(v)$. For any $m \in M$, $\zeta(m) = \zeta\theta(m) = \theta\gamma(m)$. Hence $\theta\zeta(m) = \theta^2\gamma(m) = \theta\gamma(m)$. It follows that $\gamma(m) - \zeta(m) \in Ker(\theta) = U$ for all $m \in M$.

By assumption, we see that $\{\overline{y}, \overline{\gamma}(\overline{y}), \overline{\gamma}^{\,2}(\overline{y}), \ldots\}$ spans V/U, and then $\{\theta_0(\overline{y}), \theta_0\big(\overline{\gamma}(\overline{y})\big), \theta_0\big(\overline{\gamma}^{\,2}(\overline{y})\big), \ldots\}$ spans M. Clearly, $\theta_0\big(\overline{\gamma}^{\,n}(\overline{y})\big) = \zeta^n\big(\theta_0(\overline{y})\big)$ for all $n \in \mathbb{N}$. Hence, $\{\theta_0(\overline{y}), \zeta\big(\theta_0(\overline{y})\big), \zeta^2\big(\theta_0(\overline{y})\big), \ldots\}$ spans M. In view of Lemma 7.4.2, there exists $\alpha_1 = \alpha_1^2 \in End_D(M)$ such that $\zeta - \alpha_1 \in Aut_D(M)$. By hypothesis, there exists $\alpha_2 = \alpha_2^2 \in End_D(U)$ such that $\gamma|_U - \alpha_2 \in Aut_D(U)$. Define $\alpha : V \to V$ given by $\alpha(m+u) = \alpha_1(m) + \alpha_2(u)$ for any $m \in M, u \in U$. Then $\alpha = \alpha^2 \in End_D(V)$. Let $\beta_1 = \zeta - \alpha_1$ and $\beta_2 = \gamma|_U - \alpha_2$. Let $\beta : V \to V$ be given by $\beta(m+u) = \beta_1(m) + \big(\beta_2(u) + \gamma(m) - \zeta(m)\big)$ for any $m \in M, u \in U$. One easily checks that

$$\begin{aligned}
&(\alpha+\beta)(m+u)\\
&= \alpha_1(m) + \alpha_2(u) + \beta_1(m) + \big(\beta_2(u) + \gamma(m) - \zeta(m)\big)\\
&= \big(\alpha_1(m) + \beta_1(m)\big) + \big(\alpha_2(u) + \beta_2(u)\big) + \big(\gamma(m) - \zeta(m)\big)\\
&= \zeta(m) + \gamma(u) + \big(\gamma(m) - \zeta(m)\big)\\
&= \gamma(u) + \gamma(m)\\
&= \gamma(m+u).
\end{aligned}$$

for any $m \in M, u \in U$. Thus, $\gamma - \alpha = \beta$. If $\beta(m+u) = 0$, then $\beta_1(m) = -\beta_2(u) - \gamma(m) + \zeta(m) \in M \cap U = 0$; hence, $\beta_1(m) = 0$. This implies that $m = 0$. Further, $-\beta_2(u) = 0$, and so $u = 0$. Thus β is a D-monomorphism. Obviously, $U \subseteq Im(\beta)$. It suffices to show that $M \subseteq Im(\beta)$. If $m \in M$, we can find $m_0 \in M$ such that $m = \beta_1(m_0)$. Also we have $u_0 \in U$ such that $-(\gamma(m_0) - \zeta(m_0)) = \beta_2(u_0)$. Clearly,

$$\begin{aligned}
&\beta(m_0 + u_0)\\
&= \beta_1(m_0) + \big(\beta_2(u_0) + \gamma(m_0) - \zeta(m_0)\big)\\
&= m.
\end{aligned}$$

Thus, β is a D-epimorphism, and so $\gamma - \alpha = \beta \in Aut_D(V)$. This yields the result. □

We now come to our main theorem in this section.

Theorem 7.4.4. *The endomorphism ring of all linear transformations over a countably generated right vector space over a division ring is clean.*

Proof. Let V be a countably generated right vector space over a division ring D, and let $\gamma \in End_D(V)$. Set $\Omega = \{(U, \sigma, \pi) \mid U_D \subseteq V$ is γ − invariant, $\gamma\ |_U - \sigma = \pi, \sigma = \sigma^2 \in End_D(U), \pi \in Aut_D(U)\}$. Clearly, Ω is not empty. Partially order Ω by writing $(U, \sigma, \pi) \leq (U', \sigma', \pi')$ if $U \subseteq U', \sigma = \sigma'\ |_U, \pi = \pi'\ |_U$. Given $(U_1, \sigma_1, \pi_1) \leq (U_2, \sigma_2, \pi_2) \leq \cdots$ in Ω,

then $\bigcup_{i=1}^{\infty} U_i$ is a γ-invariant vector subspace of V. Define $\sigma : \bigcup_{i=1}^{\infty} U_i \to \bigcup_{i=1}^{\infty} U_i$ given by $\sigma(x_i) = \sigma_i(x_i)$ for any $x_i \in U_i$ and $\pi : \bigcup_{i=1}^{\infty} U_i \to \bigcup_{i=1}^{\infty} U_i$ given by $\pi(x_i) = \pi_i(x_i)$ for any $x_i \in U_i$. It is easy to verify that σ and π are well defined. Furthermore, we see that

$$\gamma \mid_{\bigcup_{i=1}^{\infty} U_i} -\sigma = \pi, \sigma = \sigma^2 \in End_D\left(\bigcup_{i=1}^{\infty} U_i\right), \pi \in Aut_D\left(\bigcup_{i=1}^{\infty} U_i\right).$$

Thus, $\left(\bigcup_{i=1}^{\infty} U_i, \sigma, \pi\right) \in \Omega$, and so Ω is inductive. By using Zorn's Lemma, there exists $(W, \alpha, \beta) \in \Omega$ which is maximal in Ω. If $W \neq V$, then we can find some $y \in V - W$. Let $K = yD + \gamma(y)D + \gamma^2(y)D + \ldots$, and let $W_0 = W + K$. Obviously, W_0 is γ-invariant. In view of Lemma 7.4.3, we see that $W_0 \in \Omega$, a contradiction. Therefore $V = W$, as desired. □

Notes. Theorem 7.1.10 was proved in [297, Proposition 6]. Theorem 7.1.14 was proved in [569, Theorem 8]. Theorem 7.1.19 is due to [297, Theorem]. Theorem 7.1.24 was observed in [356, Theorem 3.2] for $n = 2$. Theorem 7.2.4 was proved by Nicholson [430, Theorem 2.1]. Theorem 7.2.7 is a direct consequence of [430, Corollary 2.6]. Theorem 7.2.22 is due to Grag et al. [276, Theorem 4.2]. Theorem 7.4.4 was first proved by Nicholson and Varadarajan [433, Theorem].

This page intentionally left blank

Chapter 8

Rings Generated by Certain Elements

The problem that motivates this chapter is to study rings generated additively by their special elements (e.g., units, idempotents, etc). We start by determining when a ring is generated by their two units. This provides various types of 2-good rings. In Section 8.2, we are concerned with perspective rings R and explore conditions on R under which every regular element is the sum of two units. In Section 8.3, we study elementary divisor rings and investigate the equivalent conditions under which a commutative ring is an elementary divisor ring. This provides a large class of 2-good matrix rings. In the final section we explore when a ring is generated by their idempotents and nilpotents. Many nil-clean and 2-nil-clean matrix rings are thereby obtained.

8.1 2-Good and 2-Clean Rings

The aim of this section is to study rings generated additively by their two units. The rings in which each element is the sum of $k (k \geq 2)$ units are called k-good rings. Clearly, every triangular matrix ring over a 2-good ring is also a 2-good ring. Furthermore, we have

Lemma 8.1.1. *Let R be a ring. Then every $n \times n$ diagonal matrix over R is the sum of two invertible matrices for any $n \geq 2$.*

Proof. Let $diag(d_1, \ldots, d_n) \in M_n(R)$. Then we easily check that

$$\begin{pmatrix} d_1 & 0 \cdots & 0 & 0 \\ 0 & d_2 \cdots & 0 & 0 \\ \vdots & \vdots \ddots & \vdots & \vdots \\ 0 & 0 \cdots & d_{n-1} & 0 \\ 0 & 0 \cdots & 0 & d_n \end{pmatrix} = \begin{pmatrix} d_1 & 1 \cdots & 0 & 0 \\ 0 & d_2 \cdots & 0 & 0 \\ \vdots & \vdots \ddots & \vdots & \vdots \\ 0 & 0 \cdots & d_{n-1} & 1 \\ 1 & 0 \cdots & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \cdots & 0 & 0 \\ 0 & 0 \cdots & 0 & 0 \\ \vdots & \vdots \ddots & \vdots & \vdots \\ 0 & 0 \cdots & 0 & -1 \\ -1 & 0 \cdots & 0 & d_n \end{pmatrix},$$

so the lemma is proved. □

By Theorem 2.4.15 and Lemma 8.1.1, every strongly clean $n \times n$ matrix over a projective-free ring is the sum of two invertible matrices for all $n \geq 2$. Furthermore, we have

Theorem 8.1.2. *Let R be a ring. Then $M_n(R)$ is a 3-good ring for any $n \geq 2$.*

Proof. We will show that every $n \times n$ matrix over R is the sum of a diagonal matrix and an invertible matrix. We proceed by induction on n. If $n = 1$, then $a = (a - 1) + 1$ for any $a \in R$, as required. Suppose the preceding claim holds for $n = k (k \geq 1)$. Let $n = k + 1$, and let $A \in M_n(R)$. Write $A = \begin{pmatrix} x & \alpha \\ \beta & X \end{pmatrix}$, where $x \in R, \alpha \in M_{1\times k}(R), \beta \in M_{k\times 1}(R)$ and $X \in M_k(R)$. Thus, we can find $u \in R$ such that $x - u = v \in U(R)$. By hypothesis, we have a diagonal matrix $U \in M_k(R)$ such that $\left(X - \beta v^{-1}\alpha\right) - U = V \in GL_k(R)$. Hence

$$A - \begin{pmatrix} u & 0 \\ 0 & U \end{pmatrix} = \begin{pmatrix} v & \alpha \\ \beta & V + \beta v^{-1}\alpha \end{pmatrix}.$$

It is easy to check that $\begin{pmatrix} v & \alpha \\ \beta & V + \beta v^{-1}\alpha \end{pmatrix} \in GL_n(R)$. Therefore we complete the proof, by induction and Lemma 8.1.1. □

In the sequel, we are concerned with rings in which every element is the sum of two units.

Proposition 8.1.3. *Let T be the ring of a Morita context (A, B, M, N, ψ, ϕ). If A and B are 2-good rings, then so is T.*

Proof. Let $A = \begin{pmatrix} a & n \\ m & b \end{pmatrix} \in T$. Since A is a 2-good ring, we have $u_1 \in U(A)$ such that $a - u_1 = u_2 \in U(A)$. Similarly, there exists $v_1 \in U(B)$ such that

$b - v_1 = v_2 \in U(B)$. Hence

$$A = \begin{pmatrix} u_1 & n \\ 0 & v_1 \end{pmatrix} + \begin{pmatrix} u_2 & 0 \\ m & v_2 \end{pmatrix}.$$

One easily checks that

$$\begin{pmatrix} u_1 & n \\ 0 & v_1 \end{pmatrix}, \begin{pmatrix} u_2 & 0 \\ m & v_2 \end{pmatrix} \in U(T).$$

So the proposition is true. □

Corollary 8.1.4. *If R is a 2-good ring, then so is $M_n(R)$.*

Proof. By induction, it is obvious from Proposition 8.1.3. □

Let R be a 2-good ring. Meehan proved that the endomorphism ring of a free R-module is 2-good (see [418]). The class of 2-good rings is an elementary class. It is closed under the formation of factor rings, direct products, etc. The next result shows that the 2-good property needs to be checked only for some special rings, e.g. indecomposable rings.

Proposition 8.1.5. *Let R be a ring. Then the following are equivalent*:

(1) *R is a 2-good ring.*
(2) *R/I is a 2-good ring, where $I \subseteq J(R)$.*
(3) *Every indecomposable factor ring of R is a 2-good ring.*

Proof. (1) $\Rightarrow$ (2) This is obvious.

(2) $\Rightarrow$ (1) Let $a \in R$. Then we can find some $\overline{u}, \overline{v} \in U(R/I)$ such that $\overline{a} = \overline{u+v}$ in R/I. As $I \subseteq J(R)$, we easily check that $u, v \in U(R)$. Set $r = a - (u + v)$. Then $r \in I \subseteq J(R)$, and that $a = u + (v + r)$. Clearly, $v + r \in U(R)$. Therefore $a \in R$ is the sum of two units, as desired.

(1) $\Rightarrow$ (3) This is clear.

(3) $\Rightarrow$ (1) Suppose that R is not a 2-good ring. Then there exists some $a \in R$ such that $a - u \notin U(R)$ for any $u \in U(R)$. We can use Zorn's Lemma to pick an ideal I of R maximal with respect to the property a is not the sum of two units in R/I. Then R/I is decomposable as a ring. Hence, $R/I \cong R/I_1 \times R/I_2$, where both ideals I_1, I_2 strictly contain I. Thus, $a + I_1 \in R/I_1$ and $a + I_2 \in R/I_2$ are sum of two units, respectively. It follows that $a + I \in R/I$ is the sum of two units, a contradiction. This completes the proof. □

Corollary 8.1.6. *Let R be a ring. Then the following are equivalent:*

(1) *R is a 2-good ring.*
(2) *$R[[x_1, \ldots, x_n]]$ is a 2-good ring.*
(3) *$T(R, M)$ is a 2-good ring, where M is an an R-R-bimodule.*

Proof. (1) $\Rightarrow$ (2) It will suffice to prove that $R[[x]]$ is a 2-good ring. Let $\varphi : R[[x]] \to R, f(x) \mapsto f(0)$. Then $R \cong R[[x]]/Ker(\varphi)$, where $Ker(\varphi) \subseteq J\big(R[[x]]\big)$. In light of Proposition 8.1.5, $R[[x]]$ is a 2-good ring.

(2) $\Rightarrow$ (1) Let $a \in R$. Then we have $u(x_1, \ldots, x_n), v(x_1, \ldots, x_n) \in U\big(R[[x_1, \ldots, x_n]]$ such that $a = u(x_1, \ldots, x_n) + v(x_1, \ldots, x_n)$. Hence, $a = u(0, \ldots, 0) + v(0, \ldots, 0)$ where $u(0, \ldots, 0), v(0, \ldots, 0) \in U(R)$. This proves (1).

(1) $\Rightarrow$ (3) Let $\varphi : T(R, M) \to R, (r, m) \mapsto r$. Then we have $R \cong T(R, M)/Ker(\varphi)$, where $Ker(\varphi) \subseteq J\big(T(R, M)\big)$. Thus, $T(R, M)$ is a 2-good ring, in terms of Proposition 8.1.5.

(3) $\Rightarrow$ (1) Let $a \in R$. Then we can find $(u, m), (v, n) \in U\big(T(R, M)\big)$ such that $(a, 0) = (u, m) + (v, n)$. Therefore $a = u + v$ and $u, v \in U(R)$, as desired. □

Theorem 8.1.7. *Let R be a semilocal ring. Then the following are equivalent*:

(1) *R is a 2-good ring.*
(2) *There exist $\alpha, \beta \in U(R)$ such that $\alpha + \beta = 1$.*
(3) *R has no homomorphic image $\mathbb{Z}/2\mathbb{Z}$.*

Proof. (1) $\Rightarrow$ (2) This is obvious.

(2) $\Rightarrow$ (3) If R has $\mathbb{Z}/2\mathbb{Z}$ as a homomorphic image, then there exists an ideal I of R such that $R/I \cong \mathbb{Z}/2\mathbb{Z}$. Clearly, $\overline{\alpha + \beta} = \overline{1}$ in R/I, a contradiction. Thus, proving (3).

(3) $\Rightarrow$ (1) Since R is semilocal, there exist division rings $D_1, \ldots, D_m$ such that $R/J(R) \cong \bigoplus_{i=1}^{m} M_{n_i}(D_i)$. If $n_i \geq 2$, then $M_{n_i}(D_i)$ is a 2-good ring by Lemma 8.1.1. If $n_i = 1$ and $D_i \cong \mathbb{Z}/2\mathbb{Z}$, then $R/J(R)$ has $\mathbb{Z}/2\mathbb{Z}$ as a homomorphic image. This gives a contradiction. If $n_i = 1$ and $D_i \ncong \mathbb{Z}/2\mathbb{Z}$, then $|D_i| \geq 3$. For any $u \in U(D_i)$, we choose $v \notin \{0, u\}$. Then $u - v \in U(D_i)$. That is, D_i is a 2-good ring. Therefore $R/J(R)$ is a 2-good ring, and then so is R. □

Corollary 8.1.8. *Let A be an artinian right R-module. If $\frac{1}{2} \in R$, then $End_R(A)$ is a 2-good ring.*

Proof. Construct an R-morphism $\varphi : A \to A$ given by $\varphi(a) = a \cdot \frac{1}{2}$ for any $a \in A$. It is easy to verify that $\varphi \in Aut_R(A)$. In addition, $1_A = \varphi + \varphi$. According to the Camps-Dicks Theorem [109, Corollary 6], $End_R(A)$ is semilocal. Therefore the result follows from Theorem 8.1.7. □

Example 8.1.9. Let $R = \{\frac{m}{n} \mid (m,n) = 1, m, n \in \mathbb{Z} \text{ and } 3, 5, 7 \nmid n\}$. Then R is a 2-good ring.

Proof. Let M be an ideal of R such that $3R \subsetneq M \subseteq R$. Choose $\frac{m}{n} \in M$ while $\frac{m}{n} \notin 3R$. Then $3 \nmid m$, and so $(3, m) = 1$. Hence $3k + lm = 1$ for some $k, l \in \mathbb{Z}$. This shows that $\frac{1}{1} = 3 \cdot \frac{k}{1} + l\frac{m}{n} \cdot \frac{n}{1} \in M$, i.e., $M = R$. Thus, $3R$ is a maximal ideal of R. Likewise, $5R$ and $7R$ are maximal ideals of R. For any $\frac{m}{n} \in 3R \cap 5R \cap 7R$ and $\frac{a}{b} \in R$, then $\frac{1}{1} - \frac{m}{n}\frac{a}{b} = \frac{nb-ma}{nb}$. Write $\frac{m}{n} = \frac{3s}{t}$. Then $3sn = mt$, and so $3 \mid mt$. Since $3 \nmid t$, we get $3 \mid m$. Obviously, $3 \nmid nb$; hence, $3 \nmid (nb - ma)$. Similarly, $5, 7 \nmid (nb - ma)$. It follows that $\frac{nb}{nb-ma} \in U(R)$. We infer that $\frac{m}{n} \in J(R)$. Therefore $3R \cap 5R \cap 7R \subseteq J(R)$. Let M be a maximal ideal of R and $M \neq 3R, 5R, 7R$. Then $3R + M = R, 5R + M = R$ and $7R + M = R$. Thus, $R = (3R+M)(5R+M)(7R+M) \subseteq 3R \cap 5R \cap 7R + M = J(R) + M \subseteq M$, hence, $R = M$, an absurd. We infer that R is a commutative ring with exactly three maximal ideals, and so R is semilocal. Obviously $2 \in R$ is invertible. Therefore R is a 2-good ring, by Theorem 8.1.7. □

Theorem 8.1.10. *Let R be an exchange ring with artinian primitive factors. Then the following are equivalent*:

(1) *R is a 2-good ring.*
(2) *There exist $\alpha, \beta \in U(R)$ such that $\alpha + \beta = 1$.*
(3) *R has no homomorphic image $\mathbb{Z}/2\mathbb{Z}$.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) These are clear.

(3) $\Rightarrow$ (1) Assume that there exists some $x \in R$ such that $x - u \notin U(R)$ for any $u \in U(R)$. Let Ω be the set of all ideals I of R such that $x - u$ is not a unit modulo A for any $\overline{u} \in U(R/A)$. Similarly to Theorem 7.1.16, Ω is a non-empty inductive set. By using Zorn's Lemma, there exists an ideal Q of R which is maximal in Ω.

Set $S = R/Q$. By the maximality of Q, one easily checks that R/Q is an indecomposable ring with $J(R/Q) = 0$. It follows by [563, Lemma 3.7] that R/Q is a simple artinian ring. That is, $R/Q \cong M_n(D)$ for a division ring D. Since R has no isomorphic image $\mathbb{Z}/2\mathbb{Z}$, we see that either $n = 1, |D| \geq 3$ or $n \geq 2$. In any case, R/Q is a 2-good ring. Hence, we have $w + Q \in U(R/Q)$

such that $(x-w)+Q \in U\big(A/Q\big)$, a contradiction. Therefore, R is a 2-good ring. □

Corollary 8.1.11. *Let R be an exchange ring of bounded index. Then the following are equivalent*:

(1) *R is a 2-good ring.*
(2) *There exist $\alpha, \beta \in U(R)$ such that $\alpha + \beta = 1$.*
(3) *R has no homomorphic image $\mathbb{Z}/2\mathbb{Z}$.*

Proof. As in the proof of Corollary 7.1.17, R is an exchange ring with artinian primitive factors. Accordingly, we complete the proof by Theorem 8.1.10. □

Corollary 8.1.12. *Let R be an exchange ring with all idempotents central. Then the following are equivalent*:

(1) *R is a 2-good ring.*
(2) *There exist $\alpha, \beta \in U(R)$ such that $\alpha + \beta = 1$.*
(3) *R has no homomorphic image $\mathbb{Z}/2\mathbb{Z}$.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) These are obvious.

(3) $\Rightarrow$ (1) In view of [493, Lemma 4.10], $R/J(R)$ is an exchange ring of bounded index 1. It follows by Corollary 8.1.11 that $R/J(R)$ is a 2-good ring, and therefore the result follows. □

Corollary 8.1.13. *Let R be an exchange ring with artinian primitive factors. Then $M_n(R)(n \geq 2)$ is a 2-good ring.*

Proof. Since R is an exchange ring with artinian primitive factors, so is $M_n(R)$ by Theorem 7.2.7. In the ring $M_n(R)$, we see that

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -1 \\ -1 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

By Theorem 8.1.10, the result follows. □

Corollary 8.1.14. *Let R be strongly nil-clean. Then*:

(1) *R is a 2-good ring if and only if 1_R is the sum of two units.*
(2) *$M_n(R)$ is a 2-good ring for all $n \geq 2$.*

Proof. (1) Since R is strongly nil-clean, it follows by Theorem 5.1.5 that $R/J(R)$ is Boolean, and so it is an exchange ring with all idempotents central. Thus proving (1), by Corollary 8.1.12 and Proposition 8.1.5.

(2) Clearly, $R/J(R)$ is an exchange ring of bounded index. Thus, $R/J(R)$ is an exchange ring with artinian primitive factors. By virtue of Corollary 8.1.13, $M_n(R/J(R))$ is 2-good for all $n \geq 2$. Therefore we complete the proof by Proposition 8.1.5. □

In view of Proposition 7.1.2, every clean ring in which 2 is invertible is a 2-good ring. Let R be a unit-regular ring or a strongly π-regular ring. If $\frac{1}{2} \in R$, we then prove that R is a 2-good ring. Furthermore, every (real or complex) Banach algebra R is a 2-good ring, for if $a \in R$ then $a - \lambda 1$ is invertible for some non-zero scalar λ by the spectral theorem. Here, we list more examples of 2-good rings.

Example 8.1.15. Let X be a topological space. Then $C(X)$ is a 2-good ring.

Proof. Let $f \in C(X)$. Then $f = \big((f \vee 0) + 1\big) + \big((f \wedge 0) - 1\big)$. Since f is continuous, then the function $|f|$ is also continuous. Since $f \vee 0 = 2^{-1}(f + |f|)$, we see that $f \vee 0$ is continuous, and so $(f \vee 0) + 1 \in C(X)$ is invertible. Likewise, $(f \wedge 0) - 1 \in C(X)$ is invertible. This completes the proof. □

Example 8.1.16. Every algebraic algebra over an infinite field is a 2-good ring.

Proof. Let R be an algebraic algebra over an infinite field F. For any $a \in R$, there exists a non-constant polynomial $p(x) \in F[x]$ such that $p(a) = 0$. Let $n = degp(x)$, and let S be the set of all roots of $p(x)$ in $F[x]$. Then $| S | \leq n$. Let $\alpha \in F - S$. Then $p(\alpha) \neq 0$. Set $q(x) = p(x + \alpha) = \sum_{i=0}^{n} q_i x^i$ with each $q_i \in F$. Then 0 is not a root of $q(x)$, and so $q_0 \neq 0$. Thus, $0 = q\big(a - \alpha \cdot 1_R\big) = q_0 \cdot 1_R + \sum_{i=1}^{n} q_i \big(a - \alpha \cdot 1_R\big)^i$. Therefore, $a - \alpha \cdot 1_R \in U(R)$, as required. □

A commutative ring R is said to be U-irreducible if the set of polynomials in $R[x]$ which represents units in R is closed under multiplication.

U-irreducible rings include many classes of rings such as commutative semilocal rings with infinite residue fields, commutative regular rings with infinite residue fields and algebraically closed integral domains having stable range 1.

Example 8.1.17. Every U-irreducible ring is a 2-good ring.

Proof. Let R be a U-irreducible ring, and let $a \in R$. Set $f(x) = 1 + ax, g(x) = x \in R[x]$. Then $f(0) = 1 = g(1)$. By the U-irreducibility of R, we can find some $r \in R$ such that $f(r)g(r) = (1 + ar)r \in U(R)$. Hence, $1 + ar, r \in U(R)$. Therefore, $a + r^{-1} \in U(R)$, as desired. □

Recall that a commutative ring R satisfies the primitive criterion if for each polynomial $f(x) = a_0 + a_1x + \cdots + a_nx^n$ $(n \geq 0)$ with $a_0R + \cdots + a_nR = R$, i.e., $f(x) \in R[x]$ is primitive, then there exists $\alpha \in R$ such that $f(\alpha) \in U(R)$. The ring of continuous cross section of a sheaf of rings of many units over a Boolean space satisfies the primitive criterion. Hence, a zero-dimensional ring satisfies the primitive criterion if and only if its residue class fields are infinite. The ring of all algebraic integers in the field of complex numbers satisfies the primitive criterion.

Let S be a multiplicatively closed subset of a commutative ring R, i.e. for any s and $t \in S$, the product st is also in S, and $0 \notin S$ and $1 \in S$. We define an equivalence relation on $R \times S$ by setting $(r_1, s_1) \sim (r_2, s_2)$ if there exists $t \in S$ such that $t(r_1s_2 - r_2s_1) = 0$. Regard the equivalence class of (r, s) as the "fraction" $\frac{r}{s}$, then the set $S^{-1}R = \{\frac{r}{s} \mid r \in R, s \in S\}$ can be turned into a ring with operations $\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{r_1s_2 + r_2s_1}{s_1s_2}$, $\frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \frac{r_1r_2}{s_1s_2}$.

Let R be any commutative ring and $S = \{f \mid f \in R[x]$ is primitive $\}$. Then the ring of fractions $R(x) := S^{-1}R[x]$ satisfies the primitive criterion (see [249]).

Example 8.1.18. Let R be a commutative ring. If R satisfies the primitive criterion, then R is a 2-good ring.

Proof. Suppose that R satisfies the primitive criterion. Let $f_1(x), f_2(x) \in R[x]$ such that $f_1(a), f_2(b) \in U(R)$. Then $f_1(x), f_2(x)$ are primitive in $R[x]$. Let $h(x) = f_1(x)f_2(x)$. Then $h(x) = \sum_{i=0}^{m} c_ix^i$ for some $c_0, \ldots, c_m \in R$. If $h(x) \in R[x]$ is not primitive, then $c_0R + \cdots + c_mR \subsetneqq R$. So we can find some maximal ideal M of R such that $c_0R + \cdots + c_mR \subseteqq M \subsetneqq R$. Write

$f_i(x) = \sum_{j=1}^{n_i} a_{ij}x^j$. Then each $\overline{f_i}(x) = \sum_{j=1}^{n_i} \overline{a_{ij}}x^j$ is primitive in $(R/M)[x]$. As R/M is a field, we know that $\overline{h}(x) = \sum_{i=0}^{m} \overline{c_i}x^i$ is primitive in $(R/M)[x]$. This implies that $\sum_{i=0}^{m} \overline{c_i}(R/M) = R/M$, a contradiction. Therefore $h(x) \in R[x]$ is primitive. By hypothesis, there exists an element $y \in R$ such that $h(y) \in U(R)$, i.e., $f_1(y)f_2(y) \in U(R)$. Thus, R is a U-irreducible ring. This completes the proof, by Example 8.1.17. □

Let R be a subring of S. We say that R is a unit closed subring if $U(R) = R \cap U(S)$. Clearly, R is a unit closed subring of $R(x)$. Thus, every commutative ring R is a unit closed subring of 2-good ring $R(x)$, by Example 8.1.18. Furthermore, we have

Theorem 8.1.19. *Every ring can be embedded as a unit closed subring in a 2-good ring.*

Proof. Let $\varphi : R \to M_2(R), r \mapsto diag(r, r)$. Then $R \cong Im(\varphi) \subseteq M_2(R)$, and so we may assume that $R \subseteq M_2(R)$. Likewise, we have

$$R \subseteq M_2(R) \subseteq M_4(R) \subseteq \cdots \subseteq M_{2^n}(R) \subseteq \cdots .$$

Set $S = \bigcup_{i=0}^{\infty} M_{2^i}(R)$. Then R is a unit closed subring of S. For any $a \in S$, then $a \in M_{2^m}(R)$ for some $m \in \mathbb{N}$, and so $diag(a, a) \in M_{2^{m+1}}(R)$. In view of Lemma 8.1.1, $diag(a, a)$ is the sum of two units in $M_{2^{m+1}}(R)$, and therefore a is the sum of two units in S, as desired. □

We next focus on 2-good endomorphism rings over countably generated infinitely dimensional right vector spaces. In [571, Theorem], Zelinsky proved that every linear transformation over a division ring is the sum of two automorphisms with the exception of the identity transformation on a space of two elements. For countably linear transformations, we will prove Zelinsky's result by means of the decompositions of matrices.

Lemma 8.1.20. *Let V be a countably generated infinite-dimensional right vector space over a division ring D, and let $\sigma \in End_D(V)$ be a shift operator. Then there exist $\alpha, \beta \in Aut_D(V)$ such that $\sigma = \alpha + \beta, \alpha^2 = \beta^2 = 1_V$.*

Proof. Let

$$C = \begin{pmatrix} 1_D & 0 \\ 0 & -1_D \end{pmatrix}, L = \begin{pmatrix} 0 & 1_D \\ 0 & 0 \end{pmatrix}, E = \begin{pmatrix} 1_D & 0 \\ 0 & 1_D \end{pmatrix},$$

and let $A = \begin{pmatrix} C & 0 & 0 & \cdots \\ L & C & 0 & \cdots \\ 0 & L & C & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$. Then A is a column-finite matrix over D and

$$C^2 = E, LC = -CL = \begin{pmatrix} 0 & -1_D \\ 0 & 0 \end{pmatrix} \text{ and } L^2 = 0.$$

Thus,

$$A^2 = \begin{pmatrix} C^2 & 0 & 0 & 0 & \cdots \\ LC + CL & C^2 & 0 & 0 & \cdots \\ L^2 & LC + CL & C^2 & 0 & \cdots \\ 0 & L^2 & LC + CL & C^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} E & 0 & 0 & \cdots \\ 0 & E & 0 & \cdots \\ 0 & 0 & E & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let

$$K = \begin{pmatrix} -1_D & 0 \\ 1_D & 1_D \end{pmatrix} \text{ and } B = \begin{pmatrix} K & 0 & 0 & \cdots \\ 0 & K & 0 & \cdots \\ 0 & 0 & K & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then B is a column-finite matrix over D and

$$B^2 = \begin{pmatrix} K^2 & 0 & 0 & \cdots \\ 0 & K^2 & 0 & \cdots \\ 0 & 0 & K^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} E & 0 & 0 & \cdots \\ 0 & E & 0 & \cdots \\ 0 & 0 & E & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let $Q = \begin{pmatrix} 0 & 0 \\ 1_D & 0 \end{pmatrix}$. Then

$$A + B = \begin{pmatrix} Q & 0 & 0 & \cdots \\ L & Q & 0 & \cdots \\ 0 & L & Q & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let $\{x_1, x_2, x_3, \ldots\}$ be a basis of V. Define $\alpha, \beta \in End_D(V)$ given by

$$\alpha(x_1, x_2, x_3, \ldots) = (x_1, x_2, x_3, \ldots)A,$$
$$\beta(x_1, x_2, x_3, \ldots) = (x_1, x_2, x_3, \ldots)B.$$

Then $\alpha^2 = \beta^2 = 1_V$. Furthermore, we have

$$\begin{aligned}
&(\alpha + \beta)(x_1, x_2, x_3, \ldots) \\
&= (x_1, x_2, x_3, \ldots)(A + B) \\
&= (x_1, x_2, x_3, \ldots) \begin{pmatrix} Q & 0 & 0 & \cdots \\ L & Q & 0 & \cdots \\ 0 & L & Q & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\
&= \sigma(x_1, x_2, x_3, \ldots),
\end{aligned}$$

and therefore $\sigma = \alpha + \beta$. □

Lemma 8.1.21. *Let D be a division ring, and let $\mid D \mid \neq 2$. Then $M_n(D)$ is a 2-good ring.*

Proof. Let $x \in D$. If $x \neq 0$, then we choose $u \notin \{0, x\}$. Hence, $x - u \in D^*$. If $x = 0$, then $x - 1_D \in D^*$. Thus, D is a 2-good ring. By Corollary 8.1.4, the result follows. □

Lemma 8.1.22. *Let V be a countably generated right vector space over a division ring D, and let $\gamma \in End_D(V)$ be such that V is spanned by $\{y, \gamma(y), \gamma^2(y), \ldots\}$. If $\mid D \mid \neq 2$, then there exists $\alpha \in Aut_D(V)$ such that $\gamma - \alpha \in Aut_D(V)$.*

Proof. We may assume that $V \neq 0$. If $\gamma^n(y) \notin yD + \gamma(y)D + \gamma^2(y)D + \cdots + \gamma^{n-1}(y)D$ for all $n \in \mathbb{N}$, then $\{y, \gamma(y), \gamma^2(y), \ldots\}$ is a basis of V. Since γ is a shift operator with respect to this basis, it follows from Lemma 8.1.20 that there exists $\alpha \in Aut_D(V)$ such that $\gamma - \alpha \in Aut_D(V)$. Assume that there exists some $n \in \mathbb{N}$ such that $\gamma^n(y) \in yD + \gamma(y)D + \gamma^2(y)D + \cdots + \gamma^{n-1}(y)D$. If n is minimal with this property, then $\{y, \gamma(y), \gamma^2(y), \ldots, \gamma^{n-1}(y)\}$ is a basis of V; hence, $V \cong nD$, and so $End_D(V) \cong M_n(D)$. According to Lemma 8.1.21, we complete the proof. □

Lemma 8.1.23. *Let V be a countably generated right vector space over a division ring D where $\mid D \mid \neq 2$, let $\gamma \in End_D(V)$, let U be a γ-invariant*

vector subspace of V, and let $V = U + K$ where $K = yD + \gamma(y)D + \gamma^2(y)D + \cdots$ for some $y \in V$. If there exists $\pi \in Aut_D(U)$ such that $\gamma\mid_U - \pi \in Aut_D(U)$, then there exists $\alpha \in Aut_D(V)$ such that $\gamma - \alpha \in Aut_D(V), \alpha\mid_U = \pi$.

Proof. Construct $M, \theta, \theta_0, \zeta$ as in the proof of Lemma 7.4.3. Then $\gamma(m) - \zeta(m) \in Ker(\theta) = U$ for all $m \in M$. In addition, $\{\theta_0(\overline{y}), \zeta\big(\theta_0(\overline{y})\big), \zeta^2\big(\theta_0(\overline{y})\big), \ldots\}$ spans M. In view of Lemma 8.1.22, $\zeta = \pi_0 + \sigma_0, \pi_0, \sigma_0 \in Aut_D(M)$. By hypothesis, we have $\gamma\mid_U = \pi + \sigma, \pi, \sigma \in Aut_D(U)$. Define $\alpha : V \to V$ given by $\alpha(m+u) = \pi_0(m) + \pi(u)$ for any $m \in M, u \in U$ and $\beta : V \to V$ given by $\beta(m+u) = \sigma_0(m) + \big(\sigma(u) + \gamma(m) - \zeta(m)\big)$ for any $m \in M, u \in U$. Clearly, $\alpha \in Aut_D(V)$ and $\alpha\mid_U = \pi$. As in the proof of Lemma 7.4.3, we have $(\alpha + \beta)(m + u) = \gamma(m + u)$. for any $m \in M, u \in U$. Hence, $\gamma = \alpha + \beta$. If $\beta(m + u) = 0$, then $\sigma_0(m) = -\sigma(u) - \gamma(m) + \zeta(m) \in M \cap U = 0$; hence, $m = 0$. This implies that $u = 0$. Obviously, $U \subseteq Im(\beta)$. It suffices to show that $M \subseteq Im(\beta)$. If $m \in M$, we can find $m_0 \in M$ such that $m = \sigma_0(m_0)$. Also we have $u_0 \in U$ such that $\zeta(m_0) - \gamma(m_0) = \sigma(u_0)$. Then $\beta(m_0 + u_0) = m$. Therefore $\beta \in Aut_D(V)$, as required. □

Theorem 8.1.24. *Let V be a countably generated right vector space over a division ring with more than two elements. Then $End_D(V)$ is a 2-good ring.*

Proof. Let V be a countably generated right vector space over a division ring D, where $\mid D \mid \neq 2$. Let $\gamma \in End_D(V)$. Set $\Omega = \{(U, \sigma, \pi) \mid U_D \subseteq V$ is $\gamma -$ invariant, $\gamma\mid_U = \sigma + \pi, \sigma, \pi \in Aut_D(U)\}$. Clearly, Ω is not empty. Partially order Ω by writing $(U, \sigma, \pi) \leq (U', \sigma', \pi')$ if $U \subseteq U', \sigma = \sigma'\mid_U, \pi = \pi'\mid_U$. Given $(U_1, \sigma_1, \pi_1) \leq (U_2, \sigma_2, \pi_2) \leq \cdots$ in Ω, then $\bigcup_{i=1}^{\infty} U_i$ is a γ-invariant vector subspace of V. Define $\sigma : \bigcup_{i=1}^{\infty} U_i \to \bigcup_{i=1}^{\infty} U_i$ given by $\sigma(x_i) = \sigma_i(x_i)$ for any $x_i \in U_i$ and $\pi : \bigcup_{i=1}^{\infty} U_i \to \bigcup_{i=1}^{\infty} U_i$ given by $\pi(x_i) = \pi_i(x_i)$ for any $x_i \in U_i$. It is easy to verify that σ and π are well defined. Furthermore, we see that

$$\gamma\mid_{\bigcup_{i=1}^{\infty} U_i} - \sigma = \pi, \sigma, \pi \in Aut_D\left(\bigcup_{i=1}^{\infty} U_i\right).$$

Thus, $\left(\bigcup_{i=1}^{\infty} U_i, \sigma, \pi\right) \in \Omega$, and so Ω is inductive. By using Zorn's Lemma, there exists $(W, \alpha, \beta) \in \Omega$ which is maximal in Ω. If $W \neq V$, then we have some $y \in V - W$. Let $K = yD + \gamma(y)D + \gamma^2(y)D + \cdots$, and let $W_0 = W + K$. Then W_0 is γ-invariant. In view of Lemma 8.1.23, $W_0 \in \Omega$, a contradiction. Thus, $V = W$, as needed. □

We use $U_2(R)$ to stand for the set of all invertible elements or those elements which is the sum of two invertible elements in R.

Lemma 8.1.25. *Let R be a ring, and let $n \geq 2$. Then $X := \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in M_n(R)$ is the sum of two invertible matrices over R if and only if*

(1) *there exists some* $\begin{pmatrix} a_1 & * & \cdots & * \\ a_2 & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ a_n & * & \cdots & * \end{pmatrix} \in GL_n(R)$, *and*

(2) $1 + \sum_{i=1}^{n} x_i a_i \in U(R)$.

Proof. $\Longrightarrow$ Write $X = U + V$ with $U, V \in GL_n(R)$. Then $XU^{-1} = I_n + VU^{-1}$, and so $I_n - XU^{-1} = -VU^{-1}$. Set $-U^{-1} = \begin{pmatrix} a_1 & * & \cdots & * \\ a_2 & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ a_n & * & \cdots & * \end{pmatrix}$.

Then $I_n + X(-U^{-1}) = \begin{pmatrix} 1 + \sum_{i=1}^{n} x_i a_i & * & \cdots & * \\ & 1 & & \\ & & & \\ & & \vdots & \\ & & & 1 \end{pmatrix} \in GL_n(R)$. Therefore, $1 + \sum_{i=1}^{n} x_i a_i \in U(R)$.

$$\Longleftarrow \text{Set } U = -\begin{pmatrix} a_1 & * & \cdots & * \\ a_2 & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ a_n & * & \cdots & * \end{pmatrix}^{-1}. \text{ Then}$$

$$-(X-U)U^{-1} = I_n + X(-U^{-1}) = \begin{pmatrix} 1+\sum_{i=1}^{n} x_i a_i & * & \cdots & * \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \in GL_n(R),$$

and so $X - U \in GL_n(R)$. This completes the proof. □

Theorem 8.1.26. *Let R be a ring and let $n \geq 2$. If there exist prime ideals $P_1, \ldots, P_n$ and elements $x_1, \ldots, x_n$ such that*

(1) $x_i \notin P_i, x_i \in \bigcap_{j \neq i} P_j$ *for* $i = 1, \ldots, n$, *and*

(2) $(P_1 + \cdots + P_n) \cap U_2(R) = \{0\}$,

then $\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in M_n(R)$ *is not the sum of two invertible matrices over R.*

Proof. Suppose that the preceding matrix is the sum of two invertible matrices. In view of Lemma 8.1.25, we can find a matrix $\begin{pmatrix} a_1 & * & \cdots & * \\ a_2 & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ a_n & * & \cdots & * \end{pmatrix} \in GL_n(R)$ such that $1 + \sum_{i=1}^{n} x_i a_i \in U(R)$. Clearly, $x_1 \in P_2 + \cdots + P_n$ and $x_2 \in P_1 + P_3 + \cdots + P_n, \ldots, x_n \in P_1 + P_2 + \cdots + P_{n-1}$. Thus, $\sum_{i=1}^{n} x_i a_i \in P_1 + \cdots + P_n$. By hypothesis, $\sum_{i=1}^{n} x_i a_i = 0$, and so $x_i a_i = -\sum_{j \neq i} x_j a_j \in P_i$. But $x_i \notin P_i$, hence, $a_i \in P_i$. Then $P_1 + \cdots + P_n = R$. This forces that $U_2(R) = 0$, a contradiction. This completes the proof. □

Example 8.1.27. Let $R = F[x_1, \ldots, x_n]$ where F is a field and $n \geq 2$. Then we claim that $M_n(R)$ is not a 2-good ring. Choose $P_i = \sum_{j \neq i} x_j R$.

Then $R/P_i \cong F[x_i]$, and so P_i is prime for each i. $x_i \notin P_i$ and $x_i \in \bigcap_{j \neq i} P_j$. Obviously, $(P_1 + \cdots + P_n) \cap U_2(R) = \{0\}$. In view of Theorem 8.1.26, $\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in M_n(R)$ is not the sum of two invertible matrices over R. Therefore $M_n(R)$ is not a 2-good ring.

Let $a, b \in R$. We say that (a, b) is an Hermite pair if there exists $Q \in GL_2(R)$ such that $(a, b)Q = (*, 0)$. We can derive

Proposition 8.1.28. *Let R be a ring, and let $a, b \in R$. If*

(1) $(aR + bR) \cap U_2(R) = \{0\}$, *and*
(2) $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ *is the sum of two invertible matrices over R,*

then (a, b) is an Hermite pair.

Proof. By Lemma 8.1.25, we can find a matrix $\begin{pmatrix} c & * \\ d & * \end{pmatrix} \in GL_2(R)$ such that $1 + ac + bd \in U(R)$. It follows by (1) that $ac + bd = 0$, and so

$$(a, b) \begin{pmatrix} c & * \\ d & * \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = (*, 0).$$

This completes the proof. □

Example 8.1.29. Let $R = \{\alpha + \beta\sqrt{-5} \mid \alpha, \beta \in \mathbb{Z}\}$. Then the only units of R are 1 and -1. Hence, $U_2(R) = \{0, 1, -1, -2, 2\}$. We check that $\left(3R + (2 + \sqrt{-5})R\right) \cap U_2(R) = \{0\}$ and $3R + (2 + \sqrt{-5})R$ is not a principal ideal of R, and thus $(3, 2 + \sqrt{-5})$ is not an Hermite pair. In light of Proposition 8.1.28, $\begin{pmatrix} 3 & 2 + \sqrt{-5} \\ 0 & 0 \end{pmatrix} \in M_2(R)$ is not the sum of two invertible matrices. Therefore $M_2(R)$ is not a 2-good ring.

Let n denote a fixed positive integer. A ring R is called an n-clean ring if every element can be written as the sum of an idempotent and n units. The class of these rings contains clean rings and n-good rings. In the next, we investigate elementary properties of n-clean rings, in particular, 2-clean rings. We have

Proposition 8.1.30. (1) *Every homomorphic image of an n-clean ring is n-clean.*

(2) *A direct product* $R = \prod_i R_i$ *of ring* R_i *is n-clean if and only if each* R_i *is n-clean.*

(3) R *is n-clean if and only if so is* $R/P(R)$.

Corollary 8.1.32. *Let* R *be a ring. Then* $R[[x]]$ *is n-clean if and only if so is* R.

Proof. $\Longrightarrow$ Let $\varphi : R[[x]] \to R, f(x) \mapsto f(0)$. Then $R \cong R[[x]]/Ker(\varphi)$. This shows that R is n-clean, by Proposition 8.1.30.

$\Longleftarrow$ Let $f(x) \in R[[x]]$. Then we can find an idempotent $e \in R$ and n units $u_i \in R$ such that $f(0) = e+u_1+\cdots+u_n$. Hence, $f(x) = e+u_1+\cdots+u_{n-1}+w$, where $w := u_n + \big(f(x) - f(0)\big)$. Since $w \in R[[x]]$ is invertible, the result follows. □

Example 8.1.33. Let R be a commutative ring. Then $R[x]$ is not 2-clean.

Proof. If $R[x]$ is 2-clean, then we can find an idempotent $f \in R[x]$ such that $x - f$ is the sum of 2 units in $R[x]$. Clearly, $f \in R$ and $U(R[x]) = \{a_0 + a_1x + \cdots + a_mx^m \mid a_0 \in U(R[x]), a_1, \ldots, a_m \in N(R)\}$. Thus, 1 is the 2 sum of nilpotents in R, and so $1 \in R$ is nilpotent. This gives a contradiction. Therefore $R[x]$ is not 2-clean. □

Proposition 8.1.34. *Let* R *be an n-clean clean ring. Then it is m-clean for all* $m \geq n$.

Proof. Let $a \in R$. Then we can find an idempotent $e \in R$ and n units $u_1, \ldots, u_n \in R$ such that $a = e + u_1 + \cdots + u_n$. Hence, $a = (1 - e) + u_1 + \cdots + u_n + (2e - 1)$ and $(2e - 1)^2 = 1$. This shows that R is $(n + 1)$-clean. By iteration of this process, we complete the proof. □

As an immediate consequence, we prove that every clean ring is an n-clean ring for all $n \in \mathbb{N}$. A ring R is weakly clean if, for any $a \in R$, either a or $-a$ is clean. We have

Corollary 8.1.35. *Every weakly clean rings is n-clean for all* $n \geq 2$.

Proof. Let R be weakly clean, and let $a \in R$. Then we have an idempotent $e \in R$ and a unit $u \in R$ such that $a = \pm e + u$. If $a = e + u$, then $a = (1 - e) + u + (2e - 1)$. If $a = -e + u$, then $a = (1 - e) + u - 1$. Thus, R is 2-clean, hence the result by Proposition 8.1.34. □

Theorem 8.1.36. *Let $T = (A, B, M, N, \psi, \varphi)$. If A and B are n-clean, then so is T.*

Proof. Let $\begin{pmatrix} a & n \\ m & b \end{pmatrix} \in T$. Then we can find an idempotent $e \in A$ and n units $u_1, \ldots, u_n \in A$ such that $a = e + u_1 + \sum_{i=2}^{n} u_i$. Furthermore, we have an idempotent $f \in R$ and n units $v_1, \ldots, v_n \in R$ such that $b - \varphi(mu_1^{-1} \otimes n) = f + v_1 + \sum_{i=2}^{n} v_i$. Hence,

$$\begin{pmatrix} a & n \\ m & b \end{pmatrix} = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} + \begin{pmatrix} u_1 & n \\ m & v_1 + \varphi(mu_1^{-1} \otimes n) \end{pmatrix} + \sum_{i=2}^{n} \begin{pmatrix} u_i & 0 \\ 0 & v_i \end{pmatrix}.$$

This completes the proof. □

Thus, every matrix ring over n-clean rings is n-clean. As two easy consequences, we have

Corollary 8.1.37. *Let R be an n-clean ring, and let $s \in C(R)$. Then $M_{(s)}(R)$ is n-clean.*

Corollary 8.1.38. *Let R be an n-clean ring, and let $m \in \mathbb{N}$. Then $T_m(R)$ is n-clean.*

Lemma 8.1.39. *Let R be a ring. Then $M_2(R)$ is 2-clean.*

Proof. Let $A = (a_{ij}) \in M_2(R)$. Choose $E = \begin{pmatrix} a_{11} - 1 & 2 - a_{11} \\ a_{11} - 1 & 2 - a_{11} \end{pmatrix}$. Then $E = E^2 \in M_2(R)$. Clearly,

$$A - E = \begin{pmatrix} 1 & a_{12} + a_{11} - 2 \\ a_{21} - a_{11} + 1 & a_{22} + a_{11} - 2 \end{pmatrix}.$$

Thus, we can find $U, V \in GL_2(R)$ and $c \in R$ such that

$$U(A - E)V = \begin{pmatrix} 1 & \\ & c \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -1 & c \end{pmatrix}.$$

Therefore A is the sum of E and two invertible matrices, as desired. □

Lemma 8.1.40. *Let R be a ring. Then $M_3(R)$ is 2-clean.*

Proof. Let $A = (a_{ij}) \in M_3(R)$. Choose

$$E = \begin{pmatrix} a_{11}-1 & a_{22}-1 & 3-a_{11}-a_{22} \\ a_{11}-1 & a_{22}-1 & 3-a_{11}-a_{22} \\ a_{11}-1 & a_{22}-1 & 3-a_{11}-a_{22} \end{pmatrix}.$$

Then $E = E^2 \in M_3(R)$. Clearly,

$$A-E = \begin{pmatrix} 1 & a_{12}-a_{22}+1 & a_{13}+a_{11}+a_{22}-3 \\ a_{21}-a_{11}+1 & 1 & a_{23}+a_{11}+a_{22}-3 \\ a_{31}-a_{11}+1 & a_{32}-a_{22}+1 & a_{33}+a_{11}+a_{22}-3 \end{pmatrix}.$$

Thus, we can find $U, V, W \in GL_3(R)$ such that

$$UV(A-E)W = \begin{pmatrix} * & 0 & * \\ * & 1 & 0 \\ 0 & * & * \end{pmatrix} = \begin{pmatrix} 0 & 1 & * \\ 0 & 0 & 1 \\ 1 & * & * \end{pmatrix} + \begin{pmatrix} * & -1 & 0 \\ * & 1 & -1 \\ -1 & 0 & 0 \end{pmatrix},$$

which is the sum of two invertible matrices. Therefore A is the sum of an idempotent matrix and two invertible matrices, as needed. □

Lemma 8.1.41. *Let R be a ring. Then $M_n(R)$ is 2-clean for all $n \geq 2$.*

Proof. Let $n \geq 2$. We only have three cases. Case I. $n = 3m-1 = 3(m-1)+2$. Case II. $n = 3m$. Case III. $n = 3m+1 = 3(m-1)+2\times 2$. In any case, $M_n(R)$ is 2-clean, by Lemma 8.4.39, Lemma 8.4.40, Theorem 8.1.36 and Proposition 8.1.34. □

Theorem 8.1.42. *Let R be a ring. Then the endomorphism ring of a free R-module of rank at least 2 is 2-clean.*

Proof. Case I. $F = nR$. Then $End_R(F) \cong M_n(R)$. In view of Lemma 8.4.41, $End_R(F)$ is 2-clean.

Case II. $F = \bigoplus_{i=1}^{\infty} R_i$ with each $R_i \cong R$. Then $F \cong 2F$, and so $End_R(F) \cong M_2\big(End_R(F)\big)$. In light of Lemma 8.1.39, $End_R(F)$ is 2-clean. □

Let R be a ring. The collection of all $\mathbb{N}\times\mathbb{N}$ matrices over R is denoted by $M_{\mathbb{N}}(R)$. In general, $M_{\mathbb{N}}(R)$ may not be a ring as the usual matrix multiplication may not be well defined. If we let $RFM_{\mathbb{N}}(R)$ be the subset of $M_{\mathbb{N}}(R)$ consisting of all matrices with many nonzero row entries and $CFM_{\mathbb{N}}(R)$ consisting of all matrices with many nonzero column entries. Then $RFM_{\mathbb{N}}(R)$ and $CFM_{\mathbb{N}}(R)$ are both rings with the point-wise addition and ordinary matrix multiplication.

Corollary 8.1.43. *Let R be a ring. Then $RFM_{\mathbb{N}}(R)$ and $CFM_{\mathbb{N}}(R)$ are 2-clean.*

Proof. This is obvious, as $RFM_{\mathbb{N}}(R) \cong CFM_{\mathbb{N}}(R) \cong End_R\Big(\bigoplus_{i\in\mathbb{N}} R_i\Big)$ with each $R_i \cong R$. □

We now use $B(R)$ to denote the ring of all matrices which are both row-finite and column-finite. Then the row-finite and column-finite matrix ring $B(R)$ is 2-clean. We refer the reader to [535, Theorem 8] for the details.

8.2 Perspective Rings

In this section we shall characterize when every regular element in a perspective ring is the sum of two units. We start by some ring-theoretic results on perspective rings that have significance for our subsequent investigations.

Lemma 8.2.1. *Let R be IC. If $eU(R)e \subseteq U(eRe)$ for any idempotent $e \in R$, then R is perspective.*

Proof. Suppose that $aR \oplus X = R$. Then $a \in R$ is regular. Write $a = axa$. Then $r(a) = fR$ where $f = 1 - xa = f^2 \in R$. Clearly, $R = xaR \oplus fR = aR \oplus X$ with $xaR \cong aR$. Since R is IC, we see that $fR \cong X$. Furthermore, we can find $u \in U(R)$ such that $X = ufR$. In view of Theorem 7.2.22, it suffices to prove that $R = X \oplus (1-f)R = r(a) \oplus (1-f)R$. If $x = ufr \in (1-f)R$, then $fx = fufr = 0$. By hypothesis, $fuf \in fU(R)f \subseteq U(fRf)$, we see that $fr = 0$. This implies that $x = 0$, and so $X \cap (1-f)R = 0$. As $fuf \in U(fRf)$, we have $fvf \in fRf$ such that $fufvf = fvfuf = f$. Hence, $f(1 - fufvf) = 0$. For any $r \in R$, we easily check that $r = ufvfr + (1 - ufvf)r$ where $ufvfr \in X$ and $(1 - ufvf)r \in (1-f)R$. Therefore $R = X + (1-f)R$, as required. □

Theorem 8.2.2. *Let R be a ring such that $eR(1-e) \subseteq J(R)$ for any idempotent $e \in R$. Then R is perspective.*

Proof. Let $e, f \in R$ be idempotents such that $eR \cong fR$. Set $\overline{R} = R/J(R)$. Then $\overline{eR} \cong \overline{fR}$. Hence, there are some $\overline{a} \in \overline{eRf}, \overline{b} \in \overline{fRe}$ such that $\overline{e} = \overline{ab}$ and $\overline{f} = \overline{b}$. By hypothesis, $\overline{e}, \overline{f} \in \overline{R}$ are central. Thus, $\overline{e} = \overline{ab} = \overline{abab} = \overline{(ba)(ab)} = \overline{b(ab)a} = \overline{ba} = \overline{f}$, and so $\overline{1-e} = \overline{1-f}$. It follows that $\overline{(1-e)R} \cong \overline{(1-f)R}$; hence, $(1-e)R/(1-e)RJ(R) \cong (1-f)R/(1-f)RJ(R)$.

By using Lemma 1.2.2, we get $(1-e)R \cong (1-f)R$. Therefore R is IC, in terms of Lemma 7.2.19.

Let $eue \in eU(R)e$ with $u \in U(R)$. Then $eu(1-e) \subseteq J(R)$; hence, $w := eu - eue \in J(R)$. This implies that $e(u-w) = eue$, and so $(eue)e(u-w)^{-1}e = e$. Thus, $eue \in eRe$ is right invertible. Likewise, $eue \in eRe$ is left invertible. Hence, $eU(R)e \subseteq U(eRe)$. In light of Theorem 8.2.1, R is perspective. □

Corollary 8.2.3. *Every right or left quasi-duo ring is a perspective ring.*

Proof. Let R be a right quasi-duo ring. If $x^2 = 0$, we shall prove that $x \in J(R)$; otherwise, $x \notin M$ for a maximal ideal M. Hence, $xR + M = R$. Write $xr + m = 1$ with $r \in R, m \in M$. Then M is an ideal, and so $x = x^2mr + xm = xm \in M$. This gives a contradiction. We infer that $x \in J(R)$. Thus, every nilpotent in R is contained in $J(R)$. For any idempotent $e \in R$, we get $eR(1-e) \subseteq J(R)$. Therefore we complete the proof, by Theorem 8.2.2. □

Lemma 8.2.4. *Let M be a right R-module, and let $E = End(M)$. Then the following are equivalent*:

(1) *M is perspective.*
(2) *For any split epimorphism $\varphi : M \to eM$ with $e^2 = e \in E$, there exists a splitting ψ such that $M = Im(\psi) \oplus (1-e)M$.*
(3) *For any split epimorphism $\varphi : M \to eM$ with $e^2 = e \in E$, there exists a splitting ψ such that $e\psi$ is an automorphism of eM.*

Proof. (1) ⇒ (2) Suppose that $\varphi : M \to eM, \psi : eM \to M, \varphi\psi = 1_{eM}$ with $e^2 = e \in E$. As $M = eM \oplus (1-e)M = eM \oplus Ker(\varphi)$, we see that $Ker(\varphi) \cong (1-e)M$. By hypothesis, we can find a right R-module D such that $M = Ker(\varphi) \oplus D = (1-e)M \oplus D$. Let $p : M \to D$ be the projection onto D. Set $\alpha = \varphi p|_{eM} : eM \to eM$. It is easy to verify that $\varphi p(eM) = eM$ and $Ker(\varphi p) \cap eM = 0$. Then α is an automorphism of eM. Set $\beta = \alpha^{-1}$. Then $\psi := p\beta$ is a splitting of φ, as $\varphi(p\beta) = 1_{eM}$. For any $x \in D$, we see that $\psi(ex) = p\beta(ex) = p(x) = x$; hence, $Im(\psi) = D$. Thus, we get $M = Im(\psi) \oplus (1-e)M$, as required.

(2) ⇒ (1) Suppose that $\alpha : eM \cong fM$ for two idempotents $e, f \in E$. Let $\pi : M \to eM$ be the projection onto eM. Then $\varphi = \alpha\pi : M \to fM$ is a split epimorphism with $Ker(\varphi) = (1-e)M$. By hypothesis, there exists a splitting ψ of φ such that $M = Im(\psi) \oplus (1-f)M$. As $\varphi\psi = 1_{eM}$, we have $M = Im(\psi) \oplus Ker(\varphi) = Im(\psi) \oplus (1-e)M$. Hence, $(1-e)M$ and $(1-f)M$

has the common complement. This implies that $(1-e)M \cong (1-f)M$. We now start with an isomorphism from $(1-e)M \cong (1-f)M$, as above, we prove that eM and fM has the common complement, as desired.

(2) $\Leftrightarrow$ (3) This is obvious, as $M = Im(\psi) \oplus (1-e)M$ if and only if $e\psi$ is an automorphism. □

Lemma 8.2.5. *Let R be a ring. Then the following are equivalent*:

(1) *R is perspective.*
(2) *${}_RR$ is a perspective left R-module.*

Proof. (1) $\Rightarrow$ (2) Let $e, f \in R$ be idempotents such that $Re \cong Rf$. Then $eR \cong fR$. By hypothesis, there exists a right R-module C such that $R = eR \oplus C = fR \oplus C$. Hence, $(1-e)R \cong R/eR \cong C \cong R/fR \cong (1-f)R$. Thus, we have a right R-module D such that $R = (1-e)R \oplus D = (1-f)R \oplus D$. Write $D = gR$ for an idempotent $g \in R$. Since $R = (1-e)R \oplus gR$, we easily check that $R = \ell(g) \oplus \ell(1-e) = R(1-g) \oplus Re$. Likewise, $R = R(1-g) \oplus Rf$. Therefore Re and Rf has the common complement $R(1-g)$, as desired.

(2) $\Rightarrow$ (1) This is symmetric. □

Theorem 8.2.6. *Let R be a ring. Then the following are equivalent*:

(1) *R is perspective.*
(2) *If $ese = e$ for some $e^2 = e, r, s \in R$, then $erte = e$ for some $t \in R$ such that $ete \in U(eRe)$.*
(3) *If $ese = e$ for some $e^2 = e, r, s \in R$, then $etse = e$ for some $t \in R$ such that $ete \in U(eRe)$.*

Proof. (1) $\Leftrightarrow$ (2) By identifying elements of $Hom(R, eR)$ with eR and that of $Hom(eR, R)$ with Re, we can prove this equivalence in terms of Lemma 8.2.4.

(1) $\Leftrightarrow$ (3) The left-right symmetry of the perspective property in rings can be seen from Lemma 8.2.5. □

As consequences of Theorem 8.2.6, we now derive

Corollary 8.2.7. *A ring R is a perspective ring if and only if so is the opposite ring R^{op}.*

Corollary 8.2.8. *Let M be a right R-module. Then $End(M)$ is a perspective ring if and only if M is a perspective R-module.*

Corollary 8.2.9. *Every corner of a perspective ring is perspective.*

Proof. Let R be perspective and $e^2 = e \in R$. It will suffice to prove that eR is a perspective right R-module. Suppose that $eR = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1 \cong A_2$. Then $R = A_1 \oplus B_1 \oplus (1-e)R = A_2 \oplus B_2 \oplus (1-e)R$. Since R is perspective, there exists a right R-module D such that $R = A_1 \oplus D = A_2 \oplus D$. Hence, $eR = eR \cap (A_1 \oplus D) = A_1 \oplus eR \cap D$. Likewise, $eR = A_2 \oplus eR \cap D$. Therefore eR is a perspective right R-module. By using Corollary 8.2.8, eRe is perspective, as asserted. □

Proposition 8.2.10. *Let I be an ideal of a ring R. Then:*

(1) *If $I \subseteq J(R)$ and R/I is a perspective ring, then R is a perspective ring.*

(2) *If $I \subseteq J(R)$ and idempotents lift modulo $J(R)$, then R/I is a perspective ring if and only if so is R.*

Proof. (1) Suppose $erse = 1$ with $e^2 = e, r, s \in R$. Then $\overline{erse} = \overline{1}$ in R/I. In virtue of Theorem 8.2.6, there exists $t \in R$ such that $\overline{erte} = \overline{1}$ and $\overline{ete} \in U(\overline{eRe})$. Hence, $1 - erte \in J(R)$, and so $u := erte \in \big(1 + J(R)\big) \cap eRe \subseteq U(eRe)$. Write $uv = vu = e$ for a $v \in eRe$. Then $er(tv)e = e$. Assume that $\overline{(ewe)(ete)} = \overline{(ete)(ewe)} = \overline{e}$ in $\overline{eRe}$. Then $e - (ewe)(ete), e - (ete)(ewe) \in J(R) \cap eRe = J(eRe)$. Hence, $ete \in U(eRe)$. This implies that $e(tv)e = (ete)v \in U(eRe)$. By using Theorem 8.2.6 again, R is perspective.

(2) One direction is clear by (1). We now assume that R is perspective. Suppose $\overline{erse} = \overline{1}$ with $\overline{e}^2 = \overline{e}, \overline{r}, \overline{s} \in R/I$. By hypothesis, we may assume that $e^2 = e$. Set $u = (1-e) + erse$. Then $u \in U(R)$; hence, $er(seu^{-1})e = 1$. Since R is perspective, it follows by Theorem 8.2.6 that $erve = 1$ and $eve \in U(eRe)$ for $v \in R$. Hence, $\overline{erve} = \overline{1}$ and $\overline{eve} \in U(\overline{eRe})$. Therefore we complete the proof by Theorem 8.2.6. □

Corollary 8.2.11. *Let R be a ring. Then the following are equivalent:*

(1) *R is a perspective ring.*
(2) *$R[[x]]$ is a perspective ring.*
(3) *$T_n(R)$ is a perspective ring.*

Proof. (1) $\Rightarrow$ (2) Let $\varphi : R[[x]] \to R, f(x) \mapsto f(0)$. Then $R \cong R[[x]]/Ker\varphi$ with $Ker\varphi \subseteq J\big(R[[x]]\big)$. By virtue of Proposition 8.2.10, $R[[x]]$ is perspective.

(2) $\Rightarrow$ (1) Let $erse = e$ with $e^2 = e, r, s \in R$. Then we can find some $t(x) \in R[[x]]$ such that $ert(x)e = e$ and $et(x)e \in U\big(R[[x]]\big)$. It follows that

$ert(0)e = e$ and $et(0)e \in U(R)$. In light of Theorem 8.2.6, R is perspective.

(1) $\Rightarrow$ (3) Set $I = \left\{ \begin{pmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ & 0 & a_{23} & \cdots & a_{2n} \\ & & & \ddots & \vdots \\ & & & & 0 \end{pmatrix} \;\middle|\; a_{ij} \in R(i<j) \right\}$. Then I is an ideal of $T_n(R)$ and $I \subseteq J\big(T_n(R)\big)$. Since $T_n(R)/I \cong R \times R \cdots \times R$ is perspective, $T_n(R)$ is perspective, by Proposition 8.2.10.

(3) $\Rightarrow$ (1) Let $erse = e$ with $e^2 = e, r, s \in R$. Then $diag(erse, \ldots, erse) = diag(e, \ldots, e)$. By hypothesis, we have some $T = (t_{ij}) \in M_n(R)$ such that $diag(er, \ldots, er)Tdiag(e, \ldots, e) = diag(e, \ldots, e)$ and

$$diag(e, \ldots, e)Tdiag(e, \ldots, e) \in U\big(diag(e, \ldots, e)T_n(R)diag(e, \ldots, e)\big).$$

Therefore, $ert_{11}e = e$ and $et_{11}e \in U(eRe)$. In light of Theorem 8.2.6, R is perspective. □

Theorem 8.2.12. *Let R be a ring. Then the following are equivalent*:

(1) *R has stable range 1.*
(2) *$M_2(R)$ is a perspective ring.*
(3) *$2R$ is a perspective right R-module.*

Proof. (1) $\Rightarrow$ (2) Since R has stable range 1, so has $M_2(R)$, as desired.

(2) $\Rightarrow$ (3) This is obvious, by Corollary 8.2.8.

(3) $\Rightarrow$ (1) Suppose that $ax + b = 1$ in R. Set $M = 2R$. Construct A_1 and B_1 as in Theorem 6.1.32, we have we have $M = A_1 \oplus B_1$ with $A_1 \cong R$. Define $f = ap_1 + p_2$ from M to R and $g = q_1x + q_2b$ from R to M. Then $M = Ker(f) \oplus g(R)$. Set $A_2 = g(R)$ and $B_2 = Ker(f)$. Then $M = A_2 \oplus B_2$ with $A_2 \cong R$. Since M is a perspective right R-module, $M = A_1 \oplus C = A_2 \oplus C$ for some $C \subseteq M$. Let $h : 2R \to A_1$ be the projection. Then $M = Ker(h) \oplus q_1(A)$. Hence, hq_1 is an isomorphism. As in the proof of Theorem 6.1.32, we prove that $x + (hq_1)^{-1}(hq_2)b \in U(R)$, as required. □

According to Theorem 8.2.12 and Corollary 8.2.9, we have

Corollary 8.2.13. *Let M be a right R-module. Then the following are equivalent:*

(1) *$End(M)$ has stable range 1.*
(2) *$2M$ is perspective.*

(3) nM *is perspective for some* $n \geq 2$.
(4) nM *is perspective for all* $n \geq 2$.

Example 8.2.14. Let $R = M_2(\mathbb{Z}_2)$. Then R is a perspective ring, but $R[x]$ is not a perspective ring.

Proof. Since $\mathbb{Z}_2$ has stable range 1, by virtue of Theorem 8.2.12, R is a perspective ring. Clearly, $R[x] = M_2(\mathbb{Z}_2)[x] \cong M_2(\mathbb{Z}_2[x])$. But $\mathbb{Z}_2[x]$ has not stable range 1. By using Theorem 8.2.12 again, $R[x]$ is not a perspective ring. □

Lemma 8.2.15. *Let M be a right R-module. Then the following are equivalent*:

(1) M *is perspective.*
(2) $M = A_1 \oplus B_1 = A_2 \oplus B_2$ *with* $A_1 \cong A_2 \Longrightarrow$ *there exists* $C \subseteq M$ *such that* $M = C \oplus B_1 = C \oplus B_2$.

Proof. (1) $\Rightarrow$ (2) Suppose that $M = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1 \cong A_2$. By hypothesis, there exists $D \subseteq M$ such that $M = A_1 \oplus D = A_2 \oplus D$. Hence, $B_1 \cong M/A_1 \cong D \cong M/A_2 \cong B_2$. Hence, we have $C \subseteq M$ such that $M = C \oplus B_1 = C \oplus B_2$, as desired.

(2) $\Longrightarrow$ (1) Given $M = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1 \cong A_2$, by hypothesis, $M = C \oplus B_1 = C \oplus B_2$ for $C \subseteq M$. Hence, $B_1 \cong B_2$. Thus, we have $D \subseteq M$ such that $M = D \oplus B_1 = D \oplus B_2$. Therefore M is perspective. □

Theorem 8.2.16. *Let M be a right R-module. Then the following are equivalent:*

(1) M *is perspective.*
(2) *For any splitting epimorphism* $\varphi : M = A \oplus H \to A$, *there exists an R-morphism* $\chi : A \to H$ *such that*

$$\begin{aligned} \theta : A &\to \quad A, \\ a &\mapsto \varphi\big(a, \chi(a)\big) \end{aligned}$$

is an isomorphism.

Proof. (1) $\Rightarrow$ (2) Let $\varphi : M = A \oplus H \to A$ be a splitting epimorphism. Set $K = Ker(\varphi)$. Then $M = A' \oplus K$ with $A' \cong A$. In view of Lemma 8.2.15, we can find $C \subseteq M$ such that $M = C \oplus H = C \oplus K$. Every $a \in A$ may be written in a unique form $a = c - h (c \in C, h \in H)$. Construct a map $\chi : A \to H$ such that $a \mapsto h$. Clearly, χ is a well defined homomorphism

from A to H. Furthermore, we see that the map $a \mapsto c = a + \chi(a)$ is a homomorphism between A and C. Therefore

$$\begin{aligned} \theta : A &\to \quad A, \\ a &\mapsto \varphi\big(a, \chi(a)\big) \end{aligned}$$

is an isomorphism, as desired.

(2) $\Rightarrow$ (1) Suppose $M = A_1 \oplus B_1 = A_2 \oplus B_2$ with $\theta : A_1 \cong A_2$. Let $p : M = A_1 \oplus B_1 \to A_1$ be the projection onto A_1. Then $\theta p : M = A_2 \oplus B_2 \to A_2$ is a splitting epimorphism. By hypothesis, there exists a splitting $\chi : A_2 \to B_2$ such that $\theta : A_2 \to A_2, a \mapsto \theta p(a + \chi(a))$ is an isomorphism. Let $\psi : A_2 \to A_2 \oplus B_2 = M, a \mapsto a + \chi(a)$. Set $C = Im(\psi)$. Then $p|_C : C \to A_1$ is isomorphic. It follows that $M = C \oplus B_1$. Next, we claim that $M = C \oplus B_2$. If $x \in C \cap B_2$, we may write $x = \psi(a) \in B_2$ for some $a \in A_2$; hence, $a + \chi(a) \in B_2$. This implies that $a \in A_2 \cap B_2$, and so $a = 0$. It follows that $x = 0$, and then $C \cap B_2 = 0$. For any $m \in M$, we may write $m = a + b$ with $a \in A_2, b \in B_2$. Then $m = \big(a + \chi(a)\big) + \big(b - \chi(a)\big) \in C + B_2$. Therefore $M = C \oplus B_2$. This completes the proof, by Lemma 8.2.15. □

Example 8.2.17. Let

$$A_1 = (1,0) \cdot \mathbb{Z}, B_1 = (0,1) \cdot \mathbb{Z}, A_2 = (7,3) \cdot \mathbb{Z} \text{ and } B_2 = (5,2) \cdot \mathbb{Z}$$

be subgroups of the abelian group $2\mathbb{Z}$. Then $2\mathbb{Z} = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1 \cong A_2 \cong \mathbb{Z} \cong B_2 \cong B_1$, while B_1 and B_2 do not have a common complement. Thus, $\mathbb{Z}$ is not perspective.

Proof. One directly verifies that $2\mathbb{Z} = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1 \cong A_2 \cong \mathbb{Z} \cong B_1 \cong B_2$. Assume that $2Z = C \oplus B_1 = C \oplus B_2$. Then $C \cong A_1 \cong \mathbb{Z}$. Assume that $\varphi : \mathbb{Z} \to C$. Then $\varphi(1) = (a, b)$ for some $a, b \in \mathbb{Z}$. Thus, $C = \mathrm{im}(\varphi) = \varphi(1) \cdot \mathbb{Z} = (a, b) \cdot \mathbb{Z}$. Let $e_1 = (1,0), e_2 = (0,1); \xi = (a,b), \eta_1 = (0,1), \eta_2 = (5,2)$. Then $(e_1, e_2), (\xi, \eta_1), (\xi, \eta_2)$ are all bases of $2\mathbb{Z}$. Observing that

$$(\xi, \eta_1) = (e_1, e_2) \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \text{ and } (\xi, \eta_2) = (e_1, e_2) \begin{pmatrix} a & 5 \\ b & 2 \end{pmatrix},$$

we see that

$$\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}, \begin{pmatrix} a & 5 \\ b & 2 \end{pmatrix} \in GL_2(\mathbb{Z}).$$

This implies that $a = \pm 1$ and $2a - 5b = \pm 1$, a contradiction. Therefore B_1 and B_2 do not have a common complement, as desired. □

We are now ready to prove the following fundamental result.

Theorem 8.2.18. *Let R be a perspective ring, and let $a \in R$ is regular. Then $Ra + Rb = R$ and $u_1, u_2 \in U(R)$ implies that there exists an idempotent $e \in R$ such that $u_i a + eb \in U(R)$ and $R = (u_i a)R \oplus eR$ for $i = 1, 2$.*

Proof. Suppose $Ra + Rb = R$ and $u_1, u_2 \in U(R)$. Write $a = axa$ and $x = xax$ for some $x \in R$. Let $K = r(a)$ and $I = aR$. Then $I_i := u_i I \cong I$. As $u_i a \in R$ is regular, each I_i is a direct summand of R. It follows from $I_1 \cong I_2$ that $R = I_1 \oplus C = I_2 \oplus C$ for a $C \subseteq R$. Set $D = xaR$. Then $R = K \oplus D = C \oplus I_1$. In view of Lemma 7.2.11, R is IC. As $I_1 \cong I \cong D$, we see that $K \cong C$.

Obviously, $K \cap r(b) = 0$, and so $b : K \to bK$ is an isomorphism. It follows that $C \cong K \cong bK$. As in the proof of Theorem 7.2.22, bK is a direct summand of R. Thus, C and bK are perspective. In view of Lemma 7.2.20, we have an idempotent $e \in R$ such that $eR = C$ and $e : bK \to C$ is an isomorphism. Then $u_i a + eb : R = K \oplus D \to R, x + y \mapsto u_i a(y) + eb(x + y)$ for any $x \in K, y \in D$. It is easy to check that the diagram of exact sequences

$$\begin{array}{ccccccc} 0 \to & K & \to & K \oplus D & \to & D & \to 0 \\ & eb \downarrow & & u_i a + eb \downarrow & & u_i a \downarrow & \\ 0 \to & C & \to & C \oplus I_i & \to & I_i & \to 0. \end{array}$$

commutes. As $eb : K \to C$ and $u_i a : D \to I_i$ are both isomorphisms, we see that $u_i a + eb \in U(R)$. Furthermore, $R = I_i \oplus C = (u_i a)R \oplus eR$, as desired. □

Theorem 8.2.19. *Let R be perspective. Then the following are equivalent:*

(1) *Every regular element in R is 2-good.*
(2) *1_R is the sum of two units.*

Proof. (1) $\Rightarrow$ (2) This is obvious.

(2) $\Rightarrow$ (1) Let $a \in R$ be regular. Write $1_R = u + v$ with $u, v \in U(R)$. Clearly, $Ra + R \cdot 1 = R$. In view of Theorem 8.2.18, there exists an idempotent $e \in R$ such that $w_1 := ua - e, w_2 := (-v)a - e \in U(R)$. Hence, $a = w_1 - w_2$, as desired. □

Corollary 8.2.20. *Let R be unit-regular. Then the following are equivalent:*

(1) *R is a 2-good ring.*
(2) *1_R is the sum of two units.*

Corollary 8.2.21. *Let R be an exchange ring having stable range 1. If 1_R is the sum of two units, then R is a 6-good ring.*

Proof. Let $a \in R$. Then we can find an idempotent $e \in R$ such that $e \in aR$ and $1-e \in (1-a)R$. This implies that $ea, (1-e)(1-a) \in R$ is regular. In view of Theorem 8.2.19, $1-e, ea$ and $(1-e)(1-a)$ are the sums of two units. It follows from $a = (1-e) + ea - (1-e)(1-a)$ that a is the sum of six units, as desired. □

Corollary 8.2.22. *Let R be strongly π-regular. If 1_R is the sum of two units, then R is a 3-good ring.*

Proof. Let $a \in R$. In view of Theorem 1.1.2, R is strongly clean. Thus, we can find an idempotent and a unit $w \in R$ such that $a = e + w$. On the other hand, R has stable range 1, by Theorem 6.1.36. So it follows from Theorem 8.2.19 that $e = u + v$ with $u, v \in U(R)$. Therefore $a = u + v + w$, and thus yielding the result. □

A ring R is said to being the summand sum property (briefly SSP) if the sum of two direct summands of R_R is also a direct summand of R. For instance, every (von Neumann) regular ring and every every abelian ring (idempotents are central, e.g., commutative rings, right duo rings) are SSP. A ring R is said to being the summand intersection property (briefly SIP) if the intersection of two direct summands of R_R is also a direct summand of R.

Lemma 8.2.23. *Every SSP ring is SIP.*

Proof. Let R be SSP, and let $e, f \in R$ be idempotents. Then $\big(eR + fR\big) \oplus C = R$ for some $C \subseteq R$. Hence, $eR \oplus (1-e)fR \oplus C = R$, and so $(1-e)fR = hR$ for an idempotent $h \in R$. Write $(1-e)f = hs$ and $h = (1-e)ft$ for some $s, t \in R$. Then $(1-e)ft(1-e)f = (1-e)f$. Set $k = t(1-e)f$. Then $k = k^2 \in R$ and $R(1-e)f = Rk$. Since $R = R(1-e)f \oplus R(1-k)$, we get $Rf = Re(1-f) \oplus \big(Rf \cap R(1-k)\big)$. Accordingly,

$$\begin{aligned} R &= R(1-f) \oplus Rf \\ &= R(1-f) \oplus R(1-e)f \oplus \big(Rf \cap R(1-k)\big) \\ &= \big(R(1-f) \oplus R(1-e)f\big) \oplus \big(Rf \cap R(1-k)\big) \\ &= \big(R(1-e) + R(1-f)\big) \oplus \big(Rf \cap R(1-k)\big). \end{aligned}$$

Write $R(1-e) + R(1-f) = Rg$ for an idempotent $g \in R$. Then

$$eR \cap fR = r(R(1-e) + R(1-f)) = r(Rg) = (1-g)R,$$

and therefore $eR \cap fR$ is a direct summand of R_R, hence the result. □

Lemma 8.2.24. *Every SSP and IC ring is perspective.*

Proof. Let R be SSP and IC. In view of Lemma 8.2.23, R is SIP. Let I, J be direct summands of R and $I \cong J$. Then $I = C_1 \oplus \big(I \cap J\big), J = C_2 \oplus \big(I \cap J\big)$ and $R = (I + J) \oplus D$ for some right ideals C_1, C_2 and D. Observing that $R = D \oplus I \oplus C_2 = D \oplus J \oplus C_1$ with $D \oplus I \cong D \oplus J$, we see that $f : C_1 \cong C_2$, as R is IC. Set $C = \{x + f(x) \mid x \in C_1\}$. Then $I + J = I \oplus C = J \oplus C$, and so $R = I \oplus \big(C \oplus D\big) = J \oplus \big(C \oplus D\big)$. Hence, I and J are perspective. This completes the proof. □

Theorem 8.2.25. *Let R be SSP. Then the following are equivalent:*

(1) *R is IC;*
(2) *For any regular $a \in R$, $Ra + Rb = R$ with $b \in R$ implies that there exists an idempotent $e \in R$ such that $a + eb \in U(R)$ and $aR \oplus eR = R$.*
(3) *Every regular element in R is special clean.*

Proof. (1) $\Rightarrow$ (2) In view of Lemma 8.2.24, R is perspective. Let $a \in R$ be regular. Then $a = axa$ for some $x \in R$. Set $X = (1 - ax)R$. Then $aR \oplus X = R$. If $aR + bR = R$, it follows by Theorem 7.2.22 that there exists an idempotent $e \in R$ such that $a + eb \in U(R)$ and $eR = X$. Thus, $aR \cap eR = 0$, as desired.

(2) $\Rightarrow$ (3) For any regular $a \in R$, we have $Ra + R(-1) = R$. Thus, we can find an idempotent $e \in R$ such that $a - e \in U(R)$ and $aR \cap eR = 0$, as required.

(3) $\Rightarrow$ (1) By virtue of Proposition 7.2.9, every regular element in R is unit-regular. Therefore R is IC, as required. □

Remark 8.2.26. (1) From a similar proof of Theorem 8.2.25, we actually derive a stronger version: An SSP ring R is IC if and only if for any regular $a, b \in R$, $r(a) \cap r(b) = 0$ implies that there exists an idempotent $e \in R$ such that $a + eb \in U(R)$ and $aR \oplus eR = R$. As SSP and IC are both left-right symmetric properties, we see that an SSP ring R is IC if and only if for any regular $a, b \in R$, $aR + bR = R$ implies that there exists an idempotent $e \in R$ such that $a + be \in U(R)$ and $Ra \oplus Re = R$, by applying Theorem 8.2.25 to the opposite ring R^{op} of R.

(2) By a more explicit checking, we prove that an SSP ring R is IC if and only if for any regular $a, b \in R$, $Ra + Rb = R$ implies that there exists an idempotent $e \in R$ such that $a + eb, a - eb \in U(R)$ and $aR \oplus eR = R$,

if and only if for any regular $a \in R$, there exists an idempotent $e \in R$ such that $a + e, a - e \in U(R)$ and $aR \cap eR = 0$.

Corollary 8.2.27. *A ring R is abelian if and only if for any regular $a \in R$, there exists a unique idempotent $e \in R$ and a unit $u \in R$ such that $a = e + u$ and $aR \cap eR = 0$.*

Proof. $\Longrightarrow$ Clearly, R is SSP and IC. Let $a \in R$ be regular. Then $a \in R$ is special clean, by Theorem 8.2.25. Assume that there exist idempotents $e, e' \in R$ and units $u, u' \in R$ such that $a = e + u = e' + u'$ and $aR \cap eR = 0 = aR \cap e'R$. Then $au^{-1} = eu^{-1} + 1$. Since R is abelian, we get $au^{-1}(1-e) = 1-e$, and so $au^{-1}(1-e)e' = e'(1-e) \in aR \cap e'R = 0$. Thus, $e' = e'e$. Similarly, $e = ee'$. Therefore $e = e'$, as required.

$\Longleftarrow$ Let $e \in R$ be an idempotent, and let $r \in R$. Choose $a = e - 1$. Then $a \in R$ is regular. Clearly, $e + ex(1-e) \in R$ is an idempotent. We easily see that $a - e, a - \big(e + ex(1-e)\big) \in U(R)$ with $aR \cap eR = aR \cap \big(e + ex(1-e)\big) = 0$. By the uniqueness, we get $e + ex(1-e) = e$, and so $ex = exe$. Likewise, $xe = exe$. Therefore $ex = xe$, the result follows. □

We now record a result of Akalan and Vas [17, Proposition 4.1]: an abelian ring R is unit-regular if and only if for any $a \in R$, there exists a unique idempotent $e \in R$ and a unit $u \in R$ such that $a = e + u$ and $aR \cap eR = 0$.

Theorem 8.2.28. *Let R be SSP and IC. Then the following are equivalent:*

(1) *Every regular element in R is 2-good.*
(2) *1_R is the sum of two units.*

Proof. In view of Lemma 8.2.24, R is perspective. Therefore we complete the proof, by Corollary 8.2.19. □

Example 8.2.29. The ring $\mathbb{Z}$ of all integers is SSP and IC, but it has not stable range 1.

Proposition 8.2.30. *Let R be a ring. Then the following are equivalent:*

(1) *R is SSP and IC.*
(2) *The product of two regular elements is unit-regular.*
(3) *The product of two regular elements is special clean.*

Proof. (1) $\Rightarrow$ (3) Let $a, b \in R$ be regular. As R is IC, there exist idempotents $e, f \in R$ and units $u, v \in R$ such that $a = ue$ and $b = fv$. Then

$ab = u(ef)v$. Since

$$(1-e)R \oplus efR = (1-e)R + fR = gR \text{ for some idempotent } g \in R,$$

we get $R = efR \oplus (1-e)R \oplus (1-g)R$. We infer that $ef \in R$ is regular, and so is $u(ef)v$. This shows that $ab \in R$ is regular. In light of Theorem 8.2.25, $ab \in R$ is special clean, as desired.

(3) $\Rightarrow$ (2) This is obvious by Lemma 7.2.9.

(2) $\Rightarrow$ (1) Obviously, every regular element in R is unit-regular. That is, R is IC. Let $e, f \in R$ be idempotents. Then $eR + fR = eR + (1-e)fR$. By hypothesis, there exists an idempotent $g \in R$ such that $(1-e)fR = gR$. Then $eR + fR = (e+g)R$ where $e+g \in R$ is an idempotent. Hence, $eR + fR$ is a direct summand of R, as required. □

By Proposition 8.2.30 and the induction, we derive

Corollary 8.2.31 *Let R be a ring. Then the following are equivalent:*

(1) *R is SSP and IC.*
(2) *The product of finitely many of regular elements is unit-regular.*
(3) *The product of finitely many regular elements is special clean.*

We note that the number of regular elements in Proposition 8.2.30 can not be reduced from two to one as the following shows.

Example 8.2.32. Let $R = \begin{pmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ 0 & \mathbb{Z}_3 \end{pmatrix}$. Then R has stable range 1. By virtue of Corollary 7.2.23, every regular element in R is special clean. Choose $e = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $f = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. Then $e, f \in R$ are idempotents, and so regular. But $ef = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \in R$ is not regular, as $ef \notin (ef)R(ef) = 0$. Therefore $ef \in R$ is not special clean.

8.3 Elementary Divisor Rings

Throughout this section, all rings are commutative with an identity. A matrix A (not necessarily square) over a ring R admits diagonal reduction if there exist invertible matrices P and Q such that PAQ is a diagonal matrix (d_{ij}), for which d_{ii} is a divisor of $d_{(i+1)(i+1)}$ for each i. A ring R is

called an elementary divisor ring provided that every matrix over R admits a diagonal reduction.

Theorem 8.3.1. *Let R be an elementary divisor ring. Then $M_n(R)$ is a 2-good ring for any $n \geq 2$.*

Proof. Let $A \in M_n(R)$. Then we have some $P, Q \in GL_n(R)$ such that

$$PAQ = diag(d_1, \ldots, d_n).$$

In light of Lemma 8.1.1, there exist $U, V \in GL_n(R)$ such that $diag(d_1, \ldots, d_n) = U + V$. Consequently, $A = P^{-1}UQ^{-1} + P^{-1}VQ^{-1}$, as desired. □

As a consequence, $M_n(D)(n \geq 2)$ is a 2-good ring for any division ring D. Moreover, we have

Example 8.3.2.

(1) $M_n(\mathbb{Z})$ is a 2-good ring for any $n \geq 2$.
(2) Let $R = M_2(\mathbb{Z}_2)$. Then $R[x]$ is 2-good.

Proof. (1) This is obvious, as $\mathbb{Z}$ is an elementary divisor ring.

(2) As $\mathbb{Z}_2[x]$ is a principal domain, it is an elementary divisor ring. In light of Theorem 8.3.1, $M_2\big(\mathbb{Z}_2[x]\big)$ is 2-good. Thus, proving (2) as $R[x] \cong M_2\big(\mathbb{Z}_2[x]\big)$. □

A ring is a Bézout ring if every finitely generated ideal is principal. Evidently, every elementary divisor ring is a Bézout ring. A ring R is an Hermite ring if every 1×2 matrix over R admits a diagonal reduction. As is well known, a ring R is Hermite if and only if for all $a, b \in R$ there exist $a_1, b_1 \in R$ such that $a = a_1 d, b = b_1 d$ and $a_1 R + b_1 R = R$ (see [566, Theorem 1.2.5]). Thus, every Hermite ring is a Bézout ring. It is attractive to explore the conditions under which a Bézout ring is an elementary divisor ring.

A ring R has stable range 2 provided that $aR + bR + cR = R$ with $a, b, c \in R \Longrightarrow \exists y, z \in R$ such that $(a + cy)R + (b + cz)R = R$. We next record the following elementary result.

Lemma 8.3.3. *A ring R is an Hermite ring if and only if it is a Bézout ring and has stable range 2.*

Proof. $\Longrightarrow$ For any $a, b \in R$, there exist $Q \in GL_2(R)$ and $d \in R$ such that $(a,b)Q = (d,0)$. Hence, $aR + bR = dR$. Thus, R is a Bézout ring. Suppose that $aR + bR + cR = R$ with $a, b, c \in R$. Then we have $P = (p_{ij}) \in GL_2(R)$ and $d \in R$ such that $(a,b)P = (d,0)$. Write $P^{-1} = (q_{ij})$. Then $a = dq_{11}$ and $b = dq_{12}$. Hence, $(a + cq_{21})p_{12} + (b + cq_{22})p_{22} = c$. It follows from $aR + bR + cR = R$ that $(a + cq_{21})R + (b + cq_{22})R + cR = R$. Therefore, $(a + cq_{21})R + (b + cq_{22})R = R$, as required.

$\Longleftarrow$ For any $a, b \in R$, there exists $d \in R$ such that $aR + bR = dR$. Write $a = ds, b = dt$ and $ax + by = d$. Set $w = sx + ty - 1$. Then $sR + tR + wR = R$ with $dw = 0$. Since R has stable range 2, we have $u, v \in R$ such that $(s + uw)R + (t + vw)R = R$. Write $(s + uw)p + (t + vw)q = 1$. Thus, $(a,b)\begin{pmatrix} p & -(t+vw) \\ q & s+uw \end{pmatrix} = (d,0)$, where $\begin{pmatrix} p & -(t+vw) \\ q & s+uw \end{pmatrix} \in GL_2(R)$, thus yielding the result. □

Lemma 8.3.4. *Let R be a ring. If $aR + bR + cR = R$ with $a, b, c \in R$, then the following are equivalent:*

(1) *The matrix $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ admits a diagonal reduction.*
(2) *There exist $p, q \in R$ such that $(pa + qb)R + qcR = R$.*

Proof. (1) $\Rightarrow$ (2) By hypothesis, there exist $P = (p_{ij}), Q = (q_{ij}) \in GL_2(R)$ such that $P\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}Q = \begin{pmatrix} u & \\ & v \end{pmatrix}$ where u is a divisor of v. Clearly, $uR + vR = R$, and so $u \in U(R)$. Further, we see that $(p_{11}a + p_{12}b)q_{11} + (p_{12}c)q_{21} = u$. Set $p = p_{11}$ and $q = p_{12}$. Then $(pa + qb)R + qcR = R$.

(2) $\Rightarrow$ (1) Write $(pa + qb)x + qcy = 1$ for some $x, y \in R$. Then we check that

$$\begin{pmatrix} 1 & \\ a(bx+cy)x - abx - acy & 1 \end{pmatrix}\begin{pmatrix} p & q \\ -(bx+cy) & a \end{pmatrix}\begin{pmatrix} a & \\ b & c \end{pmatrix}\begin{pmatrix} x & -qc \\ y & pa+qb \end{pmatrix} = \begin{pmatrix} 1 & \\ & u \end{pmatrix}$$

for some $u \in U(R)$. This completes the proof. □

The following observation is due to [346, Theorem 5.1].

Lemma 8.3.5. *A ring R is an elementary divisor ring if and only if*

(1) *R is an Hermite ring;*
(2) *Every 2×2 matrix over R admits a diagonal reduction.*

Proof. $\Longrightarrow$ This is obvious.

$\Longleftarrow$ Let A be an $m \times n$ matrix. Choose notation so that $m \geq n$. We may suppose by induction that we know diagonal reduction to be possible for smaller m and for the given m if n is smaller. We may assume that $m \geq 3$. Write A_1 for the first row of A and A_2 for the remaining $m-1$ rows. By hypothesis, there exist P_1, Q_1 such that $B := P_1 A_2 Q_1 = diag(x, \ldots)$. Hence,

$$C := \begin{pmatrix} 1 & \\ & P_1 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} Q_1 = \begin{pmatrix} A_1 Q_1 \\ B \end{pmatrix}.$$

Write D for the first two rows of C and E for the remainder. By induction again, we can find unimodular matrices P_2, Q_2 such that $F = P_2 D Q_2 = diag(y, \ldots)$. Thus,

$$H := \begin{pmatrix} P_2 & \\ & I_{m-2} \end{pmatrix} \begin{pmatrix} D \\ E \end{pmatrix} Q_2 = \begin{pmatrix} F \\ G \end{pmatrix}.$$

Let $\begin{pmatrix} z_3 \\ \vdots \\ z_m \end{pmatrix}$ be the first column of G. Then $z_i \in xR$. Further, we check that $x \in yR$, and so $z_i \in yR$. We may use elementary transformations to sweep out the fist column of H and we reach $\begin{pmatrix} y & \\ & K \end{pmatrix}$, where y is still a divisor of every element of K. Applying induction to K, we complete the proof. □

The following result is crucial for the research on elementary divisor rings.

Theorem 8.3.6. *A ring R is an elementary divisor ring if and only if*

(1) *R is an Hermite ring;*
(2) *For all $a, b, c \in R$, $aR + bR + cR = R \Longrightarrow$ there exist $p, q \in R$ such that $(pa + qb)R + qcR = R$.*

Proof. $\Longrightarrow$ Clearly, R is an Hermite ring. By Lemma 8.3.4, the condition (2) holds.

$\Longleftarrow$ In view of Lemma 8.3.5, we need only consider each 2×2 matrix over R. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then we have $U \in GL_2(R)$ such that $AU = \begin{pmatrix} x & 0 \\ y & z \end{pmatrix}$. In view of Lemma 8.3.3, R is a Bézout ring. Write $xR + yR + zR = hR$. Then $x = x'h, y = y'h, z = z'h$ and $xp + yq + zk = h$. Set $w = x'p + y'q + z'k - 1$.

Then $hw = 0$ and $AU = h\begin{pmatrix} x' & w \\ y' & z' \end{pmatrix}$ with $x'R + y'R + z'R + wR = R$. Similarly, we can find $V \in GL_2(R)$ such that $(AU)V = h\begin{pmatrix} a' & 0 \\ c' & d' \end{pmatrix}$ with $a'R + c'R + d'R = R$. By hypothesis, there exist $p, q \in R$ such that $(pa' + qc')R + qd'R = R$. In light of Lemma 8.3.4, we have $P, Q \in GL_2(R)$ such that

$$P\begin{pmatrix} a' & 0 \\ c' & d' \end{pmatrix}Q = \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix},$$

where α is a divisor of β. Therefore

$$PA(UVQ) = \begin{pmatrix} h\alpha & \\ & h\beta \end{pmatrix},$$

hence the result. □

Corollary 8.3.7. *Let R be a Bézout domain. Then the following are equivalent:*

(1) *R is an elementary divisor ring.*
(2) *For all $a, b, c \in R$, $aR + bR + cR = R \Longrightarrow$ there exist $p, q \in R$ such that $(pa + qb)R + qcR = R$.*
(3) *$aR + bR + cR = R$ with $a, b, c \in R \Longrightarrow \exists r, s, t \in R$ such that $s \mid rt$ and $ar + bs + ct = 1$.*

Proof. (1) ⇒ (2) This is obvious, by Theorem 8.3.6.

(2) ⇒ (3) Suppose $aR + bR + cR = R$ with $a, b, c \in R$. Then there exist $p, q \in R$ such that $(pa + qb)R + qcR = R$. Write $1 = (pa + qb)x + qcy$. Then $apx + bqx + cqy = 1$. Additionally, $qx \mid (px)(qy)$, as desired.

(3) ⇒ (1) For any $a, b \in R$, there exists $d \in R$ such that $aR + bR = dR$. Write $a = ds, b = dt$ and $ax + by = d$. Then $d(sx + ty - 1) = 0$. If $d = 0$, then $(a, b) = (0, 0)$. If $d \neq 0$, then $sx + ty = 1$. One easily checks that

$$(a, b)\begin{pmatrix} x & -t \\ y & s \end{pmatrix} = (d, 0).$$

Hence, R is an Hermite ring.

Let $a, b, c \in R$ be such that $aR + bR + cR = R$. By hypothesis, there exist $r, s, t \in R$ such that $s \mid rt$ and $ar + bs + ct = 1$. Since R is a Bézout ring, we can find $d \in R$ such that $dR = rR + sR$.

Case I. $d = 0$. Then $c \in U(R)$, and so $(a \cdot 0 + b \cdot c^{-1})R + (c \cdot c^{-1})R = R$.

Case II. $d \neq 0$. As $s \mid rt$, we see that $rtR \subseteq sR$. Write $r = dp$ and $s = dq$. Then $tdR = trR + tsR \subseteq sR$, and so $dqR = sR \supseteq tdR$. This implies that

$tR \subseteq qR$. Therefore $1 = ar + bs + ct \in (ar + bs)R + ctR \subseteq (ap + bq)R + cqR$; hence, $(ap + bq)R + cqR = R$. In light of Theorem 8.3.6, R is an elementary divisor ring. □

As an immediate consequence of Lemma 8.3.4 and Theorem 8.3.6, we now derive

Theorem 8.3.8. *Let R be an Hermite ring. Then the following are equivalent:*

(1) *R is an elementary divisor ring.*
(2) *Every matrix $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in M_2(R)$ with $aR + bR + cR = R$ admits an elementary reduction.*

A ring R is locally stable provided that $aR + bR = R$ with $a, b \in R \implies \exists y \in R$ such that $R/(a + by)$ has stable range 1. Following McGovern, a ring R has almost stable range 1 provided that every proper homomorphic image of R has stable range 1 (see [414]). Every ring having stable range 1 has almost stale range 1, but the converse is not true, e.g., $\mathbb{Z}$.

Proposition 8.3.9. *Every ring having almost stable range 1 is locally stable.*

Proof. Let R have almost stable range 1. Suppose that $aR + bR = R$ with $a, b \in R$. If there exists $y \in R$ such that $a + by \neq 0$. Then $R/(a + by)R$ has stable range 1. If $a + by = 0$ for all $y \in R$, then $a = 0$, and so $b \in U(R)$. Let $0 \neq z \in R$. Then $R/(a + b(b^{-1}z)R$ has stable range 1, as desired. □

Following Domsha and Vasiunyk, a ring R has adequate range 1 provided that $aR + bR = R$ with $a, b \in R \implies \exists y \in R$ such that $a + by \in R$ is adequate (see [239]). For instance, rings having stable range 1, adequate rings and VNL rings have adequate range 1.

Proposition 8.3.10. *Every Bézout ring having adequate range 1 is locally stable.*

Proof. Let R be a Bézout ring having adequate range 1. Suppose that $aR + bR = R$ for some $a, b \in R$. By hypothesis, there exists $y \in R$ such that $a + by \in R$ is adequate. In view of Theorem 7.1.14, $R/(a + by)R$ is clean. Hence, $R/(a + by)R$ has stable range 1, as required. □

A Bézout ring in which every nonzero element is adequate is called an adequate ring, e.g., $\mathbb{Z}$. A ring R is a neat ring if every proper homomorphic image is clean (see [412]).

Example 8.3.11.

(1) Every ring of stable range 1 is locally stable.
(2) Every adequate ring is locally stable.
(3) Every neat ring is locally stable.

Proof. (1) Every ring having stable range 1 has almost stable range 1. Then proving (1) by Proposition 8.3.9.

(2) As every adequate ring has adequate range 1, then by Proposition 8.3.10, it is locally stable.

(3) Let $aR + bR = R$ with $a, b \in R$. Then there exists $y \in R$ such that $a + by \neq 0$, so $R/(a + by)R$ is a proper homomorphic image of R; hence that it is clean. Thus, $R/(a + by)$ has stable range 1, as desired. □

Lemma 8.3.12. *Every locally stable ring has stable range 2.*

Proof. Let R be a locally stable ring. Suppose that $aR + bR + cR = R$ with $a, b, c \in R$. Then there exist $y, z \in R$ such that $w := a + by + cz$ and R/wR has stable range 1. Hence, $wR + bR + cR = R$, and so there is an $x \in R$ with $wR + (b + cx)R = R$. Therefore $\big(a + c(z - xy)\big)R + (b + cx)R = \big(w - (b + cx)y\big)R + (b + cx)R = R$, as desired. □

Theorem 8.3.13. *Let R be a locally stable ring. Then R is an elementary divisor ring if and only if R is a Bézout ring.*

Proof. $\Longrightarrow$ By Theorem 8.3.6, R is an Hermite ring, and then it is a Bézout ring.

$\Longleftarrow$ In view of Lemma 8.3.12, R has stable range 2. It follows from Lemma 8.3.3 that R is an Hermite ring. Suppose that $A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in M_2(R)$ with $aR+bR+cR = R$. We will suffice to prove that A admits an elementary reduction, by Theorem 8.3.8.

Write $ax + by + cz = 1$. Then $bR + (ax + cz)R = R$. As R is locally stable, there exists some $t \in R$ such that $v = b + (ax + cz)t$ and R/vR has

stable range 1. We have

$$\begin{pmatrix} 1 & 0 \\ xt & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ zt & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ v & c \end{pmatrix}.$$

Since R is an Hermite ring, there exists $P \in GL_2(R)$ such that $(v, c)P = (0, c')$, and so $\begin{pmatrix} a & 0 \\ v & c \end{pmatrix} P = \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix}$. It is easily seen that $vR \subseteq c'R$ and $a'R + b'R + c'R = R$. Since R/vR has stable range 1, we see that $R/c'R$ has stable range 1, as $R/c'R \cong (R/vR)/(c'R/vR)$. Also $\overline{a'}(R/c'R) + \overline{b'}(R/c'R) = R/c'R$, hence there exists some $w \in R$ such that $\overline{b' + a'w} \in U(R/c'R)$, and so $(\overline{b' + a'w})\overline{p} = \overline{1}$, for some $p \in R$, and then $(b' + a'w)p + c'q = 1$. One easily checks that

$$\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} c' & -(b' + a'w) \\ p & q \end{pmatrix} \begin{pmatrix} a' & b' \\ & c' \end{pmatrix} \begin{pmatrix} 1 & w \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ -pa' & 1 \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$
$$= \begin{pmatrix} 1 & \\ & a'c' \end{pmatrix}.$$

As $det \begin{pmatrix} c' & -(b' + a'w) \\ p & q \end{pmatrix} = 1$, we see that $\begin{pmatrix} c' & -(b' + a'w) \\ p & q \end{pmatrix} \in GL_2(R)$. Thus, we prove that $\begin{pmatrix} a' & b' \\ & c' \end{pmatrix}$ admits a diagonal reduction. Therefore $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ admits a diagonal reduction, hence the result. □

Corollary 8.3.14. *Let R have almost stable range 1. Then R is an elementary divisor ring if and only if R is a Bézout ring.*

Proof. In view of Proposition 8.3.9, R is a locally stable ring. This completes the proof, in terms of Theorem 8.3.13. □

Immediately, every Bézout ring having stable range 1 is an elementary divisor ring.

Corollary 8.3.15. *Let R be a ring in which every element not in $J(R)$ is adequate. Then R is an elementary divisor ring if and only if R is a Bézout ring.*

Proof. $\Longrightarrow$ This is obvious.

$\Longleftarrow$ Suppose that $aR + bR = R$ with $a, b \in R$. Case 1. $a \in J(R)$. Then $b \notin J(R)$, since $aR + bR = R$. Hence, $a + b \notin J(R)$, so it is adequate and therefore $R/(a+b\cdot 1)R$ has stable range 1, as the proof in Proposition 8.3.10.

Case 2. $a \notin J(R)$. Then $a + b \cdot 0 \notin J(R)$; hence, $a + b \cdot 0$ is adequate. Thus $R/(a + b \cdot 0)R$ has stable range 1. Accordingly, R is locally stable, and therefore R is an elementary divisor ring, by Theorem 8.3.13. □

Corollary 8.3.16. *Let R have adequate range 1. Then R is an elementary divisor ring if and only if R is a Bézout ring.*

Proof. In light of Proposition 8.3.10, R is locally stable. Therefore we complete the proof, by Theorem 8.3.13. □

Kaplansky proved that for an adequate ring being an Hermite ring was equivalent to being an elementary divisor ring. This was extended to rings with zero-divisors by Gillman and Henriksen (see [566, Theorem 1.2.13]). As an immediate consequence, we deduce that every adequate ring is an elementary divisor ring. While on Hermite rings, we also observe the following fact.

Proposition 8.3.17. *Let R be an Hermite ring, and let $A \in M_n(R)$. Then there exists $Q \in GL_n(R)$ such that AQ is a lower triangular matrix.*

Proof. For any $a_1, a_2, \ldots, a_n \in R$, we claim that there exists $U \in GL_n(R)$ such that $(a_1, a_2, \ldots, a_n)\ U = (d, 0, \ldots, 0)$. If $n = 2$, it is true. Assume that it is true for $n = k (k \geq 2)$. If $n = k + 1$, we have $U \in GL_k(R)$ such that $(a_1, a_2, \ldots, a_k)U = (d, 0, \ldots, 0)$, Hence,

$$(a_1, a_2, \ldots, a_k, a_{k+1}) \begin{pmatrix} U & \\ & 1 \end{pmatrix} = (d, 0, \ldots, 0, a_{k+1}).$$

Thus, we have $V \in GL_{k+1}(R)$ such that

$$(a_1, a_2, \ldots, a_k, a_{k+1}) \begin{pmatrix} U & \\ & 1 \end{pmatrix} V = (d, a_{k+1}, 0, \ldots, 0).$$

Clearly, there exists $W \in GL_2(R)$ such that $(d, a_{k+1})W = (c, 0)$. This implies that

$$(a_1, a_2, \ldots, a_k, a_{k+1}) \begin{pmatrix} U & \\ & 1 \end{pmatrix} V \begin{pmatrix} W & \\ & I_2 \end{pmatrix} = (c, 0, 0, \ldots, 0).$$

By induction, the claim is true.

Let $A \in M_n(R)$. Then we can find $Q_1 \in GL_n(R)$ such that $(a_{11}, a_{12}, \ldots, a_{1n})Q_1 = (d_1, 0, \ldots, 0)$. Hence,

$$AQ_1 = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ a'_{21} & a'_{22} & \cdots & a'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a'_{n1} & a'_{n2} & \cdots & a'_{nn} \end{pmatrix}.$$

Further, there exists $Q_2 \in GL_{n-1}(R)$ such that $(a'_{22}, \ldots, a'_{2n})Q_2 = (d_2, 0, \ldots, 0)$. It follows that

$$AQ_1 \begin{pmatrix} 1 & \\ & Q_2 \end{pmatrix} = \begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 \\ * & d_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a''_{n1} & a''_{n2} & a''_{n3} & \cdots & a''_{nn} \end{pmatrix}.$$

By iteration of this process, we have $Q \in GL_n(R)$ such that AQ is a lower triangular matrix. □

Lemma 8.3.18. *Let R be an Hermite ring. Then the following are equivalent:*

(1) *R is an elementary divisor ring.*
(2) *For any $a, b, c \in R$ such that $aR + bR + cR = R$ there exist $t, r, s \in R$ such that $b + ct = rs$ and $rR + sR = rR + aR = sR + cR = R$.*
(3) *For any $a, b, c \in R$ such that $aR + bR + cR = R$ there exist $t, r, s \in R$ such that $b + ct = rs$ and $rR + aR = sR + cR = R$.*

Proof. (1) $\Rightarrow$ (2) By hypothesis, the matrix $\begin{pmatrix} c & 0 \\ b & a \end{pmatrix} \in M_2(R)$ admits a diagonal reduction. In view of Lemma 8.3.4, there exist $p, q \in R$ such that $(qc + pb)R + paR = R$. Hence, $pR + (pb + qc)R = pR + cR = R$, and then $pR + (pb + qc)cR = R$. Let s be an element of R such that $sp \equiv 1(mod\ (pb + qc)c)$. Then $sp \equiv 1(mod\ c)$. We have $s(pb + qc) \equiv b(mod\ cR)$. Set $r = pb + qc$. Then $b + ct = rs$ for some $t \in R$. Additionally, $rR + sR = rR + aR = sR + cR = R$, as desired.

(2) $\Rightarrow$ (3) This is trivial.

(3) $\Rightarrow$ (1) Given $cR + bR + aR = R$ with $c, b, a \in R$, then there exist $t, r, s \in R$ such that $b + ct = rs$ and $rR + aR = sR + cR = R$. Hence, $b \equiv rs(mod\ cR)$ and $ps \equiv 1(mod\ cR)$ for some $p \in R$. Thus, $pb \equiv r(mod\ cR)$. Write $pb + qc = r$. Obviously, $cR + pR = R$. Since R is an Hermite ring, there exists $d \in R$ such that $d = px + qy, p = dp_1, q = dq_1$ and $p_1R + q_1R = R$. Clearly, $p_1R + cR = R$, and so $p_1R + q_1cR = R$. Hence, $(p_1b + q_1c)R + p_1R = R$. It follows from $rR + aR = R$ that $(pb + qc)R + aR = R$, and then $(p_1b + q_1c)R + aR = R$. Thus, $(q_1c + p_1b)R + p_1aR = R$. Accordingly, R is an elementary divisor ring, in terms of Theorem 8.3.6. □

Lemma 8.3.19. *Let R be an Hermite ring. Then the following are equivalent:*

(1) *R is an elementary divisor ring.*
(2) *For any $a, b, z \in R$ such that $aR + bR = R$, there exists $y \in R$ such that $a + by = uv$, where $uR + vR = uR + zR = vR + (1 - z)R = R$.*
(3) *For any $a, b, z \in R$ such that $aR + bR = R$, there exists $y \in R$ such that $a + by = uv$, where $uR + zR = vR + (1 - z)R = R$.*

Proof. (1) $\Rightarrow$ (2) For any $a, b, z \in R$ such that $aR + bR = R$, we get $zR + aR + b(1 - z)R = R$. By virtue of Lemma 8.3.18, we can find $y \in R$ such that $a+b(1-z)y = uv$, where $uR+vR = uR+zR = vR+(1-z)bR = R$. Hence, $vR + (1 - z)R = R$, thus proving (2).

(2) $\Rightarrow$ (3) This is obvious.

(3) $\Rightarrow$ (1) Given $aR + bR + cR = R$ with $a, b, c \in R$, we can find $d, s, t \in R$ such that $b = ds, c = dt$ and $sR + tR = R$. Furthermore, $aR + dR = R$. Write $ap + dq = 1$ for some $p, q \in R$. By hypothesis, there exist $u, v, y \in R$ such that $s+ty = uv$, where $uR+(1-dq)R = vR+dqR = R$. Hence, $b + cy = (du)v$ with $uR + aR = R$. But $dR + aR = R$, we see that $(du)R + aR = R$. Write $sx + ty = 1$ for some $x, y \in R$. Then $(uv - ty)x + ty = sx + ty = 1$; hence, $vR + tR = R$. As $vR + dR = R$, we see that $vR + cR = vR + dtR = vR + tR = R$. This completes the proof, by Lemma 8.3.18. □

Theorem 8.3.20. *Let R be a ring. Then R is an elementary divisor ring if and only if*

(1) *R is an Hermite ring;*
(2) *If $aR + bR = cR + dR = R$ with $a, b, c, d \in R \Longrightarrow \exists y \in R$ such that $a + by = \alpha\beta$, where $\alpha R + \beta R = \alpha R + cR = \beta R + dR = R$.*

Proof. $\Longrightarrow$ Clearly, R is an Hermite ring. If $aR + bR = cR + dR = R$ with $a, b, c, d \in R$, we can find some $s, t \in R$ such that $cs + dt = 1$. In view of Lemma 8.3.19, there exists $y \in R$ such that $a + by = \alpha\beta$, where $\alpha R + \beta R = \alpha R + csR = \beta R + dtR = R$. This implies that $\alpha R + cR = \beta R + dR = R$, as desired.

$\Longleftarrow$ For any $a, b, z \in R$ such that $aR+bR = R$, we have $zR+(1-z)R = R$, and so there exists $y \in R$ such that $a + by = \alpha\beta$, where $\alpha R + \beta R = \alpha R + zR = \beta R + (1 - z)R = R$. In light of Lemma 8.3.19, we obtain the result. □

Corollary 8.3.21. *Let R be a Bézout domain. Then R is an elementary divisor ring if and only if for any $x, y, z \in R$ there exist $\lambda, a, b \in R$ such that $x + \lambda y$ is a divisor of $y(1 - az)\big(1 - b(1 - z)\big)$.*

Proof. $\Longrightarrow$ Let $x, y, z \in R$. Since R is an elementary divisor ring, there exist $d, x_0, y_0 \in R$ such that $x = dx_0, y = dy_0$ and $x_0R + y_0R = R$. In view of Theorem 8.3.20, we can find $\lambda \in R$ such that $x_0 + \lambda y_0 = \alpha\beta$, where $\alpha R + \beta R = \alpha R + zR = \beta R + (1-z)R = R$. Hence, $x + \lambda y = d\alpha\beta$. Write $1 = \alpha s + az = \beta t + b(1-z)$ for some $s, t, a, b \in R$. Then $\alpha s = 1 - az$ and $\beta t = 1 - b(1-z)$; hence, $(d\alpha\beta)sty_0 = y(1-az)\big(1-b(1-z)\big)$. Therefore $x + \lambda y$ is a divisor of $y(1-az)\big(1-b(1-z)\big)$.

$\Longleftarrow$ Since R is a Bézout domain, it is an Hermite ring. For any $a, b, z \in R$ such that $aR + bR = R$, we have $\lambda \in R$ such that $a + b\lambda$ is a divisor of $b(1-az)\big(1-b(1-z)\big)$. Clearly, $(a+b\lambda)R + bR = R$, and so $a + b\lambda$ is a divisor of $(1-az)\big(1-b(1-z)\big)$. Since R is an Hermite ring, there exist $u, v, w \in R$ such that $a + b\lambda = uv, 1 - az = uw$ and $vR + wR = R$. Hence, $uR + zR = R$. Write $(a+b\lambda)s = (1-az)\big(1-b(1-z)\big)$ for some $s \in R$. Then $uvs = uw\big(1-b(1-z)\big)$. If $u = 0$, then $a \in U(R)$; hence, $a + b\cdot 0 = a\cdot 1$ where $aR + zR = R + (1-z)R = R$. If $u \neq 0$, then $vs = w\big(1-b(1-z)\big)$. It follows from $vR + wR = R$ that $vt = 1 - b(1-z)$ for some $t \in R$. This implies that $vR + (1-z)R = R$. Therefore we complete the proof by Lemma 8.3.19.

□

Theorem 8.3.22. *Let R be a Bézout domain, and let $0 \neq c \in R$. Then the following are equivalent:*

(1) *For any $a, b \in R$ such that $aR + bR = R$, there exist $r, s \in R$ such that $c = rs$ and $rR + sR = rR + aR = sR + bR = R$.*
(2) *R/cR is clean.*

Proof. (1) $\Rightarrow$ (2) Given $\overline{a}(R/cR) + \overline{b}(R/cR) = R/cR$, we have $aR + bR + cR = R$. Write $bR + cR = dR$ for some $d \in R$. Then $aR + dR = R$. By hypothesis, there exist $r, s \in R$ such that $c = rs$ and $rR + sR = rR + aR = sR + dR = R$. Since $rR + sR = R$, there exist $u, v \in R$ such that $ru + sv = 1$. Hence, $\overline{r^2u} = \overline{r}$ and $\overline{s^2v} = \overline{s}$ in R/cR. Let $\overline{e} = \overline{sv}$. Then $\overline{e} \in R/cR$ is an idempotent. As $rR + aR = R$, we see that $reR + eaR = eR$, and so $\overline{e} = \overline{a\beta e}$ for some $\beta \in R$. Since $sR + dR = R$, we see that $s(1-e)R + (1-e)dR = (1-e)R$. Thus, $(1-e)dR \equiv (1-e)R \ (\ mod\ cR)$. It follows that $(1-e)bR \equiv (1-e)R \ (\ mod\ cR)$, and then $\overline{1-e} = \overline{b\gamma(1-e)}$ for some $\gamma \in R$. Therefore we have an idempotent $\overline{e} \in \overline{a}(R/cR)$ such that $\overline{1-e} \in \overline{b}(R/cR)$. This shows that R/cR is clean.

(2) $\Rightarrow$ (1) Given $aR + bR = R$ with $a, b \in R$, we have $\overline{a}\,\overline{R} + \overline{b}\,\overline{R} = \overline{R}$, where $\overline{R} = R/cR$. Then we can find an idempotent $\overline{e} \in \overline{a}\,\overline{R}$ such that $\overline{1-e} \in \overline{b}\,\overline{R}$. Write $e - ap = cq, 1 - e - b\alpha = c\beta$ and $e(1-e) = ct$. Since

R is a Bézout domain, there exists $d \in R$ such that $eR + cR = dR$. Write $e = de_0, c = dc_0$ and $d = ex + cy$. Then $d(e_0x + c_0y - 1) = 0$, and so $e_0x + c_0y = 1$. As $de_0(1 - e) = ct = dc_0t$, we get $e_0(1 - e) = c_0t$. Thus, $e_0x(1 - e) + c_0y(1 - e) = 1 - e$; hence, $e + c_0j = 1$ for some $j \in R$. Take $r = c_0$ and $s = d$. Then $c = rs$. As $e = de_0 = se_0$, we get $se_0 + rj = 1$; whence, $rR+sR = R$. Since $e = ap+cq$, we have $ap+cq+c_0j = 1$, and then $ap + r(dq + j) = 1$. This implies that $rR + aR = R$. Since $c\beta + b\alpha + e = 1$, we get $d(c_0\beta + e_0) + b\alpha = 1$, and so $sR + bR = R$, as required. □

Corollary 8.3.23. *Let R be a Bézout domain, and let $0 \neq c \in R$ be adequate. Then for any $z \in R$ there exist $u, v \in R$ such that $c = uv$ and $uR + vR = uR + zR = vR + (1 - z)R = R$.*

Proof. This is obvious, by Theorem 7.1.14 and Theorem 8.3.22. □

Recall that a permutation matrix is a matrix obtained by permuting the rows of the same size identity matrix according to some permutation of the numbers 1 to n. Every row and column therefore contains precisely a single 1 with 0s everywhere else. A permutation matrix is invertible. In addition, a permutation matrix P satisfies $PP^T = I$. We say that an $n \times n$ matrix $A = (a_{ij})$ avoids a permutation matrix $P = (p_{ij})$ of size n if $a_{ij} = 0$, for every i, j such that $p_{ij} = 1$. We now turn our attention to rings over which sufficient large matrices admit "block" diagonal reductions.

Lemma 8.3.24. *Let R be a ring, and let $A \in M_n(R)(n \geq 2)$. If A avoids a permutation matrix, then A is the sum of two units.*

Proof. Suppose that A avoids a permutation matrix P. Then $P^TP = I_n$ and every diagonal entry of P^TA is zero. We take for U the strictly lower triangular $n \times n$ matrix, with those entries strictly below the diagonal the same as the corresponding entries of P^TA below the diagonal; the rest of the entries of U are all 0. Set $V = P^TA - U$. Then

$$P^TA = (U + I_n) + (V - I_n);$$

hence, $A = P(U + I_n) + P(V - I_n)$ and the proof is complete. □

Theorem 8.3.25. *If B is an $n \times n$ matrix in block diagonal form where the blocks have size $\leq \frac{n}{2}$ and $n \geq 2$, then B is the sum of two units.*

Proof. Let $B = (b_{ij}) = diag(B_1, \ldots, B_t)$. If $n \leq 3$, then B is diagonal. In view of Lemma 8.1.1, B is the sum of two units. Thus we may assume that $n \geq 4, t \geq 2$. If $P = (p_{ij})$ is an $n \times n$ permutation matrix, we consider the meeting number:

$$m(P, B) = |\{\text{positions}(i, j) \text{ where } b_{ij}, p_{ij} \neq 0\}|.$$

Pick a permutation matrix P. If $m(P, B) = 0$, then B avoids P, and we are done by Lemma 8.3.24. If not, we modify P by a transposition to get a permutation matrix P' with $m(P', B) < m(P, B)$. By continuing this process we finally obtain a permutation matrix Q avoiding B.

Now let $m(P, B) > 0$. Choose a particular (i, j) so that $p_{ij}, b_{ij} \neq 0$. Then (i, j) must be one of the nonempty block of B. Observe that the meeting number is unchanged if we permute the rows and columns of both P and B the same way. Thus, we may assume that (i, j) is in the first block $B_1 \in M_{p\times q}(R)$. Write $B = diag(B_1, A)$, where $A = diag(B_2, \ldots, B_t)$. Partition P into pieces of the same type. We have

$$B = \begin{pmatrix} B_1 & 0 \\ 0 & A \end{pmatrix}, P = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix},$$

where $P_1 \in M_{p\times q}(R)$ and $P_4 \in M_{(n-p)\times(n-q)}(R)$. Here, the (i, j)-entry of B_1 is nonzero and the entry of P_1 is 1.

We claim that some entry of P_4 is 1. If no entry of P_4 is 1, then $P_4 = 0$. Hence, P_3 has 1 in each row. Thus, $n - p$ ones must fit into $q - 1$ columns of P. Thus, $n - p \leq q - 1 < \frac{n}{2}$. As $p \leq \frac{n}{2}$, we see that $n - p \geq \frac{n}{2}$, a contradiction.

We now assume that P has 1 in the (r, c) entry, where $p < r \leq n, q < c \leq n$. Denote E_{ir} the elementary matrix that is the outcome of switching rows i and r applied to I_n. Set $W = (w_{ij}) = E_{ir}P$. Then $w_{ic} = w_{rj} = 1$. Clearly, $b_{ic} = b_{rj} = 0$. Thus, $m(W, B) < Q(P, B)$, as desired. □

Corollary 8.3.26. *Let R be a Dedekind domain with class group of size h. Then $M_n(R)$ is 2-good for all $n \geq 2h$.*

Proof. Let $A \in M_n(R)(n \geq 2h)$. Then A is equivalent to a block diagonal matrix B: all entries of B are zero except for rectangular blocks along a "block diagonal", where both dimensions of the nonzero blocks are at most h (see [390]). Thus, B is the sum of two units, by Theorem 8.3.25. This completes the proof. □

Example 8.3.27. Let $R = \{\alpha+\beta\sqrt{-5} \mid \alpha, \beta \in \mathbb{Z}\}$. Then R is a Dedekind domain with class group of size 2. Therefore it follows from Corollary 8.3.26 that $M_n(R)$ is 2-good for all $n \geq 4$.

The quadratic integers, which belong to a quadratic fields $\mathbb{Q}(\sqrt{d})$, form an integral domain called ring of integers of $\mathbb{Q}(\sqrt{d})$. They may be written $a + b\omega$, where a and b are integers, and where ω is defined by:

$$\omega = \begin{cases} \sqrt{d} & \text{if } d \not\equiv 1 \ mod\ 4 \\ \frac{1+\sqrt{d}}{2} & \text{if } d \ \equiv 1 \ mod\ 4. \end{cases}$$

We are concerned with the conditions under which such rings of quadratic integers are generated by their units. The following two examples are due to Tichy and Ziegker (see [508, Proposition 1 and Proposition 2]).

Example 8.3.28. Let $d \in \mathbb{Z}$ be square free and $d \not\equiv 1$ $(mod\ 4)$, then $R = \mathbb{Z}[\sqrt{d}]$ is generated by its units if and only if $d = a^2 \pm 1$ for some $a \in \mathbb{Z}$.

Proof. $\Longrightarrow$ By the known theorem in number theory, every unit in R is in the form

$$\pm(m_0 + n_0\sqrt{d})^k, k = \pm 1, \pm 2, \ldots,$$

where (m_0, n_0) is the smallest positive solution of the Pell equation $m^2 - dn^2 = \pm 1, m, n \in \mathbb{Z}, n \neq 0$. Set $v = m_0 + n_0\sqrt{d}$. Then $v^2 = 2m_0v + (n_0^2d - m_0^2)$, hence, every positive power of v is a linear combination of $1, v$. Write $v^{-1} = x + y\sqrt{d}$. Then $\frac{x}{m_0} = \frac{-y}{n_0} = p$; hence, $p(m_0^2 - n_0^2d) = 1$. Thus, $v^{-1} = \pm(m_0 - n_0\sqrt{d}) = \pm 2m_0 \mp v$. Hence, we can express $v^{-1}, v^{-2}, \ldots$ as integral linear combinations of $1, v$. This shows that every element in R is generated by $1, v$. Write $\sqrt{d} = s + tv$. Then $tn_0 = 1$, and so $n_0 = \pm 1$. It follows that $m_0^2 - d = \pm 1$, and therefore $d = m_0^2 \mp 1$, as desired.

$\Longleftarrow$ Choose $u = a + \sqrt{d}$. It is easy to verify that $u^{-1} = a - \sqrt{d}$, as $a^2 - d = \mp 1$. That is, $u \in U(R)$. For any $x + y\sqrt{d} \in R$, we see that $x + y\sqrt{d} = (x - ya) \cdot 1 + yu$. Therefore every element in R is generated by its units, as required. □

Example 8.3.29. Let $d \in \mathbb{Z}$ be square free, then the ring of integers of $\mathbb{Q}[\sqrt{d}]$ is generated by its units if and only if $d = a^2 \pm 1, d \not\equiv 1(mod\ 4)$ or $d = a^2 \pm 4, d \equiv 1(mod\ 4)$.

Proof. As in the proof of Example 8.4.28, we replace $\sqrt{d}$ by $\frac{1+\sqrt{d}}{2}$. As the ring of integers is generated by $1, \sqrt{d}$ if $d \not\equiv 1(mod\ 4)$ and by 1 and $\frac{1+\sqrt{d}}{2}$, otherwise. We easily obtain the result. □

8.4 Nil-clean and 2-Nil-clean Rings

The motivation of this section is to consider a subclass of clean rings, and completely determine the structure of such rings. An element $a \in R$ is called nil-clean if there exists an idempotent $e \in R$ such that $a - e \in R$ is a nilpotent. A ring R is nil-clean provided that every element in R is nil-clean. A ring R is 2-nil-clean provided that every element in R is the sum of two idempotents and a nilpotent. We are concerned with nil-clean and 2-nil-clean matrix rings over rings. We begin with

Lemma 8.4.1. *Let F be a field. Then the following are equivalent:*

(1) *$M_n(F)$ is nil-clean for all $n \in \mathbb{N}$.*
(2) *$M_n(F)$ is nil-clean for some $n \geq 2$.*
(3) *$F \cong \mathbb{Z}_2$.*

Proof. (1) $\Rightarrow$ (2) This is obvious.

(2) $\Rightarrow$ (3) Let $0 \neq a \in F$. Then $aI_n = E + W$, where $E^2 = E$ and $W \in M_n(F)$ is nilpotent. As R is commutative, we see that $E = aI_n - W = aI_n\big(I_n - a^{-1}W\big) \in GL_n(F)$. It follows that $E = I_n$. Thus, $aI_n - E \in M_n(F)$ is nilpotent. This shows that $a - 1 \in F$ is nilpotent. Hence, $a = 1$. Therefore $F \cong \mathbb{Z}_2$.

(3) $\Rightarrow$ (1) As every matrix over a field has a Frobenius normal form which is a block diagonal matrix in which the diagonal elements are companion matrices, and that nil-matrix is invariant under the similarity, we may assume

$$A = \begin{pmatrix} 0 & & & c_0 \\ 1 & 0 & & c_1 \\ & 1 & 0 & c_2 \\ & & \ddots & \vdots \\ & & \ddots & 0 \ c_{n-2} \\ & & & 1 \ c_{n-1} \end{pmatrix}.$$

Case I. $c_{n-1} = 1$. Choose

$$W = \begin{pmatrix} 0 & & & 0 \\ 1 & 0 & & 0 \\ & 1 & 0 & 0 \\ & & \ddots & \vdots \\ & & \ddots & 0 \ 0 \\ & & & 1 \ 0 \end{pmatrix}, E = \begin{pmatrix} 0 & & & c_0 \\ 0 & 0 & & c_1 \\ & 0 & 0 & c_2 \\ & & \ddots & \vdots \\ & & \ddots & 0 \ c_{n-2} \\ & & & 0 \ \ 1 \end{pmatrix}.$$

Case II. $c_{n-1} = 0, c_{n-2} = 1$. Choose

$$W = \begin{pmatrix} 0 & & & 0 \\ 1 & 0 & & 0 \\ & \ddots & & \vdots \\ & \ddots & 0 \ 0 \ 0 & \\ & \ddots & 1 \ 1 \ 1 & \\ & & 0 \ 1 \ 1 & \end{pmatrix}, E = \begin{pmatrix} 0 & & & c_0 \\ 0 & 0 & & c_1 \\ & \ddots & & \vdots \\ & \ddots & 0 \ 0 & c_{n-3} \\ & \ddots & 0 \ 1 & 0 \\ & & 0 \ 0 & 1 \end{pmatrix}.$$

Case III. $c_{n-1} = 0$ and $c_{n-2} = 0$. Choose

$$W = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E = 0$$

or

$$W = \begin{pmatrix} 0 & & & 0 \\ 1 & 0 & & 0 \\ & \ddots & & \vdots \\ & \ddots & 0 \ 1 \ 1 & \\ & \ddots & 1 \ 1 \ 0 & \\ 0 & & 0 \ 1 \ 1 & \end{pmatrix}, E = \begin{pmatrix} 0 & & & c_0 \\ 0 & 0 & & c_1 \\ & \ddots & & \vdots \\ & \ddots & 0 \ 1 & c_{n-3}+1 \\ & \ddots & 0 \ 1 & 0 \\ 0 & & 0 \ 0 & 1 \end{pmatrix}.$$

In any case, $E^2 = E$, $W \in M_n(F)$ is nilpotent and $A = E + W$, as desired. □

Lemma 8.4.2. *Let R be a Boolean ring. Then $M_n(R)$ is nil-clean for all $n \in \mathbb{N}$.*

Proof. Let $A \in M_n(R)$, and let S be the subring of R generated by the entries of A. That is, S is formed by finite sums of monomials of the form: $a_1 a_2 \dots a_m$, where $a_1, \dots, a_m \in A$. Then S is a finite Boolean ring. Thus,

S is isomorphic to finite direct product of $\mathbb{Z}_2$. In view of Lemma 8.4.1, $M_n(S)$ is nil-clean. But $A \in M_n(S)$, we see that A can be written as the sum of an idempotent matrix and a nilpotent matrix over S, as required. □

A ring R is 2-primal if its prime radical $P(R)$ coincides with the set of nilpotent elements of R, e.g., commutative rings, reduced rings, etc.

Theorem 8.4.3. *Let R be a strongly nil-clean ring. If R is 2-primal, then $M_n(R)$ is nil-clean for all $n \in \mathbb{N}$.*

Proof. Suppose that R is 2-primal. In view of Theorem 5.1.5, $R/J(R)$ is Boolean and $J(R)$ is nil. Then $M_n(R/J(R))$ is nil-clean, by Lemma 8.4.2. Clearly, $M_n(R/J(R)) \cong M_n(R)/M_n(J(R))$. As R is 2-primal, we see that $J(R) \subseteq N(R) = P(R) \subseteq J(R)$, and so $J(R) = P(R)$. Thus, $M_n(J(R)) = M_n(P(R)) = P(M_n(R))$ is nil. It follows that $M_n(R)$ is nil-clean, as asserted. □

Lemma 8.4.4. *Let R be of bounded index. If $J(R)$ is nil, then $J(M_n(R))$ is nil for all $n \in \mathbb{N}$.*

Proof. Obviously, $P(R) \subseteq J(R)$. Let $x \in J(R)$. Then $RxR \subseteq J(R) \subseteq N(R)$. Since R is of bounded index, we have $(RxR)^m = 0$ for some $m \in \mathbb{N}$. Let P be a prime ideal of R. Then $RxR \subseteq P$, and so $x \in P$. This shows that $J(R) \subseteq P(R)$; hence, $J(R) = P(R)$. As argued in the proof of Theorem 8.4.3, we prove that $M_n(J(R))$ is nil, as required. □

Theorem 8.4.5. *Let R be of bounded index. If R is strongly nil-clean, then $M_n(R)$ is nil-clean for all $n \in \mathbb{N}$.*

Proof. By virtue of Theorem 5.1.5, $R/J(R)$ is Boolean and $J(R)$ is nil. Then $M_n(R/J(R))$ is nil-clean from Lemma 8.4.2. According to Lemma 8.4.4, $M_n(J(R))$ is nil. As $M_n(R/J(R))/J(M_n(R)) \cong M_n(R/J(R))$, $M_n(R)$ is nil-clean, as asserted. □

Theorem 8.4.6. *Let R be a commutative ring, and let $n \in \mathbb{N}$. Then the following are equivalent:*

(1) *$M_n(R)$ is nil-clean.*
(2) *$R/J(R)$ is Boolean and $J(R)$ is nil.*

Proof. (1) $\Rightarrow$ (2) Clearly, $M_n(J(R))$ is nil, and then so is $J(R)$. Let M be a maximal ideal of R. Since $M_n(R)$ is nil-clean, then so is $M_n(R/M)$.

But R/M is a filed. It follows by Lemma 8.4.1 that $R/M \cong \mathbb{Z}_2$. Let $\varphi_M : R/J(R) \to R/M, r + J(R) \mapsto r + M$. Then $\bigcap_M Ker\varphi_M = \bigcap_M \{r + J(R) \mid r \in M\} = 0$. Therefore $R/J(R)$ is isomorphic to a subdirect product of some $\mathbb{Z}_2$. Hence, $R/J(R)$ is Boolean, as desired.

(2) $\Rightarrow$ (1) In light of Theorem 5.1.5, R is strongly nil-clean. This completes the proof, by Theorem 8.4.3. □

We note that the "commutative" condition is necessary in Theorem 8.4.6. Let $R = M_n(\mathbb{Z}_2)(n \geq 2)$. Then R is nil-clean. But $R/J(R)$ is not Boolean. Here, R is not commutative.

Corollary 8.4.7. *A commutative ring R is nil-clean if and only if $M_n(R)$ is nil-clean.*

Proof. Suppose that R is nil-clean. Then $R/J(R)$ is Boolean and $J(R)$ is nil. In view of Theorem 8.4.6, $M_n(R)$ is nil-clean. Conversely, assume that $M_n(R)$ is nil-clean. In light of Theorem 8.4.6, $R/J(R)$ is Boolean and $J(R)$ is nil. Then we obtain the result, by Theorem 5.1.5. □

Example 8.4.8. Let F be a field, and let $R = F[x, y]/(x, y)^2$. Then $M_n(R)$ is nil-clean if and only if $F \cong \mathbb{Z}_2$. Clearly, $J(R) = (x, y)/(x, y)^2$, and so $J(R)$ is nil. Therefore R is nil-clean if and only if $R/J(R)$ is nil-clean. But $R/J(R) \cong F$, we see that R is nil-clean if and only if $F \cong \mathbb{Z}_2$. Thus, we are done by Corollary 8.4.7.

Example 8.4.9. Let $m \geq 2, n \in \mathbb{N}$. Then $M_n(\mathbb{Z}_m)$ is nil-clean if and only if $m = 2^r$ for some $r \in \mathbb{N}$. Write $m = p_1^{r_1} \cdots p_s^{r_s}$ ($p_1, \ldots, p_s$ are distinct primes, $r_1, \ldots, r_s \in \mathbb{N}$). Then $Z_m \cong Z_{p_1^{r_1}} \times \cdots \times Z_{p_s^{r_s}}$. In light of Corollary 8.4.7, $M_n(\mathbb{Z}_m)$ is nil-clean if and only if so is $\mathbb{Z}_m$. Clearly, $Z_{p_i^{r_i}}/J(Z_{p_i^{r_i}}) \cong Z_{p_i}$ and $J(Z_{p_i^{r_i}}) = p_i Z_{p_i^{r_i}}$ is nil. Thus, $M_n(\mathbb{Z}_m)$ is nil-clean if and only if $s = 1$ and $p_1 = 2$, and we are through.

In the next, we construct examples of 2-nil-clean rings and investigate certain subclass of such rings. We have

Example 8.4.10. The class of 2-nil-clean rings contains many familiar examples.

(1) Every nil-clean ring is 2-nil-clean, e.g., strongly nil-clean rings, Boolean rings.

(2) $\mathbb{Z}_3 \times \mathbb{Z}_3$ is 2-nil-clean, while it is not nil-clean.
(3) A local ring R is 2-nil-clean if and only if $R/J(R) \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$, and $J(R)$ is nil.

We also provide some examples illustrating which ring-theoretic extensions of 2-nil-clean rings produce 2-nil-clean rings.

Example 8.4.11.

(1) Any quotient of a 2-nil-clean ring is 2-nil-clean.
(2) Any finite product of 2-nil-clean rings is 2-nil-clean. But $R = \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8 \times$ is an infinite product of 2-nil-clean rings, which is not 2-nil-clean. Here, the element $(0, 2, 2, 2, \ldots) \in R$ can not written as the sum of two idempotents and a nilpotent.
(3) The triangular matrix ring $T_n(R)$ over a 2-nil-clean ring R is 2-nil-clean.
(4) The quotient ring $R[[x]]/(x^n)(n \geq 1)$ of a 2-nil-clean ring R is 2-nil-clean.

Theorem 8.4.12. *Let I be a nil ideal of the ring R. Then R is 2-nil-clean if and only if R/I is 2-nil-clean.*

Proof. $\Longrightarrow$ It is obtained from Example 8.4.11(1).

$\Longleftarrow$ Let $a \in R$, there exist two idempotents $\overline{e}, \overline{f} \in R/I$ and a nilpotent $\overline{w} \in R/I$ such that $\overline{a} = \overline{e} + \overline{f} + \overline{w}$. As idempotents and nilpotents lift modulo every nil ideal, we can assume that $e, f \in R$ are idempotents and $w \in N(R)$. Then $a = e + f + w + r$ for some $r \in I$. Since $w \in N(R)$, write $w^k = 0$ for some $k \in \mathbb{N}$. Then $(w + r)^k \in I$, and so $w + r \in N(R)$. This completes the proof. □

Corollary 8.4.13. *A ring R is 2-nil-clean if and only if $R/P(R)$ is 2-nil-clean.*

Proof. As $P(R)$ is a nil ideal of R, the result follows from Theorem 8.4.12. □

Corollary 8.4.14. *Let R be a ring. Then the following are equivalent:*

(1) *R is 2-nil-clean.*
(2) *$T_n(R)$ is 2-nil-clean for all $n \in \mathbb{N}$.*
(3) *$T_n(R)$ is 2-nil-clean for some $n \in \mathbb{N}$.*

Proof. (1) $\Rightarrow$ (2) Let $I = \left\{ \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \middle| \, a_{ij} \in R \right\}$. Then I is a nil ideal of $T_n(R)$. Clearly, $T_n(R)/I \cong R \times R \times \cdots \times R$. In light of Example 8.4.11 and Theorem 8.4.12, $T_n(R)$ is 2-nil-clean.

(2) $\Rightarrow$ (3) This is trivial.

(3) $\Rightarrow$ (1) Straightforward. □

A ring is strongly 2-nil-clean if every element is the sum of two idempotents and a nilpotent that commute. Strongly 2-nil-clean rings form a subclass of 2-nil-clean rings.

Lemma 8.4.15. *Let R be a ring. Then the following are equivalent:*

(1) *R is strongly 2-nil-clean.*
(2) *For any $a \in R$, there exist two idempotents $e, f \in R$ and a nilpotent $w \in R$ that commute such that $a = e - f + w$.*

Proof. (1) $\Rightarrow$ (2) Let $a \in R$. Then $1-a = e+f+w$ for two idempotents e, f and $w \in N(R)$. Here, e, f and w commute. Then $a = (1-e) - f + (-w)$, as desired.

(2) $\Rightarrow$ (1) For any $a \in R$, there exist two idempotents $e, f \in R$ and $w \in N(R)$ that commute such that $1-a = e-f+w$. Then $a = (1-e)+f+(-w)$, as required. □

Theorem 8.4.16. *Let R be a ring. Then the following are equivalent:*

(1) *R is strongly 2-nil-clean.*
(2) *For all $a \in R$, $a - a^3 \in N(R)$.*
(3) *For all $a \in R$, $a^2 \in R$ is strongly nil-clean.*

Proof. (1) $\Rightarrow$ (3) Let $a \in R$. By Lemma 8.4.15, there exist two idempotents f, g and $w \in N(R)$ that commute such that $a = f - g + w$. Set $e = f - g$. Then $e^3 = e$ and $ae = ea$. Hence, $a^2 - a^4 = (e+w)^2 - (e+w)^4 \in N(R)$, as desired.

(3) $\Rightarrow$ (2) Let $a \in R$. If $a^2 \in R$ is strongly nil-clean, $a^2 - a^4 \in N(R)$. Thus, $a(a - a^3) \in N(R)$; hence, $(a - a^3)^2 = a(a - a^3)(1 - a^2) \in N(R)$. Hence, $a - a^3 \in N(R)$, as needed.

(2) $\Rightarrow$ (1) By hypothesis, $6 = 2^3 - 2 \in N(R)$. Further, $(a^2 - a) - (a^2 - a)^3, a^3 - a \in N(R)$, and so $3a^2 - 3a \in N(R)$. This shows that

$(-2a^2)^2-(-2a^2)=4a^4+2a^2=(6a^4-2a^4)+2a^2=6a^4+2a(a-a^3)\in N(R)$. Moreover, $(a+2a^2)^2-(a+2a^2)=a^2+4a^3+4a^4-a-2a^2=(3a^2-3a)+4(a^3-a)+6a+4a(a^3-a)\in N(R)$. As in the proof of Theorem 5.1.1, there exist $f(t),g(t)\in\mathbb{Z}[t]$ such that $(-2a^2)-f(a),(a+2a^2)-g(a)\in N(R)$, $f(a)=f^2(a)$ and $g(a)=g^2(a)$. Therefore $a-\big(f(a)+g(a)\big)=\big((a+2a^2)-g(a)\big)+\big((-2a^2)-f(a)\big)\in N(R)$. One easily checks that $af(a)=f(a)a$ and $ag(a)=g(a)a$. This proving (1). □

We note that strong 2-nil-cleanness is not Morita invariant as the following shows.

Example 8.4.17. Let $n\geq 2$. Then $M_n(R)$ is not strongly 2-nil-clean for any ring R.

Proof. Let R be a ring, and let $A=\begin{pmatrix}1_R & 1_R\\ 1_R & 0\end{pmatrix}$. Then $A^3-A=\begin{pmatrix}2 & 1_R\\ 1_R & 1_R\end{pmatrix}$. One checks that $\begin{pmatrix}2 & 1_R\\ 1_R & 1_R\end{pmatrix}^{-1}=\begin{pmatrix}1_R & -1_R\\ -1_R & 2\end{pmatrix}$, and therefore $M_n(R)$ is not strongly 2-nil-clean, by Theorem 8.4.16. □

An element $e\in R$ is a tripotent if $e^3=e$. A ring R is tripotent if $a^3=a$ for all $a\in R$.

Theorem 8.4.18. *Let R be a ring. Then the following are equivalent:*

(1) *R is strongly 2-nil-clean.*
(2) *For any $a\in R$, there exists a tripotent $e\in R$ such that $a-e\in R$ is nilpotent and $ea=ae$.*

Proof. (1) $\Rightarrow$ (2) Let $a\in R$. By Lemma 8.4.15, we have two idempotents f,g and $w\in N(R)$ that commute such that $a=f-g+w$. Set $e=f-g$. Then $e\in R$ is a tripotent, $ae=ea$ and $a-e=w\in N(R)$, as desired.

(2) $\Rightarrow$ (1) Let $a\in R$. Then we have a tripotent e and nilpotent w such that $a=e+w$ and $ae=ea$. Hence, $a^3-a=w(3e^2+3ew+w^2-1)\in N(R)$. This completes the proof by Theorem 8.4.16. □

Lemma 8.4.19. *Let R be a ring, and let $n\geq 2$. Then the following are equivalent:*

(1) *$N(R)$ forms an ideal whenever $a-a^n\in N(R)$ for all $a\in R$.*
(2) *$n\not\equiv 1\ (mod\ 3)$ and $n\not\equiv 1\ (mod\ 8)$.*

Proof. See [317, Theorem A.1]. □

Lemma 8.4.20. *A ring R is tripotent if and only if R is a commutative ring in which every element is the sum of two idempotents.*

Proof. $\Longrightarrow$ Clearly, $N(R) = 0$. By virtue of Theorem 8.4.16, every element in R is the sum of two commuting idempotents. Let $e^2 = e, x \in R$. Then $ex(1-e) = (ex(1-e))^3 = 0$; hence, $ex = exe$. Likewise, $xe = exe$. Thus, $ex = xe$, and so R is abelian. This shows that R is commutative, as required.

$\Longleftarrow$ Let $a \in R$. Then $1 - a = e + f$, and so $a = (1-e) - f$. Therefore $a^3 = a$, hence the result. □

Theorem 8.4.21. *A ring R is strongly 2-nil-clean if and only if*

(1) *$J(R)$ is nil;*
(2) *$R/J(R)$ is tripotent.*

Proof. $\Longrightarrow$ Let $x \in J(R)$. Then $x^3 - x \in N(R)$ by Theorem 8.4.16. Hence, $w := x(1-x^2) \in N(R)$. This implies that $x = w(1-x^2)^{-1} \in N(R)$; hence, $J(R)$ is nil.

Let $a \in R$. In light of Theorem 8.4.16, $a^3 - a \in N(R)$. According to Lemma 8.4.19, $N(R)$ forms an ideal of R. Hence, $N(R) \subseteq J(R)$. This shows that every element in $R/J(R)$ is the sum of two idempotents that commute. By virtue of Lemma 8.4.20, $R/J(R)$ is tripotent, as desired.

$\Longleftarrow$ As $R/J(R)$ is tripotent, then by Lemma 8.4.20, it is strongly 2-nil-clean ring. Since $J(R)$ is nil, we obtain the result by Theorem 8.4.12. □

A ring R is 2-Boolean if $a^2 \in R$ is an idempotent for all $a \in R$. Clearly, every tripotent ring is 2-Boolean. But the converse is not true, e.g., $\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, 0, 1 \in \mathbb{Z}_2 \right\}$.

Lemma 8.4.22. *A ring R is strongly 2-nil-clean if and only if*

(1) *$J(R)$ is nil;*
(2) *$R/J(R)$ is 2-Boolean.*

Proof. $\Longrightarrow$. As R is strongly 2-nil-clean, by Theorem 8.4.21, $J(R)$ is nil and $R/J(R)$ is tripotent, then it is 2-Boolean.

$\Longleftarrow$ Let $\overline{a} \in R/J(R)$, then $\overline{a}^2 = \overline{e}$ for some idempotent $\overline{e} \in R/J(R)$. This implies that $\overline{a}^2$ is strongly nil-clean, then by Theorem 8.4.16, $R/J(R)$ is strongly 2-nil-clean. This completes the proof by Theorem 8.4.12. □

A ring R is a Yaqub ring if it is isomorphic to a subdirect product of $\mathbb{Z}_3$'s.

Lemma 8.4.23. *Let R be a ring. Then the following are equivalent:*

(1) *R is tripotent.*
(2) *R is a Boolean ring, a Yaqub ring or the product of such rings.*

Proof. (1) $\Rightarrow$ (2) Since R is tripotent, $2^3 = 2$; hence, $6 = 0$. Then $R \cong R_1, R_2$ or $R_1 \times R_2$. Here, $R_1 = R/2R$ and $R_2 = R/3R$. By Birkhoff's Theorem, $R_i (i = 1, 2)$ is the subdirect product of some subdirectly irreducible rings S_{ij}. Here, a ring is subdirectly irreducible if and only if the intersection of all its non-zero ideals is non-zero. As a homomorphic image of R_i, each S_{ji} is tripotent. Thus, S_{ji} is a commutative ring in which every element is the sum of two idempotents, by Lemma 8.4.20. Since S_{ji} is subdirectly irreducible, it has no central idempotents except for 0 and 1. Thus, $S_{ji} = \{0, 1, 2\}$. As $2 \in N(R_1)$ and $3 \in N(R_2)$, we see that $S_{j1} \cong \mathbb{Z}_2$ and $S_{j2} \cong \mathbb{Z}_3$. Therefore R_1 is a Boolean ring and R_2 is a Yaqub ring, as desired.

(2) $\Rightarrow$ (1) Clearly, Boolean rings and Yaqub rings are tripotent. Accordingly, R is tripotent, as asserted. □

Theorem 8.4.24. *A ring R is strongly 2-nil-clean if and only if*

(1) *$J(R)$ is nil;*
(2) *$R/J(R)$ is isomorphic to a Booelan ring, a Yaqub ring, or the product of such rings.*

Proof. Combining Theorem 8.4.21 and Lemma 8.4.23, we complete the proof. □

Lemma 8.4.25. *A ring R is Boolean if and only if*

(1) *$2 \in R$ is nilpotent;*
(2) *R is tripotent.*

Proof. $\Longrightarrow$ This is obvious.

$\Longleftarrow$ Let $a \in R$. In virtue of Lemma 8.4.23, R is the subdirect product of $\mathbb{Z}_2$'s and $\mathbb{Z}_3$'s. As $2 \in R$ is nilpotent, we see that R is the subdirect product of $\mathbb{Z}_2$'s. Therefore R is Boolean, as asserted. □

Lemma 8.4.26. *A ring R is a Yaqub ring if and only if*

(1) $3 \in R$ *is nilpotent;*
(2) R *is tripotent.*

Proof. $\Longrightarrow$ Since $3 \in \mathbb{Z}_3$ is nilpotent, we see that $3 \in N(R)$. As $\mathbb{Z}_3$ is tripotent, so is R, as desired.

$\Longleftarrow$ In light of Lemma 8.4.23, R is the subdirect product of $\mathbb{Z}_2$'s and $\mathbb{Z}_3$'s. As $3 \in N(R)$, we have R is the subdirect product of $\mathbb{Z}_3$'s. Therefore R is a Yaqub ring, as desired. □

Theorem 8.4.27. *A ring R is strongly 2-nil-clean if and only if $R \cong R_1, R_2$ or $R_1 \times R_2$, where*

(1) $R_1/J(R_1)$ *is a Boolean ring and $J(R_1)$ is nil;*
(2) $R_2/J(R_2)$ *is a Yaqub ring and $J(R_2)$ is nil.*

Proof. $\Longrightarrow$ In view of Theorem 8.4.16, $6 \in N(R)$. Write $6^n = 0 (n \in \mathbb{N})$. Then $2^n R + 3^n R = R$; hence, $R \cong R_1, R_2$ or $R_1 \times R_2$, where $R_1 = R/2^n R$ and $R_2 = R/3^n R$. As the homomorphic images of R, R_1 and R_2 are strongly 2-nil-clean. By using Theorem 8.4.21, $R_1/J(R_1)$ is tripotent. As $2 \in N(R_1/J(R_1))$, it follows by Lemma 8.4.25 that $R_1/J(R_1)$ is Boolean. Likewise, $R_2/J(R_2)$ is tripotent and $3 \in N(R_2/J(R_2))$. In view of Lemma 8.4.26, $R_2/J(R_2)$ is a Yaqub ring. By virtue of Theorem 8.4.24, $J(R)$ is nil; hence, $J(R_1)$ and $J(R_2)$ are both nil, as desired.

$\Longleftarrow$ By hypothesis, $R/J(R)$ is a Boolean ring $R_1/J(R_1)$, a Yaqub ring $R_2/J(R_2)$, or the direct product of such rings. In light of Lemma 8.4.23, $R/J(R)$ is tripotent. On the other hand, $J(R) \cong J(R_1) \times J(R_2)$ is nil. Therefore R is strongly 2-nil-clean, by Theorem 8.4.21. □

We now investigate certain strongly 2-nil-clean rings over which every matrix is 2-nil-clean. We have

Lemma 8.4.28. *$M_n(\mathbb{Z}_3)$ is 2-nil-clean.*

Proof. As in the proof of Lemma 8.4.1, we may assume that

$$A = \begin{pmatrix} 0 & & & c_0 \\ 1 & 0 & & c_1 \\ & 1 & 0 & c_2 \\ & & \ddots & \vdots \\ & & \ddots & 0 \; c_{n-2} \\ & & & 1 \; c_{n-1} \end{pmatrix}.$$

Case I. $c_{n-1} = 1$. Choose

$$W = \begin{pmatrix} 0 & & & 0 \\ 1 & 0 & & 0 \\ & 1 & 0 & 0 \\ & & \ddots & \vdots \\ & & \ddots & 0 \; 0 \\ & & & 1 \; 0 \end{pmatrix}, E = \begin{pmatrix} 0 & & & c_0 \\ 0 & 0 & & c_1 \\ & 0 & 0 & c_2 \\ & & \ddots & \vdots \\ & & \ddots & 0 \; c_{n-2} \\ & & & 0 \quad 1 \end{pmatrix}.$$

Then $E^2 = E$, and so $A = E + 0 + W$ is 2-nil-clean.

Case II. $c_{n-1} = -1$. Choose

$$W = \begin{pmatrix} 0 & & & 0 \\ 1 & 0 & & 0 \\ & 1 & 0 & 0 \\ & & \ddots & \vdots \\ & & \ddots & 0 \; 0 \\ & & & 1 \; 0 \end{pmatrix}, E_1 = \begin{pmatrix} 0 & & & c_0 \\ 0 & 0 & & c_1 \\ & 0 & 0 & c_2 \\ & & \ddots & \vdots \\ & & \ddots & 0 \; c_{n-2} \\ & & & 0 \quad 1 \end{pmatrix}, E_2 = diag(0, \ldots, 0, 1).$$

Then $A = E_1 + E_2 + W$ is 2-nil-clean.

Case III. $c_{n-1} = 0$.

If $n = 2$, then

$$\begin{pmatrix} 0 & c_0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & c_0 - 1 \\ 0 & 0 \end{pmatrix} \text{ is 2-nil-clean.}$$

If $n = 3$, then

$$\begin{pmatrix} 0 & 0 & c_0 \\ 1 & 0 & c_1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & c_0 \\ 0 & 0 & c_1 - 1 \\ 0 & 0 & 0 \end{pmatrix}$$

is 2-nil-clean.

If $n \geq 4$, we have

$$A = \begin{pmatrix} 0&0&\cdots&0&0&0&0\\ 0&0&\cdots&0&0&0&0\\ 0&0&\cdots&0&0&0&0\\ \vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\vdots\\ 0&0&\cdots&0&0&0&0\\ 0&0&\cdots&0&0&1&0\\ 0&0&\cdots&0&1&0&1 \end{pmatrix} + \begin{pmatrix} 0&0&\cdots&0&0&0&0\\ 0&0&\cdots&0&0&0&0\\ 0&0&\cdots&0&0&0&0\\ \vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\vdots\\ 0&0&\cdots&0&0&0&0\\ 0&0&\cdots&0&1&-1&1\\ 0&0&\cdots&0&-1&1&-1 \end{pmatrix}$$
$$+ \begin{pmatrix} 0&0&\cdots&0&0&0&c_0\\ 1&0&\cdots&0&0&0&c_1\\ 0&1&\cdots&0&0&0&c_2\\ \vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\vdots\\ 0&0&\cdots&1&0&0&c_{n-3}\\ 0&0&\cdots&0&0&0&c_{n-2}-1\\ 0&0&\cdots&0&0&0&0 \end{pmatrix}$$

is the sum of two idempotents and a nilpotent. This implies that $A \in M_n(\mathbb{Z}_3)$ is 2-nil-clean. Therefore $M_n(\mathbb{Z}_3)$ is 2-nil-clean. □

Lemma 8.4.29. *Let R be tripotent. Then $M_n(R)$ is 2-nil-clean for all $n \in \mathbb{N}$.*

Proof. Let $A \in M_n(R)$, and let S be the subring of R generated by the entries of A. That is, S is formed by finite sums of monomials of the form: $a_1 a_2 \cdots a_m$, where $a_1, \ldots, a_m$ are entries of A. Since R is a commutative ring in which $6 = 0$, S is a finite ring in which $x = x^3$ for all $x \in S$. Thus, S is isomorphic to finite direct product of $\mathbb{Z}_2$ and/or $\mathbb{Z}_3$. In terms of Lemma 8.4.1 and Lemma 8.4.28, $M_n(S)$ is 2-nil-clean. As $A \in M_n(S)$, A is the sum of two idempotent matrices and a nilpotent matrix over S, as desired. □

Theorem 8.4.30. *Let R be 2-primal. If R is strongly 2-nil-clean, then $M_n(R)$ is 2-nil-clean for all $n \in \mathbb{N}$.*

Proof. Since R is strongly 2-nil-clean, it follows by Theorem 8.4.21 that $J(R)$ is nil and $R/J(R)$ is tripotent. In virtue of Lemma 8.4.29, $M_n(R/J(R))$ is 2-nil-clean. Furthermore, $J(R) \subseteq N(R) = P(R) \subseteq J(R)$, we get $J(R) = P(R)$. Hence, $M_n(J(R)) = M_n(P(R)) = P(M_n(R))$ is nil.

Since $M_n\big(R/J(R)\big) \cong M_n(R)/M_n(J(R))$, it follows by Theorem 8.4.12 that $M_n(R)$ is 2-nil-clean. This completes the proof. □

Corollary 8.4.31. *Let R be a commutative 2-nil-clean ring. Then $M_n(R)$ is 2-nil-clean for all $n \in \mathbb{N}$.*

Example 8.4.32. Let $m = 2^k 3^l (k, l \in \mathbb{N})$. Then $M_n(\mathbb{Z}_m)$ is 2-nil-clean for all $n \in \mathbb{N}$.

Proof. Clearly, $\mathbb{Z}_m \cong R_1 \times R_2$. Here, $R_1 = \mathbb{Z}_{2^k}$ and $R_2 = \mathbb{Z}_{3^l}$. One easily checks that $R_1/J(R_1) \cong \mathbb{Z}_2$ and $R_2/J(R_2) \cong \mathbb{Z}_3$. In light of Theorem 8.4.27, $\mathbb{Z}_m$ is a commutative 2-nil-clean ring, hence the result by Corollary 8.4.31. □

Theorem 8.4.33. *Let R be of bounded index. If R is strongly 2-nil-clean, then $M_n(R)$ is 2-nil-clean for all $n \in \mathbb{N}$.*

Proof. By virtue of Lemma 8.4.4, $M_n(J(R))$ is nil. In view of Theorem 8.4.21, $R/J(R)$ is tripotent. Thus, $M_n(R/J(R))$ is 2-nil-clean, in terms of Corollary 8.4.31. Since $M_n(R/J(R))/J(M_n(R)) \cong M_n(R/J(R))$, according to Theorem 8.4.12, $M_n(R)$ is 2-nil-clean. □

Corollary 8.4.34. *Let R be a ring, and let $m \in \mathbb{N}$. If $(a - a^3)^m = 0$ for all $a \in R$, then $M_n(R)$ is 2-nil-clean for all $n \in \mathbb{N}$.*

Proof. Let $x \in J(R)$. Then $(x - x^3)^m = 0$, and so $x^m = 0$. This implies that $J(R)$ is nil. In light of Lemma 8.4.19, $N(R)$ forms an ideal of R, and so $N(R) \subseteq J(R)$. Hence, $J(R) = N(R)$ is nil. Furthermore, $R/J(R)$ is tripotent. In light of Theorem 8.4.21, R is strongly 2-nil-clean. If $a^k = 0 (k \in \mathbb{N}$, then $1 - a, 1 + a \in U(R)$, and so $1 - a^2 = (1 - a)(1 + a) \in U(R)$. By hypothesis, $a^m(1 - a^2)^m = 0$. Hence, $a^m = 0$, and so R is of bounded index. This complete the proof, by Theorem 8.4.33. □

Corollary 8.4.35. *Let R be a 2-Boolean ring. Then $M_n(R)$ is 2-nil-clean for all $n \in \mathbb{N}$.*

Proof. Let $a \in R$. Then $a^2 = a^4$. Hence, $a^2(1 - a^2) = 0$. This shows that $(1 - a^2)^2 a^2 (1 - a^2)a = 0$, i.e., $(a - a^3)^3 = 0$. In light of Corollary 8.4.34, the result follows. □

Notes. Theorem 8.1.2 was first proved by Henriksen [308, Theorem 3]. Theorem 8.1.24 is due to Zelinsky [571, Theorem]. Theorem 8.1.26 was observed in [308, Theorem 6]. Theorem 8.1.42 is due to Wang and Chen [535, Theorem 7]. Perspective rings were carried out by Grag et al. [276]. Theorem 8.2.2, Theorem 8.2.6 and Theorem 8.2.12 are [276, Theorem 4.7, Theorem 3.4 and Theorem 5.12]. The proof of Theorem 8.2.18 was based on [278, Theorem 3.5] and [537, Theorem 3.1]. Theorem 8.3.6 was observed by Kaplansky [346, Theorem 5.2]. Theorem 8.3.20 was first proved for Bézout domains by Shchedryk [467, Theorem 5]. Theorem 8.3.22 was proved by Zabavsky [567, Theorem 31]. Theorem 8.3.25 is due to Vámos and Wiegand [519, Corollary 4.6]. Theorem 8.4.3 and Theorem 8.4.5 are due to Kosan et al. [373, Theorem 6.1 and Corollary 6.8].

Chapter 9

Weakly Clean and Weakly Nil-clean Properties

A ring R is weakly clean provided that every element in R is the sum or difference of a unit and an idempotent. The motivation of this chapter is to explore structure of such rings. In Section 9.1, we establish some elementary properties of commutative weakly clean rings. Then we study some subclasses of weakly clean rings which involve nilpotents. In Section 9.2, we are concerned with weakly nil-clean rings. On the basis of these results, we are able to describe weakly nil-clean matrix rings over a field. In Section 9.3, we are focus on strongly weakly nil-clean rings. Finally, in the last section, we determine necessary and sufficient conditions under which a ring is uniquely weakly nil-clean.

9.1 Weakly Clean Rings

Throughout this section, all rings are commutative. Let S be a nonempty subset of $Id(R)$ of a ring R. We say that R is S-weakly clean provided that every element $a \in R$ can be written in the form $a = e + u$ or $a = -e + u$ where $e \in S(R)$ and $u \in U(R)$. Equivalently, a commutative ring R is weakly clean if and only if R is $Id(R)$-weakly clean. We begin with

Lemma 9.1.1. *If R is S-weakly clean, then the homomorphic image $\overline{R}$ is $\overline{S}$-weakly clean where $\overline{S}$ is the image of S.*

Theorem 9.1.2. *Let R be S-weakly clean where $|S| < \infty$.*

(1) *R has at most $2|S| - 1$ maximal ideals.*
(2) *If $0 \in S$,then R has at most $2|S| - 2$ maximal ideals.*

Then R is semilocal. In particular, $|Id(R)| < \infty$ and R is a finite direct product of indecomposable rings.

Proof. (1) Let $S = \{e_1, e_2, \ldots, e_n\}$ where each $e_i \neq 0$. Suppose that R has more than $2n - 1$ maximal ideals. Say $M_1, \ldots, M_{2n}$ are distinct maximal ideals of R. Then

$$\overline{R} := R/M_1 \cdots M_{2n} \cong R/M_1 \times \cdots \times R/M_{2n}.$$

In view of Lemma 9.1.1, $\overline{R}$ is $\overline{S}$-weakly clean. Let $f_1, \ldots, f_{2n}$ be the idempotents of $\overline{R}$ corresponding to the standard basis for $R/M_1 \times \cdots \times R/M_{2n}$. Let $x = -\overline{e_1}f_1 - \cdots - \overline{e_n}f_n + \overline{e_1}f_{n+1} + \cdots + \overline{e_n}f_{2n}$. Then for each $i = 1, 2, \ldots, n$ we have $x \pm \overline{e_i} \notin U(\overline{R})$. This gives a contradiction. Therefore R has at most $2n - 1$ maximal ideals.

(2) Let $S = \{e_1, e_2, \ldots, e_n\}$ with $e_n = 0$. Suppose that R has more than $2(n-1)$ maximal ideals. Say $M_1, \ldots, M_{2(n-1)}, M_{2(n-1)+1}$ are distinct maximal ideals of R. Then

$$\overline{R} := R/M_1 \cdots M_{2(n-1)+1} \cong R/M_1 \times \cdots \times R/M_{2(n-1)+1}.$$

Clearly, $\overline{R}$ is $\overline{S}$-weakly clean. Let $f_1, \ldots, f_{2(n-1)}, f_{2(n-1)+1}$ be the idempotents of $\overline{R}$ corresponding to the standard basis for $R/M_1 \times \cdots \times R/M_{2(n-1)+1}$. Let $x = -\overline{e_1}f_1 - \cdots - \overline{e_{n-1}}f_{n-1} + \overline{e_1}f_n + \cdots + \overline{e_{n-1}}f_{2(n-1)}$. Then for each $i = 1, 2, \ldots, n$ we have $x \pm \overline{e_i} \notin U(\overline{R})$, a contradiction. Thus, R has at most $2n - 2$ maximal ideals.

Since R has only finitely many maximal ideals, it is a semilocal ring.

If R has infinitely many idempotents, the we choose $0, 1 \neq e \in R$ is an idempotent. Then $R = eR \times (1 - e)R$. Clearly, either eR or $(1 - e)R$ has infinitely many idempotents. Say eR has infinitely many idempotents. Then we have an idempotent $0, e \neq f \in eR$ such that $eR = f(eR) \oplus (e - f)(eR)$. By iteration of this process, we can find $R_1, \ldots, R_m$ such that $R = R_1 \oplus \cdots \oplus R_m$, where $m = 2|S|$. But R has at least m maximal ideals. This gives a contradiction. Therefore $|Id(R)| < \infty$.

For any S-weakly clean ring R, we now prove that $|Id(R)| < \infty \Longrightarrow R$ is a finite direct product of indecomposable rings. The proof is by induction on $|Id(R)| = n (n \geq 2)$. Suppose that $n = 2$. Then the only idempotents

of R is 0 and 1. Hence, R is indecomposable. Now suppose the claim is true for any S-weakly clean ring with the number of idempotents strictly less than $k+1 (k \leq 2)$. We now assume that $n = k+1$. Then choose an idempotent $e \in R - \{0,1\}$. Then $R = eR \oplus (1-e)R$. Obviously, eR is $\overline{S}$-weakly clean as a homomorphic image of R. As $|\overline{S}| \leq k$, by hypothesis, we see that eR is a finite direct product of indecomposable rings. Likewise, $(1-e)R$ is a finite direct product of indecomposable rings. Accordingly, R is a finite direct product of indecomposable rings. □

Let R be S-weakly clean where $|S| < \infty$. As a consequence of Theorem 9.1.2, we see that R is a finite direct product of $\{0,1\}$-weakly clean rings. Clearly, a ring R is $\{0,1\}$-weakly clean if and only if for $a \in R$ at least one of the elements $a, 1-a$ and $1+a$ is invertible.

Lemma 9.1.3. *Let R be a commutative ring. If R has two maximal ideals and $2 \in U(R)$, then R is $\{0,1\}$-weakly clean.*

Proof. Suppose that R has two maximal ideals M_1 and M_2. Let $a \in R$. If $a \notin M_1 \cup M_2$, then $a \in U(R)$. If $a \in M_1 \backslash M_2$, then $a+1, a-1 \notin M_1$. Suppose that $a+1, a-1 \in M_2$, then $2a = (a+1)+(a-1) \in M_2$, and so $a \in M_2$. This gives a contradiction. Therefore, $a+1$ or $a-1$ is not in M_2. Thus, either $a+1 \in U(R)$ or $a-1 \in U(R)$. If $a \in M_2 \backslash M_1$, similarly to the proceeding discussion, $a+1 \in U(R)$ or $a-1 \in U(R)$. If $a \in M_1 \cap M_2$, then $a \in J(R)$; hence, $a-1 \in U(R)$. Therefore R is $\{0,1\}$-weakly clean. □

Lemma 9.1.4. (1) *If R is $\{0,1\}$-weakly clean, then R has at most two maximal ideals.*
(2) *Let F_1 and F_2 be fields. Then $F_1 \times F_2$ is $\{0,1\}$-weakly clean if and only if both F_1 and F_2 have characteristic not equal to 2.*

Proof. (1) This is obvious in terms of Theorem 9.1.2.

(2) Suppose that $F_1 \times F_2$ is $\{0,1\}$-weakly clean. If $char(F_1) = 2$, then $(1,0), (1,1)-(1,0) = (0,-1)$ and $(1,1)+(1,0) = (0,1)$ are all not invertible. This implies that $F_1 \times F_2$ is not $\{0,1\}$-weakly clean, a contradiction. Hence, F_1 has characteristic not equal to 2. Likewise, F_2 has characteristic not equal to 2.

Conversely, assume that F_1 and F_2 have characteristic not equal to 2. Then $F_1 \times F_2$ has exactly two maximal ideals with $2 \in U(F_1 \times F_2)$. Therefore the result follows by Lemma 9.1.3. □

Let $R = \mathbb{Z}_{(3)} \times \mathbb{Z}_{(3)}$. In view of Lemma 9.1.6, R is $\{0,1\}$-weakly clean. In this case, $S = \{(0,0),(1,1)\} \subsetneq Id(R) = \{(0,0),(0,1),(1,0)(1,1)\}$.

Theorem 9.1.5. *A ring R is $\{0,1\}$-weakly clean if and only if either* (1) *R is local, or* (2) *R has two maximal ideals and $2 \in U(R)$.*

Proof. $\Longrightarrow$ In view of Lemma 9.1.4, R has at most two maximal ideals. Case I. R has only one maximal ideal. Then R is local. Case II. R has exactly two maximal ideals, say M_1 and M_2. Then $R/M_1M_2 \cong R/M_1 \times R/M_2$. Since R is $\{0,1\}$-weakly clean, then so is $R/M_1 \times R/M_2$. By virtue of Lemma 9.1.3, R/M_1 and R/M_2 have characteristic not equal to 2. This shows that $2 \notin M_1 \cup M_2$. Therefore $2 \in U(R)$, as asserted.

$\Longleftarrow$ If R is local, then R is $\{0,1\}$-weakly clean. We now assume that R has two maximal ideals M_1, M_2 and $2 \in U(R)$. In light of Lemma 9.1.4, R is $\{0,1\}$-weakly clean, hence the result. □

Corollary 9.1.6. *An indecomposable weakly clean ring is either local, or an indecomposable ring with exactly two maximal ideals in which 2 is a unit.*

Proof. Let R be an indecomposable weakly clean ring. then it is $\{0,1\}$-weakly clean. Therefore we complete the proof by Theorem 9.1.5. □

Let p, q be distinct primes. Set $\mathbb{Z}_{(p)} \cap \mathbb{Z}_{(q)} = \{\frac{m}{n} \mid (m,n) = 1, m, n \in \mathbb{Z}, p, q \nmid n\}$. Then R is a commutative ring with identity $\frac{1}{1}$.

Theorem 9.1.7. *Let $p, q \neq 2$ be distinct primes. Then $\mathbb{Z}_{(p)} \cap \mathbb{Z}_{(q)}$ is $\{0,1\}$-weakly clean, but it is not clean.*

Proof. Set $R = \mathbb{Z}_{(p)} \cap \mathbb{Z}_{(q)}$. If M is an ideal of R such that $pR \subsetneq M \subseteq R$. Choose $\frac{m}{n} \in M$ while $\frac{m}{n} \notin pR$. Then $p \nmid m$; hence, $(p,m) = 1$. Thus, we can find some $k, l \in \mathbb{Z}$ such that $kp + lm = 1$. Clearly, $\frac{m}{1} = \frac{m}{n}\frac{n}{1} \in M$, hence, $\frac{1}{1} = p \cdot \frac{k}{1} + l\frac{m}{1} \in M$. This implies that $M = R$. Therefore pR is a maximal ideal of R. Likewise, qR is a maximal ideal of R. As $p, q \neq 2$, we see that $p, q \nmid 2$, and so $\frac{1}{2} \in R$. Obviously, $J(R) \subseteq pR \cap qR$. For any $\frac{m}{n} \in pR \cap qR$ and $\frac{a}{b} \in R$, then $\frac{1}{1} - \frac{m}{n}\frac{a}{b} = \frac{nb-ma}{nb}$. Write $\frac{m}{n} = \frac{ps}{t}$. Then $psn = mt$; hence, $p \mid mt$. As $p \nmid t$, we see that $p \mid m$. Likewise, $q \mid m$. Clearly, $p \nmid nb$, and so $p \nmid (nb - ma)$. Similarly, we see that $q \nmid (nb - ma)$. This yields that $\frac{nb}{nb-ma} \in R$.

Thus, $\left(\frac{1}{1} - \frac{m}{n}\frac{a}{b}\right) \cdot \frac{nb}{nb-ma} = \frac{1}{1}$. This means that $\frac{1}{1} - \frac{m}{n}\frac{a}{b} \in U(R)$, and so $\frac{m}{n} \in J(R)$. Therefore $J(R) = pR \cap qR$. Assume that M is a maximal

ideal of R and $M \neq pR, qR$. Then $pR + M = R$ and $qR + M = R$. It follows that $R = (pR + M)(qR + M) \subseteq pR \cap qR + M = J(R) + M \subseteq M$; hence, $R = M$. This gives a contradiction. Therefore R be a commutative ring with exactly two maximal ideals. According to Theorem 9.1.5, R is $\{0, 1\}$-weakly clean.

As $(p, q) = 1$, we see that $\frac{p(q+1)}{p+q} \in R$. Observing that R is an integral domain, the set of all idempotents in R is $\{\frac{0}{1}, \frac{1}{1}\}$. Since $p \mid p(q + 1)$, we see that $\frac{p(q+1)}{p+q} \notin U(R)$. Also we have $\frac{p(q+1)}{p+q} - \frac{1}{1} \notin U(R)$. This shows that $\frac{p(q+1)}{p+q} \in R$ is not strongly clean, as required. □

Example 9.1.8. Let F be a field with $char(F) \neq 2$, and let $R = F[[x, y]]$. Let $S = R \setminus (x) \cup (y)$. Then R_S is $\{0, 1\}$-weakly clean, but it is not clean.

Proof. As $F[[x, y]]/(x) \cong F[[y]]$ is an integral domain, we see that (x) is a prime ideal of $F[[x, y]]$. Likewise, (y) is a prime ideal of R. Set $S = R \setminus (x) \cup (y)$. Then S is a multiplicative closed subset of R. Let P be a maximal ideal of R_S. Then we have an ideal Q of R such that $P = Q_S$ such that $Q \cap S = \emptyset$. Hence, $Q \subseteq (x) \cup (y)$. Assume that $Q \nsubseteq (x)$ and $Q \nsubseteq (y)$. Then we can find some $b \in Q$, but $b \notin (x)$. Likewise, we have some $c \in Q$, but $c \notin (y)$. Set $a = b + c$. Then $a \in Q$, but $a \notin (x) \cup (y)$. This gives a contradiction. We infer that $Q \subseteq (x)$ or $Q \subseteq (y)$. Hence, $Q_S \subseteq (x)_S$ or $Q_S \subseteq (y)_S$. By the maximality of P, we get $P = (x)_S$ or $(y)_S$. Therefore R_S has exactly two maximal ideals $(x)_S$ and $(y)_S$. As $char(F) \neq 2$, we see that $2 \in U(R_S)$. Therefore R is $\{0, 1\}$-weakly clean, by Theorem 9.1.5. Clearly, R_S is an integral domain; and so it is indecomposable. This implies that R_S is not clean; otherwise, it is local, which is impossible. □

Example 9.1.9. Let $T = \mathbb{Q}[x, y, z]$, and let $S = R \setminus (x, y) \cup (x, z)$. Then $R = T_S/(xyz)_S$ is $\{0, 1\}$-weakly clean, but it is not clean.

Proof. Clearly, T has exactly two maximal ideals $(x, y)_S$ and $(x, z)_S$. Thus, R has exactly two maximal ideals $(x, y)_S/(xyz)_S$ and $(x, z)_S/(xyz)_S$. As $2 \in U(R)$, R is $\{0, 1\}$-weakly clean, by Theorem 9.1.5.

As $R/(x)_S/(xyz)_S \cong T_S/(x)_S$ is an integral domain, we see that $(x)_S/(xyz)_S$ is a prime ideal of R. Let $e \in R$ be an idempotent. Then $e \in (x)_S/(xyz)_S$ or $1 - e \in (x)_S/(xyz)_S$. But $(x)_S/(xyz)_S \subseteq \big((x, y)_S/(xyz)_S\big) \cap \big((x, z)_S/(xyz)_S\big) = J(R)$. We get $e = 0$ or 1. Therefore R is indecomposable. This implies that R is not clean; otherwise, R is local, an absurd. □

Example 9.1.10. A weakly clean is not necessarily $\{0,1\}$-weakly clean ring. Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2$. Then R is weakly clean. But R is not $\{0,1\}$-weakly clean, as $(1,0), (1,0) - (1,1) = (0,1), (1,0) + (1,1) = (0,1)$ are all not invertible.

Theorem 9.1.11. *Let $\{R_i\}$ be a family of commutative rings. Then the direct product $R = \prod R_i$ of rings is a weakly clean ring if and only if each R_i is a weakly clean ring and at most one is not a clean ring.*

Proof. $\Longrightarrow$ In view of Lemma 9.1.1, each R_i is weakly clean. Suppose R_{i_1} and $R_{i_2} (i_1 \neq i_2)$ are not clean. Then there exist some $x_{i_j} \in R_{i_j} (j = 1, 2)$ such that $x_{i_1} \in R_{i_1}$ and $-x_{i_2} \in R_{i_2}$ are not clean. Choose $x = (x_i)$ where $x_i = 0$ whenever $i \neq i_j (j = 1, 2)$. Then $x \pm e \notin U(R)$ for all idempotents $e \in R$. This gives a contradiction. Therefore each R_i is a weakly clean ring and at most one is not a clean ring.

$\Longleftarrow$ Suppose that R_{i_0} is weakly clean and all the other R_i is clean. For any $x = (x_i) \in R$, we can find an idempotent $e_{i_0} \in R$ such that $x_{i_0} - e_{i_0} \in U(R_{i_0})$ or $x_{i_0} + e_{i_0} \in U(R_{i_0})$.

Case I. $x_{i_0} - e_{i_0} \in U(R_{i_0})$. For each $i \neq i_0$, there exists an idempotent e_i such that $x_i - e_i \in U(R_i)$. Set $e = (e_i)$. Then $x - e \in U(R)$ and $e \in R$ is an idempotent.

Case II. $x_{i_0} + e_{i_0} \in U(R_{i_0})$. For each $i \neq i_0$, there exists an idempotent e_i such that $(-x_i) - e_i \in U(R_i)$. Set $e = (e_i)$. Then $x + e \in U(R)$ and $e \in R$ is an idempotent.

Therefore R is weakly clean. □

Corollary 9.1.12. *Let R be a commutative ring. Then the following are equivalent:*

(1) *R is an S-weakly clean ring with $|S| < \infty$.*
(2) *R is a weakly clean ring that is finite direct product of indecomposable rings.*
(3) *R is a finite direct product of local rings or R is a finite direct product of local rings and an indecomposable ring with exactly two maximal ideals in which 2 is a unit.*

Proof. (1) $\Rightarrow$ (2) This is obvious by Theorem 9.1.2.

(2) $\Rightarrow$ (3) Write $R = R_1 \times \cdots \times R_n$ where each R_i is a indecomposable ring. In view of Theorem 9.1.11, each R_i is weakly clean and at most one is not a clean ring. If all R_i are clean, then they are local, as every indecomposable clean ring is local. Otherwise, we may assume that $R_1, \ldots, R_{n-1}$

are clean and R_n is not clean. Then $R_1, \ldots, R_{n-1}$ are local. As R_n is indecomposable, it is a $\{0,1\}$-weakly clean ring. In light of Theorem 9.1.5, R has exactly two maximal ideals in which 2 is a unit.

(3) $\Rightarrow$ (1) Clearly, every local ring is $\{0,1\}$-weakly clean. By virtue of Theorem 9.1.5, every commutative ring with exactly two maximal ideals in which 2 is invertible is $\{0,1\}$-weakly clean. Therefore R an S-weakly clean ring with $|S| < \infty$. □

Proposition 9.1.13. *Let R be a commutative ring. Then the following are equivalent:*

(1) *R is weakly clean.*
(2) *$R[[X]]$ is weakly clean.*
(3) *$R/N(R)$ is weakly clean.*

Proof. (1) $\Rightarrow$ (2) For any $f(x) \in R[[x]]$, there exists an idempotent $e \in R$ such that $f(0) \pm e \in U(R)$. Hence, $f(x) \pm e \in U\big(R[[x]]\big)$. Therefore $R[[x]]$ is weakly clean.

(2) $\Rightarrow$ (1) This is obvious.

(1) $\Rightarrow$ (3) This is clear by Lemma 9.1.1.

(3) $\Rightarrow$ (1) Let $a \in R$. There exists an idempotent $\overline{e}$ and a unit $\overline{u}$ in $R/N(R)$ such that $\overline{a} = \pm\overline{e} + \overline{u}$. As idempotents and units lift modulo $N(R)$, we may assume that $e = e^2 \in R$ and $u \in U(R)$. Further, $a = \pm e + u + r$ for some $r \in N(R)$. Clearly, $u + r \in U(R)$. Therefore R is weakly clean. □

Proposition 9.1.14. *Let R be a commutative ring and M an R-module. Then $T(R,M)$ is weakly clean if and only if so is R.*

Proof. $\Longrightarrow$ For any $a \in R$, there exists an idempotent $(e,m) \in T(R,M)$ such that $(a,0) \pm (e,m) \in T(R,M)$ is invertible. This implies that $e = e^2 \in R$ and $a \pm e \in U(R)$, and so R is weakly clean.

$\Longleftarrow$ Let $(a,m) \in T(R,M)$. Then we have an idempotent $e \in R$ such that $a \pm e \in U(R)$. Hence, $(a,m) \pm (e,0) = (a \pm e, m) \in T(R,M)$ is invertible. Therefore $T(R,M)$ is weakly clean. □

Proposition 9.1.15. *Let R be a ring. Then the following are equivalent:*

(1) *R is unit-regular.*
(2) *For any $a \in R$, there exists an idempotent $e \in R$ such that $a + e$ or $a - e \in U(R)$ and $aR \cap eR = 0$.*

Proof. (1) $\Rightarrow$ (2) This is obvious by Corollary 7.2.26.

(2) $\Rightarrow$ (1) Let $a \in R$. Then there exists an idempotent $e \in R$ and a unit $u \in R$ such that $a = e \pm u$ and $aR \cap eR = 0$. Hence, $au^{-1} = eu^{-1} \pm 1$. Thus, $au^{-1}e = eu^{-1}e \pm e \in aR \cap eR = 0$. Hence, $au^{-1}(a \mp u) = 0$, and so $a(\pm u^{-1})a = a$. Therefore $a \in R$ is unit-regular. □

We record a result of Chin and Qua [195, Theorem 2.1] as follows.

Proposition 9.1.16. *Let R be a commutative ring. Then the following are equivalent:*

(1) *R is weakly clean.*
(2) *For any $a \in R$, there exists an idempotent $e \in aR$ such that $1 - e \in (1 - a)R$ or $1 - e \in (1 + a)R$.*

Proof. (1) $\Rightarrow$ (2) For any $a \in R$, there exists an idempotent $f \in R$ such that $a = \pm f + v$, where $v \in U(R)$. Set $e = 1 - f$. Then $e = e^2 \in R$. Clearly, $a \mp e = (a^2 \mp a)v^{-1}$. Hence, $e \in aR$ and $1 - e \in (1 - a)R$ or $1 - e \in (1 + a)R$, as desired.

(2) $\Rightarrow$ (1) For any $a \in R$, there exists an idempotent $e \in aR$ such that $1 - e \in (1 - a)R$ or $1 - e \in (1 + a)R$. If $1 - e \in (1 - a)R$, then $a - (1 - e) \in U(R)$ as in the proof of Corollary 4.4.8. If $1 - e \in (1 + a)R$, then $ae \in U(eRe)$ and $(1 + a)(1 - e) \in U\big((1 - e)R(1 - e)\big)$. Thus, $a = -(1 - e) + \big(ea + (1 - e)(1 + a)\big)$. Therefore R is weakly clean. □

Theorem 9.1.17. *Let R be a commutative ring with at most three maximal ideals. If $2 \in U(R)$ and $J(R)$ is nil, then R is weakly clean.*

Proof. Case I. R has only maximal ideal. Then R is local, and so it is clean.

Case II. R has two maximal ideals. Then R is weakly clean, by Theorem 9.1.5.

Case III. R has three maximal ideals M_1, M_2 and M_2. Let $a \in R$. If $a \notin M_1, M_2, M_3$, then $aR = R$; hence, $a \in U(R)$. So we may assume that $a \in M_1$. If $1 - a \notin M_1, M_2, M_3$, then $1 - a \in U(R)$, and so $a \in R$ is clean. If $1 - a \in M_1$, then $1 = a + (1 - a) \in M_1$, a contradiction. Thus, $1 - a \in M_2$ or $1 - a \in M_3$. If $1 + a \notin M_1, M_2, M_3$, then $1 + a \in U(R)$; hence, $a \in R$ is weakly clean. If $1 + a \in M_1$, then $1 = (1 + a) - a \in M_1$, a contradiction. Hence $1 + a \in M_2$ or $1 + a \in M_3$. There are only two possible cases: $1 - a \in M_2, 1 + a \in M_3$ or $1 - a \in M_3, 1 + a \in M_2$, as

$2 \in U(R)$. Thus, we may assume that $1-a \in M_2, 1+a \in M_3$. Hence, $a(1-a)(1+a) \in M_1M_2M_3 \subseteq J(R)$, and then $\overline{a} = \overline{a}^3 \in R/J(R)$. Set $\overline{e} = \overline{1-a^2}$ and $\overline{u} = \overline{a^2+a-1}$. Then $\overline{e} \in R/J(R)$ is an idempotent and $\overline{u}^2 = \overline{1}$. Further, we have $\overline{a} = \overline{e} + \overline{u}$. By hypothesis, $J(R)$ is nil, and then every unit and every idempotent lift modulo $J(R)$. So we may assume that $e \in R$ is an idempotent and $u \in U(R)$. Set $w := a - e - u$. Then $a = e + u + w$ where $w \in J(R)$. Clearly, $u + w = u(1 + u^{-1}w) \in U(R)$, and so $a \in R$ is clean. Therefore R is weakly clean. □

Example 9.1.18. Let $R = \{\frac{m}{n} \mid (m,n) = 1, m, n \in \mathbb{Z} \text{ and } 3,5,7 \nmid n\}$. Then R has three maximal ideals and $2 \in U(R)$, but R is not weakly clean.

Proof. Obviously $2 \in R$ is invertible. Let M be an ideal of R such that $3R \subsetneq M \subseteq R$. Choose $\frac{m}{n} \in M$ while $\frac{m}{n} \notin 3R$. Then $3 \nmid m$, and so $(3,m) = 1$. So $3k + lm = 1$ for some $k, l \in \mathbb{Z}$. This shows that $\frac{1}{1} = 3 \cdot \frac{k}{1} + l\frac{m}{n} \cdot \frac{n}{1} \in M$, i.e., $M = R$. Thus, $3R$ is a maximal ideal of R. Likewise, $5R$ and $7R$ are maximal ideals of R. For any $\frac{m}{n} \in 3R \cap 5R \cap 7R$ and $\frac{a}{b} \in R$, then $\frac{1}{1} - \frac{m}{n}\frac{a}{b} = \frac{nb-ma}{nb}$. Write $\frac{m}{n} = \frac{3s}{t}$. Then $3sn = mt$, and so $3 \mid mt$. Since $3 \nmid t$, we get $3 \mid m$. Obviously, $3 \nmid nb$; hence, $3 \nmid (nb - ma)$. Similarly, $5,7 \nmid (nb - ma)$. It follows that $\frac{nb}{nb-ma} \in U(R)$. We infer that $\frac{m}{n} \in J(R)$. Therefore $3R \cap 5R \cap 7R \subseteq J(R)$. Let M be a maximal ideal of R and $M \neq 3R, 5R, 7R$. Then $3R + M = R, 5R + M = R$ and $7R + M = R$. Thus, $R = (3R+M)(5R+M)(7R+M) \subseteq 3R \cap 5R \cap 7R + M = J(R) + M \subseteq M$, hence, $R = M$, an absurd. We infer that R is a commutative ring with exactly three maximal ideals.

If R is weakly clean, then it is weakly $\{0,1\}$-clean, as R is an integral domain. It follows by Theorem 9.1.5, R has two maximal ideal, a contradiction. This completes the proof. □

Lemma 9.1.19. *Let G be a cyclic group of order q generated by a, and let p be prime. Then $\alpha = \sum_{i=0}^{q-1} k_i a^i \in U\big(\mathbb{Z}_{(p)}G\big)$ if and only if*

$$p \nmid \begin{vmatrix} k_0 & k_{q-1} & k_{q-2} & \cdots & k_1 \\ k_1 & k_0 & k_{q-1} & \cdots & k_2 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ k_{q-1} & k_{q-2} & k_{q-3} & \cdots & k_0 \end{vmatrix}.$$

Proof. $\alpha = \sum_{i=0}^{q-1} k_i a^i \in U\big(\mathbb{Z}_{(p)}G\big)$ if and only if there exists $\beta = \sum_{i=0}^{q-1} l_i a^i \in \mathbb{Z}_{(p)}G$ such that $\alpha\beta = 1$, i.e.,

$$\begin{aligned} k_0l_0 + k_{q-1}l_1 + \cdots + k_1l_{q-1} &= 1 \\ k_1l_0 + k_0l_1 + \cdots + k_2l_{q-1} &= 0 \\ &\vdots \\ k_{q-1}l_0 + k_{q-2}l_1 + \cdots + k_0l_{q-1} &= 0 \end{aligned}$$

The determinant of the coefficient matrix is

$$\begin{vmatrix} k_0 & k_{q-1} & k_{q-2} & \cdots & k_1 \\ k_1 & k_0 & k_{q-1} & \cdots & k_2 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ k_{q-1} & k_{q-2} & k_{q-3} & \cdots & k_0 \end{vmatrix}.$$

Therefore, by Crammer's Rule, the result follows. □

Let G be a cyclic group of order 2 generated by a. Then $\alpha = k + la \in U\big(\mathbb{Z}_pG\big)$ if and only if $p \nmid k^2 - l^2$. Let G be a cyclic group of order 3 generated by a. Then $\alpha = k + la + ma^2 \in U\big(\mathbb{Z}_pG\big)$ if and only if $p \nmid k^3 + l^3 + m^3 - 3klm$.

Lemma 9.1.20. *Let G be a cyclic group of order 3 generated by a, and let p be prime with $p \neq 3$. If $e \in \mathbb{Z}_{(p)}G$ is an idempotent, then $e = 0, 1, \frac{1}{3} + \frac{1}{3}a + \frac{1}{3}a^2, \frac{2}{3} - \frac{1}{3}a - \frac{1}{3}a^2$.*

Proof. Write $e = x + ya + za^2 \in \mathbb{Z}_{(p)}G$. Then $e = e^2$, and so

$$\begin{aligned} x^2 + 2yz &= x \quad (1) \\ z^2 + 2xy &= y \quad (2) \\ y^2 + 2xz &= z \quad (3) \end{aligned}$$

$(2) \cdot z - (3) \cdot y$, we get $y^3 = z^3$ in $\mathbb{Z}_{(p)}$. As y, z are both rational numbers, we get $y = z$. Hence,

$$\begin{aligned} x^2 + 2y^2 &= x \\ y^2 + 2xy &= y. \end{aligned}$$

This implies that $(x - y)(x - y - 1) = 0$, and so $x = y$ or $x = y + 1$. If $x = y$, then $y^2 + 2y^2 = y$; whence, $y = z = 0$ or $y = z = \frac{1}{3}$. If $x = y + 1$, then $(y+1)^2 + 2y^2 = y + 1$; whence, $y = z = 0$ or $y = z = -\frac{1}{3}$. Accordingly, $e = 0, 1, \frac{1}{3} + \frac{1}{3}a + \frac{1}{3}a^2, \frac{2}{3} - \frac{1}{3}a - \frac{1}{3}a^2$. □

Theorem 9.1.21. *If G is a cyclic group of order 3 and p is prime, then $\mathbb{Z}_{(p)}G$ is weakly clean.*

Proof. Let $G = \{1, a, a^2\}$ with $a^3 = 1$. Then

$$\mathbb{Z}_{(p)}G = \left\{ \frac{m_1}{n_1} + \frac{m_2}{n_2}a + \frac{m_3}{n_3} \;\middle|\; m_i, n_i \in \mathbb{Z}, (m_i, n_i) = 1 \text{ and } p \nmid \text{ each } n_i \right\}.$$

Let $\alpha \in \mathbb{Z}_{(p)}G$. Then we can find $n \in \mathbb{Z}$ such that $(p, n) = 1$ and $\alpha = \frac{k+la+ma^2}{n}$.

Case I. $p \neq 2$. One easily checks that

$$\begin{aligned} \alpha &= 0 + \tfrac{k+la+ma^2}{n} \\ &= 1 + \tfrac{(k-n)+la+ma^2}{n} \\ &= -1 + \tfrac{(k+n)+la+ma^2}{n}. \qquad (*) \end{aligned}$$

In view of Lemma 9.1.19, we will suffice to prove that p does not divide at least one of the following:

$$\begin{aligned} &k^3 + l^3 + m^3 - 3klm \quad (1) \\ &(k-n)^3 + l^3 + m^3 - 3(k-n)lm \quad (2) \\ &(k+n)^3 + l^3 + m^3 - 3(k+n)lm \quad (3) \end{aligned}$$

Suppose that p divides all of (1) − (3). As p divides (1) and (2), we get $p| - 3k^2n + 3kn^2 - n^3 + 3nlm$. Since p divides (1) and (3), we get $p|3k^2n + 3kn^2 + n^3 - 3nlm$. It follows that $p|6kn^2$. As $p \nmid n^2$, we get $p|6k$. Likewise, $p|6l, 6m$. Since $p \neq 2$, we see that $p|3k, 3l, 3m$. Hence, $p|n^3$, a contradiction. Therefore $\mathbb{Z}_{(p)}G$ is weakly clean for $p \neq 2$.

Case II. $p = 2$. As $2 \nmid n$, n is odd. Thus, $\frac{k+la+ma^2}{n} \in U\big(\mathbb{Z}_{(p)}G\big)$ if and only if $2 \nmid \big(k^3 + l^3 + m^3 - 3klm\big)$.

(1) If k, l, m are all even, then

$$\frac{k+la+ma^2}{n} = 1 + \frac{(k-n)+la+ma^2}{n}.$$

Here, l and m are even, while $k - n$ is odd. Thus, $(k-n)^3 + l^3 + m^3 - 3(k-n)lm$ is odd. This implies that $\frac{(k-n)+la+ma^2}{n} \in U\big(\mathbb{Z}_{(p)}G\big)$.

(2) If one of k, l, m is even and the other two are odd. We may assume that k is even, l and m are odd. Then

$$\frac{k+la+ma^2}{n} = \frac{1+a+a^2}{3} + \frac{(3k-n)+(3l-n)a+(3m-n)a^2}{3n}.$$

Here, $3l - n$ and $3m - n$ are even, while $3k - n$ is odd. Thus, $(3k-n)^2 + (3l-n)^2 + (3m-n)^2 - 3(3k-n)(3l-n)(3m-n)$ is odd. This implies that $\frac{(3k-n)+(3l-n)a+(3m-n)a^2}{3n} \in U\big(\mathbb{Z}_{(p)}G\big)$.

(3) If k, l, m are all odd, then

$$\frac{k+la+ma^2}{n} = \frac{2-a-a^2}{3} + \frac{(3k-2n)+(3l+n)a+(3m+n)a^2}{3n}.$$

Here, $3k-2n$ is odd, $3l+n$ and $3m+n$ are even. Thus, $(3k-2n)^3+(3l+n)^3+(3m+n)^3-3(3k-2n)(3l+n)(3m+n)$ is odd. This implies that $\frac{(3k-2n)+(3l+n)a+(3m+n)a^2}{3n} \in U\big(\mathbb{Z}_{(p)}G\big)$.

In any case, $\frac{k+la+ma^2}{n}$ is the sum of an idempotent and a unit. Therefore $\mathbb{Z}_{(p)}G$ is clean for $p=2$, and we are done. □

Let G be a cyclic group of order 3 and $p \neq 2$ be prime. As in the proof of Theorem 9.1.21, we prove that $\mathbb{Z}_{(p)}G$ is 2-good.

We come now to record an example which is due to [556, Example 3.1].

Example 9.1.22. If G is a cyclic group of order 3, then $\mathbb{Z}_{(7)}G$ is weakly clean, while it is not clean.

Proof. Let $G=\{1,a,a^2\}$. In view of Theorem 9.1.21, $\mathbb{Z}_{(7)}G$ is weakly clean. If $2+3a \in \mathbb{Z}_{(7)}G$ is clean, then there exists an idempotent $e \in \mathbb{Z}_{(7)}G$ such that $2+3a-e \in U\big(\mathbb{Z}_{(7)}G\big)$. In view of Lemma 9.1.20, $e=0,1,\frac{1}{3}+\frac{1}{3}a+\frac{1}{3}a^2$ or $\frac{2}{3}-\frac{1}{3}a-\frac{1}{3}a^2$. Observing that

$$\begin{aligned} 2+3a &= 0+(2+3a)\\ &= 1+(1+3a)\\ &= \big(\tfrac{1}{3}+\tfrac{1}{3}a+\tfrac{1}{3}a^2\big)+\tfrac{5+8a-a^2}{3}\\ &= \big(\tfrac{2}{3}-\tfrac{1}{3}a-\tfrac{1}{3}a^2\big)+\tfrac{4+10a+a^2}{3}, \quad (*) \end{aligned}$$

we need to show that each element on the second column in $(*)$ is not invertible. It is easy to verify that 7 divides

$$\begin{aligned} &2^3+3^3-3\cdot 2\cdot 3\cdot 0=35,\\ &1^3+3^3-3\cdot 1\cdot 3\cdot 0=28,\\ &\big(\tfrac{5}{3}\big)^3+\big(\tfrac{8}{3}\big)^3+\big(-\tfrac{1}{3}\big)^3-3\cdot\tfrac{5}{3}\cdot\tfrac{8}{3}\cdot\big(-\tfrac{1}{3}\big)=\tfrac{756}{27},\\ &\big(\tfrac{4}{3}\big)^3+\big(\tfrac{10}{3}\big)^3+\big(\tfrac{1}{3}\big)^3-3\cdot\tfrac{4}{3}\cdot\tfrac{10}{3}\cdot\tfrac{1}{3}=\tfrac{945}{27}, \end{aligned}$$

and so none of them is invertible, by Lemma 9.1.19. Accordingly, $2+3a \in \mathbb{Z}_{(7)}G$ is not clean. □

Let G be a cyclic group of order 3, and let p be prime. Then the group ring $\mathbb{Z}_{(p)}G$ is n-clean for all $n \geq 2$. An element a in a ring is periodic if $a^m=a^n$ for some distinct $m,n \in \mathbb{N}$. By virtue of Theorem 9.1.21, every element in $\mathbb{Z}_{(p)}G$ is the sum of a periodic element and a unit. This was also proved by Ye (see [556, Theorem 3.1]). Thus, every element in $M_2(\mathbb{Z}_{(p)}G)$ is the sum of a periodic matrix and an invertible matrix.

In the next, we give more examples of weakly clean subrings of $\mathbb{R}$ and $\mathbb{C}$ which need not be clean, and determine when they are clean. By $\mathbb{Z}_{(p)}[\sqrt{q}]$, where p is prime and $q \in \mathbb{N}$ is not a square, denote the set $\{a_0 + a_1\sqrt{q} \mid a_0, a_1 \in \mathbb{Z}_{(p)}\}$. Then it is a subring of $\mathbb{R}$.

Lemma 9.1.23. *$\frac{k_0+k_1\sqrt{q}}{m} \in U\big(\mathbb{Z}_{(p)}[\sqrt{q}]\big)$ if and only if $p \nmid k_0^2 - k_1^2 q$.*

Proof. The inverse of $\frac{k_0+k_1\sqrt{q}}{m}$ in $\mathbb{R}$ is $\frac{m(k_0-k_1\sqrt{q})}{k_0^2-k_1^2q}$. Thus, $\frac{k_0+k_1\sqrt{q}}{m} \in U\big(\mathbb{Z}_{(p)}[\sqrt{q}]\big)$ if and only if p does not divide the denominator. □

Theorem 9.1.24. *Let p be a prime. Then $\mathbb{Z}_{(p)}[\sqrt{q}]$ is weakly clean.*

Proof. Suppose, on the contrary, there exists $a = \frac{k_0+k_1\sqrt{q}}{m}, p \nmid n$, such that none of $a, a-1, a+1$ is a unit. Then we have

$$\begin{aligned} p &\mid k_0^2 - k_1^2 q && (1)\\ p &\mid (k_0 - n)^2 - k_1^2 q && (2)\\ p &\mid (k_0 + n)^2 - k_1^2 q && (3). \end{aligned}$$

Subtracting (2) from (1) and (1) from (3), we get

$$\begin{aligned} p &\mid n(2k_0 - n)\\ p &\mid n(2k_0 + n). \end{aligned}$$

Since $p \nmid n$, we see that $p|2k_0 - n, 2k_0 + n$, and then $p|2n$. This implies that $p|2$, and so $p = 2$. It follows from $p|2k_0 + n$ that $p|n$, a contradiction. Therefore every element in $\mathbb{Z}_{(p)}[\sqrt{q}]$ is the sum or difference of a unit and an idempotent. This complestes the proof. □

Example 9.1.25. Let $q \in \mathbb{N}$ be not a square. Then $\mathbb{Z}_{(2)}[\sqrt{q}]$ is clean.

Proof. Suppose $a = \frac{k_0+k_1\sqrt{q}}{m}(m \text{ odd})$ is such that a and $a-1$ are not units in $\mathbb{Z}_{(2)}[\sqrt{q}]$. By virtue of Lemma 9.1.23, we get

$$\begin{aligned} 2 &\mid k_0^2 - k_1^2 q && (1)\\ 2 &\mid (k_0 - n)^2 - k_1^2 q && (2) \end{aligned}$$

Subtracting (2) from (1), we see that $2|n(2k_0 - n)$. Since $2 \nmid n$, we see that $2|2k_0 - n$, an absurd. Therefore $\mathbb{Z}_{(2)}[\sqrt{q}]$ is clean. □

Theorem 9.1.26. *Let p be an odd prime, $q \in \mathbb{N}$ not a square, and let $R = \mathbb{Z}_{(p)}[\sqrt{q}]$.*

(1) *If $p|q$, then R is clean.*
(2) *If $p \nmid q$, then R is clean if and only if q is a quadratic nonresidue of p.*

Proof. (1) Suppose $a = \frac{k_0+k_1\sqrt{q}}{n}(p \nmid n)$ is such that a and $a-1$ are not units in R. In virtue of Lemma 9.1.23, we have

$$\begin{aligned} p \mid k_0^2 - k_1^2 q \\ p \mid (k_0 - n)^2 - k_1^2 q. \end{aligned}$$

As $p|q$, we get $p \mid k_0$ and $p \mid k_0 - n$. Thus, $p|n$, a contradiction. Hence, R is clean.

(2) $\Longrightarrow$ Suppose q is a quadratic residue of p, i.e., $q \equiv k^2 (\bmod p)$ for $1 \leq k \leq p-1$. Take $a = \frac{k+\sqrt{q}}{2k} \in R$. We easily check that a and $a-1$ are not invertible. This shows that R is not clean.

$\Longleftarrow$ Suppose R is not clean. Then we can find $a = \frac{k_0+k_1\sqrt{q}}{n}, p \nmid n$ such that $a, a-1$ are not units. Hence,

$$\begin{aligned} p \mid k_0^2 - k_1^2 q \\ p \mid (k_0 - n)^2 - k_1^2 q. \end{aligned}$$

Subtracting these two, we have $p \mid n(2k_0 - n)$. Since $p \nmid n$, we get $p \nmid k_0$. It follows from (1) that $p \nmid k_1$; hence, $(p, k_1) = 1$. By Fermat's little theorem, $(k_1)^{p-1} \equiv 1 (\bmod p)$. Thus, we have some $l \in \mathbb{Z}$ such that $k_1 l \equiv 1 (\bmod p)$. Write $k_0^2 - k_1^2 q = pt$ for some $t \in R$. Then $k_0^2 \equiv k_1^2 q (\bmod p)$. Hence, $q \equiv (k_0 l)^2 (\bmod p)$. Therefore q is a quadratic residue of p, as required. □

Let $R = \mathbb{Z}_{(7)}[\sqrt{2}]$. Then R is not clean. Clearly, $2 \equiv 3^2 (\bmod 7)$, i.e., 2 is a quadratic residue of 7. Thus we are through by Theorem 9.1.26.

Lemma 9.1.27. $\frac{k_0+k_1 i}{m} \in U\big(\mathbb{Z}_{(p)}[i]\big)$ *if and only if* $p \nmid k_0^2 + k_1^2$.

Proof. The inverse of $\frac{k_0+k_1 i}{m}$ in $\mathbb{C}$ is $\frac{m(k_0-k_1 i)}{k_0^2+k_1^2}$. We see that $\frac{k_0+k_1 i}{m} \in U\big(\mathbb{Z}_{(p)}[i]\big)$ if and only if $p \nmid k_0^2 + k_1^2$. □

Theorem 9.1.28. *Let p be a prime. Then $\mathbb{Z}_{(p)}[i]$ is weakly clean.*

Proof. Suppose $\mathbb{Z}_{(p)}[i]$ is not weakly clean. The we have an element $a = \frac{k_0+k_1 i}{n}, p \nmid n$ such that $a, a-1, a+1$ are not invertible. In light of Lemma 9.1.19, we have

$$\begin{aligned} p \mid k_0^2 + k_1^2 \\ p \mid (k_0 - n)^2 + k_1^2 \\ p \mid (k_0 + n)^2 + k_1^2. \end{aligned}$$

As in the proof of Theorem 9.1.24, we derive

$$\begin{aligned} p \mid n(2k_0 - n) \\ p \mid n(2k_0 + n), \end{aligned}$$

and then $p|n$. This gives a contradiction, whence the result. □

As in the proof of Example 9.1.25, we have

Example 9.1.29. The ring $\mathbb{Z}_{(2)}[i]$ is clean.

Theorem 9.1.30. *Let p be an old prime. Then $\mathbb{Z}_{(p)}[i]$ is clean if and only if $p \equiv 3(mod\ 4)$.*

Proof. Analogously to Theorem 9.1.26, we show that $\mathbb{Z}_{(p)}[i]$ is clean if and only if -1 is a quadratic nonresidue of p. By Euler's criterion, this will happen whenever $(-1)^{\frac{p-1}{2}} \not\equiv 1(\text{mod } p)$, i.e., $p \equiv 3(\text{mod } 4)$. □

According to Theorem 9.1.30 and Theorem 9.1.28, we easily derive

Example 9.1.31. The ring $\mathbb{Z}_{(5)}[i]$ is not clean, but it is weakly clean.

As is well known, the matrix ring over a clean ring is clean. But the matrix ring over a weakly clean ring may be not weakly clean. We shall close this section by providing an example, due to Kosan et al. (see [372, Example 6]). We need first an elementary lemma.

Lemma 9.1.32. *Let R be a commutative ring, $A \in M_n(R)$ and $x \in R$. Then $det(xE_{ij} + A) = xa + det(A)$ for some $a \in R$, where E_{ij} is the square matrix with (i, j)-entry 1 and all other entries 0.*

Theorem 9.1.33. *Let R be a commutative ring, $n \geq 2$. Then $M_n(R)$ is weakly clean if and only if R is clean.*

Proof. $\Longrightarrow$ Similarly to Theorem 7.1.6, a ring R is clean if and only if so is its every indecomposable homomorphic image. Thus, we may assume that R is indecomposable. Suppose that R is not clean. Take $x \in R$ which is not clean.

Choose $A = xE_{11} - xE_{22}$. Then we have $E^2 = E, U \in GL_2(R)$ such that $A = E \pm U$. Clearly, $det(E) = 0$ or 1 as R is indecomposable. If $det(E) = 1$, then $E = I_2$, and so $x \in R$ is clean, a contradiction. Thus $det(E) = 0$. If $A = E + U$, then $det(U) = det(xE_{11} - xE_{22} - E) = xa + det(-xE_{22} - E) = xa + xb + (-1)^n det(E) = x(a + b)$. This shows that $x \in U(R)$, a contradiction. If $A = -E + U$, analogously, we get a contradiction. Therefore R is clean, as desired.

$\Longleftarrow$ This is obvious by Corollary 7.1.20. □

Example 9.1.34. Let $R = \mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)}$. Then R is weakly clean, but $M_2(R)$ is not weakly clean.

Proof. This is obvious by Theorem 9.1.7 and Theorem 9.1.33. □

9.2 Weakly Nil-clean Rings

A ring R is weakly nil-clean provided that for any $a \in R$ there exists an idempotent $e \in R$ such that $a - e$ or $a + e$ is nilpotent. Clearly, every homomorphic image of a weakly nil-clean ring is weakly nil-clean. Further, we have

Proposition 9.2.1. *Let I be a nil ideal of a ring R. Then R is weakly nil-clean if and only if R/I is weakly nil-clean.*

Proof. One direction is obvious. Conversely, assume that R/I is weakly nil-clean. Let $a \in R$. Then $\overline{a} \in R/I$, and so we can find an idempotent $\overline{e} \in R/I$ and a unit $\overline{u} \in R/I$ such that $\overline{a} = \overline{u} \pm \overline{e}$. As idempotents and units lift modulo I, we may assume that $e \in R$ is an idempotent and $u \in U(R)$. Thus, $a = (u + r) \pm e$ for some $r \in I$. Clearly, $u + r \in I$ is invertible, hence the result. □

Corollary 9.2.2. *Let R be a ring. Then the following are equivalent:*

(1) *R is weakly nil-clean.*
(2) *$R/P(R)$ is nil-clean.*

Proof. In view of Lemma 4.3.1, $P(R)$ is nil. This completes the proof by Proposition 9.2.1. □

Corollary 9.2.3. *A ring R is weakly nil-clean if and only if $R[[x]]/(x^n)(n \in \mathbb{N})$ is weakly nil-clean.*

Proof. Clearly, $R[[x]]/(x^n) = \{a_0 + a_1x + \cdots + a_{n-1}x^{n-1} \mid a_0, \ldots, a_{n-1} \in R\}$. Let $\alpha : R[[X]]/(x^n) \longrightarrow R$ be a morphism such that $\alpha(f) = f(0)$. It is easy to check that α is an R-epimorphism and $Ker\alpha$ is a nil ideal of R, and therefore the result follows from Proposition 9.2.1. □

Lemma 9.2.4. *Let R be weakly nil-clean and S be nil-clean. Then $R \times S$ is weakly nil-clean.*

Proof. Set $A = R \times S$. Let $(a, b) \in A$. Then there exists an idempotent $e \in R$ and a nilpotent $v \in R$ such that $a = e + v$ or $a = -e + v$.

Case I. $a = e + v$. Then we have an idempotent $f \in S$ and a nilpotent $w \in S$ such that $b = f + w$. Thus, $(a, b) = (e, f) + (v, w)$, where $(e, f) \in A$ is an idempotent and $(v, w) \in A$ is nilpotent.

Case II. $a = -e+v$. Then we have an idempotent $f \in S$ and a nilpotent $w \in S$ such that $-b = f + w$. Thus, $(a, b) = -(e, f) + (v, -w)$, where $(e, f) \in A$ is an idempotent and $(v, -w) \in A$ is nilpotent.

Therefore (a, b) is the sum or difference of a nilpotent and an idempotent in A, hence the result. □

Theorem 9.2.5. *Let $\{R_i\}$ be a family of rings. Then the direct product $R = \prod R_i$ of rings R_i is weakly nil-clean if and only if each R_i is weakly nil-clean and at most one is not nil-clean.*

Proof. $\Longrightarrow$ Obviously, each R_i is weakly nil-clean. Suppose R_{i_1} and $R_{i_2} (i_1 \neq i_2)$ are not nil-clean. Then there exist some $x_{i_j} \in R_{i_j} (j = 1, 2)$ such that $x_{i_1} \in R_{i_1}$ and $-x_{i_2} \in R_{i_2}$ are not nil-clean. Choose $x = (x_i)$ where $x_i = 0$ whenever $i \neq i_j (j = 1, 2)$. Then x and $-x$ are not the sum of an idempotent and a nilpotent. This gives a contradiction. Therefore each R_i is a weakly nil-clean and at most one is not nil-clean.

$\Longleftarrow$ Suppose that R_{i_0} is weakly nil-clean and all the others R_i are nil-clean. Then $\prod\limits_{i \neq i_0} R_{i_0}$ is nil-clean. In light of Lemma 9.2.4, we conclude that R is weakly nil-clean. □

Corollary 9.2.6. *Let $L = \prod\limits_{i \in I} R_i$ be the direct product of rings $R_i \cong R$ and $|I| \geq 2$. Then L is weakly nil-clean if and only if L is nil-clean if and only if R is nil-clean.*

Proposition 9.2.7. *Let R be a ring. Then the following are equivalent:*

(1) *R is nil-clean.*
(2) *$T_n(R)$ is nil-clean for all $n \in \mathbb{N}$.*
(3) *$T_n(R)$ is nil-clean for some $n \in \mathbb{N}$.*
(4) *$T_n(R)$ is weakly nil-clean for some $n \geq 2$.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) These are proved as in the proof of Corollary 8.4.14.

(3) $\Rightarrow$ (4) This is trivial.

(4) $\Rightarrow$ (1) Let $a \in R$. Choose $A = \begin{pmatrix} a & & & & \\ & -a & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} \in T_n(R)$. By hypothesis, we can find an idempotent $\begin{pmatrix} e_1 & & & & \\ & e_2 & & & \\ & & \ddots & & \\ & & & & e_n \end{pmatrix}$ and a nilpotent $\begin{pmatrix} w_1 & & & & \\ & w_2 & & & \\ & & \ddots & & \\ & & & & w_n \end{pmatrix}$ such that

$$A = \pm \begin{pmatrix} e_1 & & & & \\ & e_2 & & & \\ & & \ddots & & \\ & & & & e_n \end{pmatrix} + \begin{pmatrix} w_1 & & & & \\ & w_2 & & & \\ & & \ddots & & \\ & & & & w_n \end{pmatrix}.$$

It follows that $a = e_1 + w_1$ or $a = e_2 - w_2$. Clearly, e_1, e_2 are idempotents and w_1, w_2 are nilpotent. Therefore proving (1). □

Theorem 9.2.8. *Let R be a ring. Then the following are equivalent:*

(1) *R is weakly nil-clean.*
(2) *$R/6R$ is weakly nil-clean and $6 \in N(R)$.*
(3) *$R/J(R)$ is weakly nil-clean and $J(R)$ is nil.*

Proof. (1) $\Rightarrow$ (2) Write $2 = e + w$ or $2 = -e + w$ with $e = e^2, w \in N(R)$. If $2 = e + w$, then $1 - e = w - 1 \in U(R)$, and so $e = 0$. This implies that $2 = w \in N(R)$. If $2 = -e + w$, then $3 = (1 - e) + w$. Obviously, $ew = we$, and so $6 = (-e + w)\big((1 - e) + w\big) = -ew + w(1 - e) + w^2 \in N(R)$. Clearly, $R/6R$ is weakly nil-clean, as desired.

(2) $\Rightarrow$ (1) This is obvious, by Proposition 9.2.1.

(1) $\Rightarrow$ (3) Since R is weakly nil-clean, then so is $R/J(R)$. Let $x \in J(R)$. Then there exists an idempotent $e \in R$ such that $w := x - e \in N(R)$. Write $w^m = 0 (m \in \mathbb{N})$. Then $e = x - w$, and so $e = (x - w)^m \in J(R)$. We infer that $e = 0$; hence, $x = w \in N(R)$. Therefore $J(R)$ is nil.

(3) $\Rightarrow$ (1) This is clear from Proposition 9.2.1. □

Lemma 9.2.9. *Let R be a ring. Then the following are equivalent:*

(1) *R is nil-clean.*
(2) *R is weakly nil-clean and $2 \in J(R)$.*

Proof. (1) $\Rightarrow$ (2) Clearly, R is weakly nil-clean. Write $2 = e + w$ with $e = e^2 \in R$ and $w \in N(R)$. Then $1 - e = w - 1 \in U(R)$, and so $e = 0$. This implies that $2 \in N(R)$. Thus, $1 - 2r \in U(R)$ for any $r \in R$; hence, $2 \in J(R)$.

(2) $\Rightarrow$ (1) In view of Theorem 9.2.8, $J(R)$ is nil, and so $2 \in N(R)$. So the lemma is true. □

Theorem 9.2.10. *Let R be a ring. Then the following are equivalent:*

(1) *R is weakly nil-clean.*
(2) *$R = A, B$ or $A \times B$ where A is a nil-clean ring of characteristic 2^k and B is a weakly nil-clean ring of characteristic of 3^l.*

Proof. In view of Theorem 9.2.8, $6 \in N(R)$. Write $6^m = 0 (m \in \mathbb{N})$. As $2^m R + 3^m R = R$ and $(2^m R)(3^m R) = (3^m R)(2^m R)$, we easily check that $(2^m R) \cap (3^m R) = (2^m R)(3^m R) = 0$. Hence, $R = R/\big(2^m R \cap 3^m R\big) \cong R/(2^m R) \times R/(3^m R)$. Set $A = R/(2^m R)$ and $B = R/(3^m R)$. Then $R = A, B$ or $A \times B$ where A is a weakly nil-clean ring of characteristic 2^k and B is a weakly nil-clean ring of characteristic of 3^l. As $2 \in N(A)$, by Lemma 9.2.9, A is nil-clean, as desired.

(2) $\Rightarrow$ (1) This is obvious. □

Example 9.2.11. Let $n \geq 2$. Then $\mathbb{Z}_n$ is weakly nil-clean if and only if $n = 2^k 3^l (k, l \geq 0)$.

Theorem 9.2.12. *The center of a weakly nil-clean ring is weakly nil-clean.*

Proof. Let R be weakly nil-clean, and let $a \in C(R)$. Then there exists an idempotent $e \in R$ and a nilpotent $w \in R$ such that $a = e + w$ or $a = -e + w$. We just deal with the first equality. Hence, $ea = e + ew$, and so $e = ea(1+w)^{-1} = e(1+w)^{-1}a$. This implies that $e \in Ra^k$ for all $k \in \mathbb{N}$. Obviously, $(1-e)a = (1-e)w$. Write $w^t = 0$. Then $(1-e)a^t = (1-e)w^t = 0$.

Let $x \in R$. Then $ex(1-e) \in Ra^t x(1-e) = Rx(1-e)a^t = 0$. Hence, $ex = exe$. Likewise, $xe = exe$. Therefore $ex = xe$, and so $e \in C(R)$. It follows that $w \in C(R)$. Therefore $a = e + w$ is a nil-clean decomposition, as desired. □

Corollary 9.2.13. *The center of a nil-clean ring is nil-clean.*

Proof. Let R be nil-clean. In view of Lemma 9.2.9, R is weakly nil-clean and $2 \in J(R)$. It follows by Theorem 9.2.12 that $C(R)$ is weakly nil-clean. As $2 \in J(R)$, we easily check that $2 \in J\big(C(R)\big)$. By using Lemma 9.2.9 again, $C(R)$ is nil-clean. □

An element a in a ring R is a unipotent if $a - 1$ is a nilpotent. The following lemma is due to Šter (see [490, Proposition 1]).

Lemma 9.2.14. *Let R be a ring, and let $a \in R$ be the sum of an idempotent and a nilpotent. If $a^2 = 1$, then a is a unipotent.*

Proof. Write $a = e + w$ with $e^2 = e \in R$, $w \in N(R)$. Set $f = 1 - e$ and $r = w(w + 1)$. Then $fw = f(a - e) = fa$ and $fr = fw(w + 1) = f(a - e)(a - e + 1) = fa(a - e + 1) = fa^2 + faf = faf$. Similarly, $wf = faf$. Hence, $fw = wf$. Thus, r is a nilpotent which commutes with f, e, w and a. It is easy to check that $f = fa^2 = f(e + w)a = fwa = fr(w + 1)^{-1}a = f(w + 1)^{-1}ra = f(w + 1)^{-1}ar \in N(R)$. This implies that $f = 0$, and then $e = 1$. Therefore $a = 1 + w \in R$ is a unipotent. □

An element u in a ring R is an involution if $u^2 = 1$. A ring R is said to be an IU ring if for any $a \in R$, either a or $-a$ is the sum of an involution and a unipotent. Clearly, a ring R is an IU ring if and only if every element in R is the sum or difference of an involution and a unipotent.

Theorem 9.2.15. *Let R be a ring. Then the following are equivalent:*

(1) *R is an IU ring.*
(2) *R is weakly nil-clean and $2 \in U(R)$.*
(3) *$N(R)$ forms an ideal of R and $R/N(R) \cong \mathbb{Z}_3$.*
(4) *$J(R)$ is nil and $R/J(R) \cong \mathbb{Z}_3$.*

Proof. (1) ⇒ (2) By hypothesis, there exist an involution $u \in R$ an a nilpotent $w \in R$ such that $3 = u \pm 1 + w$. Clearly, $uw = wu$. If $3 = u + 1 + w$, then $1 = u - 1 + w$, and then $3 = (u + 1)(u - 1) + s = s \in N(R)$. Hence, $2 = 3 - 1 \in U(R)$. If $3 = u - 1 + w$, then $5 = u + 1 + w$. This shows that $15 = (u - 1)(u + 1) + t = t \in N(R)$, and so $14 = 15 - 1 \in N(R)$. It follows that $2 \in U(R)$. In any case, $2 \in U(R)$.

Let $a \in R$. Then there exist $u, v \in R$ such that $-2a = u \pm v$, where $u^2 = 1$ and $w := 1 - v \in N(R)$. Then $v = 1 - w$, and so $-2a = u \pm (1 - w)$. Hence, $a = -\frac{1+u}{2} + \frac{w}{2}$ or $\frac{1-u}{2} - \frac{w}{2}$. As $u^2 = 1$, we easily check that $\frac{1+u}{2}, \frac{1-u}{2} \in R$ are idempotents. This shows that R is weakly nil-clean.

(2) $\Rightarrow$ (3) Let $e^2 = e \in R$, and let $a = 1 - 2e$. Then $a^2 = 1$. Since R is weakly nil-clean, either a or $-a$ is nil-clean. By virtue of Lemma 9.2.14, $a \in 1 + N(R)$ or $-1 + N(R)$. If $a \in 1 + N(R)$, then $2e \in N(R)$, and so $e \in N(R)$. This implies that $e = 0$. If $a \in -1+N(R)$, then $2(1-e) \in N(R)$; hence, $1 - e \in N(R)$. This implies that $e = 1$. Therefore R has only trivial idempotents. Since R is weakly nil-clean, $R = \{q, 1+q, -1+q \mid q \in N(R)\}$. Clearly, R is an abelian strongly π–regular ring. In view of Theorem 5.3.2, $N(R)$ forms an ideal of R. Clearly, $R/N(R)$ has only 3 elements, and so $R/N(R) \cong \mathbb{Z}_3$.

(3) $\Rightarrow$ (4) Obviously, $N(R) \subseteq J(R)$. Let $x \in J(R)$. Then $\overline{x} \in J(R/N(R))$; hence, $\overline{x} = \overline{0}$, i.e., $x \in N(R)$. This implies that $J(R) \subseteq N(R)$, and so $J(R) = N(R)$, as desired.

(4) $\Rightarrow$ (1) Let $a \in R$. Then $\frac{a}{2} \in R/N(R)$. Hence, $\frac{a}{2} = \overline{0}, \overline{1}$ and $-\overline{1}$. Therefore we can find a nilpotent $w \in R$ such that $a = (-1)+(1+2w), a = 1 + (1 + 2w)$ or $a = (-1) - (1 - 2w)$. Accordingly, R is an IU ring. □

Corollary 9.2.16. *Every IU ring is strongly clean.*

Proof. Let R be an IU ring, and let $a \in R$. Then $a, a - 1$ or $a + 1$ in $N(R)$ by Theorem 9.2.15. If $a \in N(R)$, then $a = 1 + (a - 1)$ with $a - 1 \in U(R)$; otherwise, $a = 0 + a$ with $a \in U(R)$. Therefore R is strongly clean. □

Theorem 9.2.17. *Let R be a ring. Then the following are equivalent:*

(1) *R is weakly nil-clean.*
(2) *R is a nil-clean ring, an IU ring, or the direct product of a nil-clean ring and an IU ring.*

Proof. (1) $\Rightarrow$ (2) By virtue of Theorem 9.2.10, $R = A, B$ or $A \times B$ where A is nil-clean and B is weakly nil-clean with $char(B) = 3^l (l \in \mathbb{N})$. As $(3^l, 2) = 1$, we see that $2 \in B$ is invertible. It follows by Theorem 9.2.15, B is an IU ring, as needed.

(2) $\Rightarrow$ (1) In light of Theorem 9.2.15, every IU ring is weakly nil-clean. Therefore R is weakly nil-clean, in terms of Theorem 9.2.5. □

As a consequence of Theorem 9.2.17 and Corollary 9.2.16, we conclude that every weakly nil-clean ring is clean.

Lemma 9.2.18. $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \in M_2(\mathbb{Z}_3)$ *is not nil-clean.*

Proof. Assume that $A = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \in M_2(\mathbb{Z}_3)$ is nil-clean. Then there exists an idempotent $E \in M_2(\mathbb{Z}_3)$ and a nilpotent $W \in M_2(\mathbb{Z}_3)$ such that $A = E + W$. As $E = E^2$, we see that $det(E) = 1$ or $det(E) = 0$. If $det(E) = 1$, then $E = I_2$. Hence, $A \pm I_2 \in M_2(\mathbb{Z}_3)$ is nilpotent, a contradiction.

If $det(E) = 0$, then $det(I_2 - E) = 1 - tr(E) + det(E) = 1 - tr(E)$. Clearly, $tr(E) = \pm\big(tr(A) - tr(W)\big) = 0$, as $tr(W) = 0$. It follows that $det(I_2 - E) = 1$, and then $E = 0$. This implies that $A = W \in M_2(\mathbb{Z}_3)$ is nilpotent, a contradiction.

Therefore $A \in M_2(\mathbb{Z}_3)$ is not nil-clean. □

Lemma 9.2.19. $\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \in M_3(\mathbb{Z}_3)$ *is not nil-clean.*

Proof. Assume that $A = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \in M_3(\mathbb{Z}_3)$ is nil-clean. Then there exists an idempotent $E \in M_3(\mathbb{Z}_3)$ and a nilpotent $W \in M_3(\mathbb{Z}_3)$ such that $A = E + W$. Clearly, $det(E) = 1$ or $det(E) = 0$. If $det(E) = 1$, then $E = I_3$. Hence, $A - I_3 \in M_2(\mathbb{Z}_3)$ is nilpotent, a contradiction.

We now assume that $det(E) = 0$. It is easy to check that $tr(E) = tr(A) - tr(W) = 0$, as $tr(W) = 0$. Obviously, we can find an invertible $U \in GL_2(\mathbb{Z}_3)$ such that $U^{-1}EU = \begin{pmatrix} e_1 & & \\ & e_2 & \\ & & e_3 \end{pmatrix}$, where each e_i is 0 or 1. As $det(E) = 0$ and $tr(E) = 0$, we may assume that $e_1 = e_2 = e_3 = 0$ or $e_1 = 1, e_2 = -1, e_3 = 0$. Thus, $E = 0$, a contradiction. Otherwise, $U^{-1}EU = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$, which yields $E^2 \neq E$, an absurd. Therefore, $A \in M_3(\mathbb{Z}_3)$ is not nil-clean. □

We now pass on consideration of the weakly nil-clean matrix rings.

Theorem 9.2.20. *Let R be a field, and let $n \in \mathbb{N}$. Then $M_n(R)$ is weakly nil-clean if and only if*

(1) $R \cong \mathbb{Z}_2$; *or*
(2) $R \cong \mathbb{Z}_3$ *and* $n = 1$.

Proof. $\Longrightarrow$ Let $0 \neq a \in R$. Then we can find an idempotent $E \in M_n(R)$ and a nilpotent $W \in M_n(R)$ such that $aI_n = \pm E + W$. Hence, $\pm E = aI_n - W \in GL_n(R)$. This shows that $E = I_n$. It follows that $(a \mp 1)I_n = W$, and then $a = \pm 1$. Hence $R \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$. Let $A = \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix} \in M_n(\mathbb{Z}_3)$ where $A_{11} = diag(1, -1)$. Let $Q = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & I_{n-2} \end{pmatrix}$. Then $QAQ^{-1} = -A$. It suffices to prove $A \in M_n(\mathbb{Z}_3)$ is not nil-clean. By virtue of Lemma 9.2.18 and Lemma 9.2.19, $M_2(\mathbb{Z}_3)$ and $M_3(\mathbb{Z}_3)$ are not weakly nil-clean.

Suppose that A is nil-clean for some $n \geq 4$. Then we have a nilpotent $W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$ with $W_{11} \in M_2(\mathbb{Z}_3)$ such that $A - W$ is an idempotent.

Case I. $rank(W_{21}) = 0$. Then $W_{21} = 0$. This implies that $A_{11} \in M_2(\mathbb{Z}_3)$ is nil-clean, a contradiction, by Lemma 9.2.18.

Case II. $rank(W_{21}) = 1$. Then there exists $U \in GL_n(R)$ such that $UW_{21} = \begin{pmatrix} * & * \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$. We check that

$$\begin{pmatrix} I_2 & \\ & U \end{pmatrix} A \begin{pmatrix} I_2 & \\ & U \end{pmatrix}^{-1} - \begin{pmatrix} W_{11} & * & * \\ * & * & * \\ & & * \end{pmatrix}$$

is an idempotent. We infer that $\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \in M_3(\mathbb{Z}_3)$ is nil-clean. This gives a contradiction, by Lemma 9.2.19.

Case III. $rank(W_{21}) = 2$. Then there exists $U \in GL_{n-2}(R)$ such that $UW_{21} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$. Thus, we have a nilpotent $\begin{pmatrix} I_2 & \\ & U \end{pmatrix} W \begin{pmatrix} I_2 & \\ & U \end{pmatrix}^{-1}$ such that $B = (b_{ij}) := \begin{pmatrix} I_2 & \\ & U \end{pmatrix} A \begin{pmatrix} I_2 & \\ & U \end{pmatrix}^{-1} - \begin{pmatrix} I_2 & \\ & U \end{pmatrix} W \begin{pmatrix} I_2 & \\ & U \end{pmatrix}^{-1}$ is an

idempotent. That is,

$$\begin{pmatrix} * & * & & & * & \\ * & * & & & & \\ -1 & 0 & b_{33} & b_{34} & \cdots & b_{3n} \\ 0 & -1 & b_{43} & b_{44} & \cdots & b_{4n} \\ 0 & 0 & b_{53} & b_{54} & \cdots & b_{5n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & b_{n3} & b_{n4} & \cdots & b_{nn} \end{pmatrix}$$

is an idempotent. This implies that $b_{53} = \cdots = b_{n3} = 0$ and $b_{54} = \cdots = b_{n4} = 0$. So, we may write

$$\begin{pmatrix} I_2 & \\ & U \end{pmatrix} W \begin{pmatrix} I_2 & \\ & U \end{pmatrix}^{-1} = \begin{pmatrix} W_{11} & * & * & * \\ 1\,0 & * & * & * \\ 0\,1 & * & * & * \\ & & & * \end{pmatrix}$$

Hence, there is a nilpotent

$$V := \begin{pmatrix} W_{11} & * & * \\ 1\,0 & * & * \\ 0\,1 & * & * \end{pmatrix} = \begin{pmatrix} 1-e_{11} & -e_{12} & -e_{13} & -e_{14} \\ -e_{21} & -1-e_{22} & -e_{23} & -e_{24} \\ 1 & 0 & -e_{33} & -e_{34} \\ 0 & 1 & -e_{43} & -e_{44} \end{pmatrix}$$

such that $E = (e_{ij}) := diag(1,-1,0,0) - V$ is an idempotent. Clearly, $E = \begin{pmatrix} E_{11} & E_{12} \\ -I_2 & E_{22} \end{pmatrix}$. Then we have

$$\begin{aligned} V^2 &= diag(1,1,0,0) - diag(1,-1,0,0)E - Ediag(1,-1,0,0) + E \\ &= diag(1,1,0,0) - diag(1,-1,0,0)E - Ediag(1,-1,0,0) + E \\ &= \begin{pmatrix} 1-e_{11} & e_{12} & 0 & 0 \\ e_{21} & 1 & -e_{23} & -e_{24} \\ 0 & 0 & e_{33} & e_{34} \\ 0 & 1 & e_{43} & e_{44} \end{pmatrix}. \end{aligned}$$

Since $V^4 = 0$, we get $e_{21} = 0$. It follows from $e_{21}e_{12} + 1 - e_{24} = 0$ that $e_{24} = 1$. As $e_{21}e_{12} + (-1-e_{22})^2 - e_{24} = 1$, we get $(1+e_{22})^2 = -1$. But the equation $x^2 = -1$ is not solvable in $\mathbb{Z}_3$. This gives a contradiction.

Therefore we conclude that A is not weakly nil-clean, as desired.

$\Longleftarrow$ This is obvious by Lemma 8.4.1. □

Corollary 9.2.21. *Let R be a commutative ring. Then $M_n(R)$ is weakly nil-clean if and only if $M_n(R)$ is nil-clean for all $n \geq 2$.*

Proof. $\Longrightarrow$ Let M be a maximal ideal of R and $n \geq 2$. Then $M_n(R/M)$ is weakly nil-clean. Since R/M is a field, it follows by Theorem 9.2.20 that $R/M \cong \mathbb{Z}_2$. Thus, $R/J(R)$ is isomorphic to the subdirect product of some $\mathbb{Z}_2$'s; hence, $R/J(R)$ is Boolean. By Lemma 8.4.2, $M_n(R/J(R))$ is nil-clean. Clearly, $J(M_n(R))$ is nil. Accordingly, $M_n(R)$ is nil-clean, by Proposition 9.2.1.

$\Longleftarrow$ This is obvious. □

An element $e \in R$ is a weak idempotent if e or $-e$ is an idempotent. A ring R is weakly Boolean if every element in R is a weak idempotent. We have

Lemma 9.2.22. *Every weakly Boolean ring is commutative.*

Proof. Let R be a weakly Boolean ring. Let $e \in R$ be an idempotent, and let $x \in R$. Then $ex(1-e) \in R$ is nilpotent. By hypothesis, $ex(1-e)$ or $-ex(1-e)$ is an idempotent. Hence, $ex(1-e) = 0$, and so $ex = exe$. Likewise, $xe = exe$; hence, $ex = xe$. Let $-e \in R$ be an idempotent, and let $x \in R$. Then $(-e)x = x(-e)$, and so $ex = xe$. Therefore R is commutative. □

Theorem 9.2.23. *A ring R is weakly Boolean if and only if R is isomorphic to one of the following:*

(a) $\mathbb{Z}_3$,
(b) *a Boolean ring, or*
(c) $\mathbb{Z}_3 \times B$ *where B is Boolean.*

Proof. Suppose that R is weakly Boolean. Then R is commutative, by Lemma 9.2.22.

Case I. R is indecomposable. Then $Id(R) = \{0, 1\}$. Hence, $R = Id(R) \cup -Id(R) = \{0, 1, -1\}$. This shows that $R \cong \mathbb{Z}_2, \mathbb{Z}_3$.

Case II. R is decomposable. Let $e \in R$ be a non-trivial idempotent. Then $R = Re \times R(1-e)$. If Re is not Boolean, then we have some $re \in Re$ which is not an idempotent. For any $s(1-e) \in R(1-e)$, we see that $-\big(re, -s(1-e)\big) \in R$ is an idempotent; hence, $s(1-e) \in R(1-e)$ is an idempotent. Thus, $R(1-e)$ is Boolean. Therefore Re or $R(1-e)$ is Boolean. This implies that $2e = 0$ or $2(1-e) = 0$.

If $2 = 0$, then R is Boolean.

If $2 \neq 0$, then $r(2) \neq R$, then we can find a maximal ideal M of R such that $r(2) \subseteq M$. Choose $x \in M \backslash r(2)$. Then $-x \in M \backslash r(2)$. By hypothesis,

x or $-x$ is an idempotent. Hence there exists an idempotent $f \in M \backslash r(2)$. By the preceding discussion, $2f = 0$ or $2(1-f) = 0$. If $2f = 0$, then $f \in r(2)$, a contradiction. If $2(1-f) = 0$, then $1 - f \in r(2) \subseteq M$, and so $1 = (1-f) + f \in M$, a contradiction. Therefore $r(2) = M$ is a maximal ideal of R. Hence, $R/r(2)$ is a field. Furthermore, $R/r(2) \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$.

Obviously, $2 \notin r(2)$, and so $2R + r(2) = R$. As $2R \cap r(2) = 0$, we get $R \cong R/2R \times R/r(2)$. As $2 = 0$ in $R/2R$, we see that $R/2R$ is Boolean. This completes the proof. □

Corollary 9.2.24. *Let R be a ring. Then $R = U(R) \cup Id(R) \cup -Id(R)$ if and only if R is isomorphic to one of the following:*

(a) *a division ring,*
(b) *a Boolean ring,*
(c) *$\mathbb{Z}_3 \times B$ where B is a Boolean, or*
(d) *$\mathbb{Z}_3 \times \mathbb{Z}_3$.*

Proof. $\Longrightarrow$ As every nilpotent in R is not a unit, as in the proof of Lemma 9.2.22, R is abelian.

Case I. R is indecomposable. Then $Id(R) = \{0, 1\}$. Hence, $R = U(R) \cup \{0, 1, -1\}$. This shows that R is a division ring.

Case II. R is decomposable. Then $R = S \times T$. For any $t \in T$, $(0, t) \in R$ is not invertible. Hence, $(0, t)$ or $-(0, t)$ is an idempotent. Thus, every element in T is a weak idempotent. In light of Theorem 9.2.23, T is isomorphic to one of the following: $\mathbb{Z}_3$, a Boolean ring, or $\mathbb{Z}_3 \times B$ where B is Boolean. Likewise, S is also isomorphic to one of the preceding. Thus, $R = S \times T$ is one of the following:

(i) a Boolean ring,
(ii) $\mathbb{Z}_3 \times B$ where B is Boolean,
(iii) $\mathbb{Z}_3 \times \mathbb{Z}_3$, or
(iv) $\mathbb{Z}_3 \times \mathbb{Z}_3 \times B$ where B is Boolean.

But in the case (iv), $R \neq U(R) \cup Id(R) \cup -Id(R)$, as $(1, 2, 0) \notin U(R) \cup Id(R) \cup -Id(R)$. Therefore R is isomorphic to one of $(a) - (d)$.

$\Longleftarrow$ If R is a division ring, then $R = U(R) \cup Id(R)$. If R is a Boolean ring, then $R = Id(R)$. If $R = \mathbb{Z}_3 \times B$ where B is Boolean, then $Id(R) = \{(0, b), (1, b) | b \in B\}$ and $-Id(R) = \{(2, b) \mid b \in B\}$. Hence, $R = Id(R) \cup -Id(R)$. If $R = \mathbb{Z}_3 \times \mathbb{Z}_3$, then $U(R) = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$, $Id(R) = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ and $-Id(R) = \{(0, 2), (2, 0)\}$. Therefore $R = U(R) \cup Id(R) \cup -Id(R)$, as asserted. □

Corollary 9.2.25. *Let R be a ring. Then $R = U(R) \cup Id(R)$ if and only if R is isomorphic to one of the following:*

(a) *a division ring, or*
(b) *a Boolean ring.*

Proof. Suppose that $R = U(R) \cup Id(R)$. By virtue of Corollary 9.2.24, R is isomorphic to one of the following:

(a) a division ring,
(b) a Boolean ring,
(c) $\mathbb{Z}_3 \times B$ where B is Boolean, or
(d) $\mathbb{Z}_3 \times \mathbb{Z}_3$.

In Cases (c) and (d), $(2,0) \notin U(R) \cup Id(R)$. Therefore R is a division ring, or a Boolean ring. The converse is obvious. □

We are now ready to prove:

Theorem 9.2.26. *Let R be abelian. Then R is weakly nil-clean if and only if*

(1) *$J(R)$ is nil;*
(2) *$R/J(R)$ is $\mathbb{Z}_3$, B or $\mathbb{Z}_3 \times B$ where B is a Boolean ring.*

Proof. $\Longrightarrow$ In view of Theorem 9.2.15, $J(R)$ is nil. Let $a \in R$. Then $\pm a$ is nil-clean, and so $a \pm a^2 \in N(R)$. Thus, we can find some $n \in \mathbb{N}$ such that $a^n = a^{n+1}f(a)$ for some $f(t) \in \mathbb{Z}[t]$. Hence, R is periodic, and so it is strongly π-regular. By virtue of Theorem 5.3.2, $N(R)$ forms an ideal of R; hence, $N(R) = J(R)$. This implies that $R/J(R)$ is weakly Boolean. In light of Theorem 9.2.23, we prove (2).

$\Longleftarrow$ Clearly, $R/J(R)$ is weakly nil-clean, and therefore R is weakly nil-clean, by Proposition 9.2.1. □

As in the proof of Corollary 8.2.3, applying Theorem 9.2.26 to $R/J(R)$, we now derive

Corollary 9.2.27. *Let R be (left) right quasi-duo. Then R is weakly nil-clean if and only if*

(1) *$J(R)$ is nil;*
(2) *$R/J(R)$ is $\mathbb{Z}_3$, B or $\mathbb{Z}_3 \times B$ where B is a Boolean ring.*

It is of interest to investigate when a ring is a union of subsets containing of special elements. Next we explore when a ring consists entirely of units, weak idempotents, and nilpotents.

Lemma 9.2.28. *Let R be a decomposable ring. Then R consists entirely of units, weak idempotents, and nilpotents if and only if R is isomorphic to one of the following:*

(1) *a Boolean ring;*
(2) $\mathbb{Z}_3 \times \mathbb{Z}_3$*;*
(3) $\mathbb{Z}_3 \times B$ *where B is Boolean.*

Proof. $\Longrightarrow$ Write $R = A \times B$ with $A, B \neq 0$. Then A and B are rings consisting entirely of weak idempotents, units, and nilpotents. If $0 \neq x \in N(A)$, then $(x, 1) \notin Id(R) \cup -Id(R) \cup U(R) \cup N(R)$. This shows that $A = U(A) \cup Id(A) \cup -Id(A)$. Likewise, $B = U(B) \cup Id(B) \cup -Id(B)$. In light of Corollary 9.2.24, R is one of the following:

(a) a Boolean ring;
(b) $B \times D$ where B is a Boolean ring and D is a division ring;
(c) $\mathbb{Z}_3 \times \mathbb{Z}_3 \times B$ where B is a Boolean ring;
(d) $\mathbb{Z}_3 \times B$ where B is a Boolean ring;
(e) $D \times D'$ where D and D' are division rings; or
(f) $\mathbb{Z}_3 \times B \times D$ where B is a Boolean ring and D is a division ring.
(g) $\mathbb{Z}_3 \times \mathbb{Z}_3 \times D$;
(h) $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$;
(i) $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times B$ where B is a division ring;
(j) $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$;

In Case (b). If $0, \pm 1 \neq x \in D$, then $(0, x) \notin U(R) \cup Id(R) \cup -Id(R)$. This forces $D \cong \mathbb{Z}_2, \mathbb{Z}_3$. Hence, (b) forces R is in (1) or (3). (c) does not occur. (e) forces $D, D' \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$. Hence, R is in $(1) - (3)$. (f) does not occur except $D \cong \mathbb{Z}_2$. Thus, R is in $(1) - (3)$. Cases (g)-(j) do not occur as $(1, -1, 0), (1, -1, 0, 0) \notin I(R) \cup -Id(R) \cup N(R)$, as desired.

$\Longleftarrow$ This is obvious. □

Theorem 9.2.29. *Let R be an abelian ring. Then R consists entirely of units, weak idempotents, and nilpotents if and only if R is isomorphic to one of the following:*

(1) $\mathbb{Z}_3$;
(2) *a Boolean ring;*
(3) $\mathbb{Z}_3 \times \mathbb{Z}_3$;
(4) $\mathbb{Z}_3 \times B$ *where* B *is a Boolean ring; or*
(5) *local ring with nil Jacobson radical.*

Proof. $\Longrightarrow$ Case I. R is indecomposable. Then $R = U(R) \cup N(R)$. This shows that R is local. Let $x \in J(R)$, then $x \in N(R)$, and so $J(R)$ is nil.

Case II. R is decomposable. In view of Lemma 9.2.28, R is isomorphic to one of the following:

(1) a Boolean ring;
(2) $\mathbb{Z}_3 \times \mathbb{Z}_3$;
(3) $\mathbb{Z}_3 \times B$ where B is a Boolean ring.

This shows that R is isomorphic to one of $(1) - (5)$, as desired.

$\Longleftarrow$ This is clear. □

Lemma 9.2.30. *Let R be any ring that consists entirely of units, weak idempotents, and nilpotents. Then eRe is a division ring for any noncentral idempotent $e \in R$.*

Proof. Let $e \in R$ be a noncentral idempotent, and let $f = 1 - e$. Then $R \cong \begin{pmatrix} eRe & eRf \\ fRe & fRf \end{pmatrix}$. The subring $\begin{pmatrix} eRe & 0 \\ 0 & fRf \end{pmatrix}$ consists entirely of units, weak idempotents, and nilpotents. That is, $eRe \times fRf$ consists entirely of units, weak idempotents, and nilpotents. Set $A = eRe$ and $B = fRf$. As in the proof of Lemma 9.2.28, A is isomorphic to one of the following:

(1) $\mathbb{Z}_3$;
(2) a Boolean ring;
(3) a division ring;
(4) $\mathbb{Z}_3 \times B$ where B is a Boolean ring.

That is, A is a division ring or a ring in which every element is weak idempotent. Suppose that eRe is not a division ring. Then eRe must contains a nontrivial idempotent, say $a \in R$. Let $b = e - a$. Let $x \in eRf$ and $y \in fRe$. Choose

$$X_1 = \begin{pmatrix} a & x \\ 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} b & x \\ 0 & 0 \end{pmatrix}, Y_1 = \begin{pmatrix} a & 0 \\ y & 0 \end{pmatrix}, Y_2 = \begin{pmatrix} b & 0 \\ y & 0 \end{pmatrix}.$$

Then X_1, X_2, Y_1, Y_2 are not invertible. As $a, b \in eRe$ is nontrivial idempotents, we see that X_1, X_2, Y_1, Y_2 are all not nilpotent matrices. This shows

that X_1 and X_2 are both weak idempotents. It follows that $X_1 = \pm X_1^2$ or $X_2^2 = \pm X_2$. As $x \in eRf, y \in fRe$, we see that $ex = x$ and $fy = y$.

Case I. $X_1 = X_1^2, X_2 = X_2^2$. Then $ax = x, bx = x$, and so $x = ex = 2x$; hence, $x = 0$.

Case II. $X_1 = X_1^2, X_2 = -X_2^2$. Then $ax = x, bx = -x$, and so $x = ex = 0$.

Case III. $X_1 = -X_1^2, X_2 = X_2^2$. Then $ax = -x, bx = x$, and so $x = ex = 0$.

Case IV. $X_1 = -X_1^2, X_2 = -X_2^2$. Then $a = -a^2, ax = -x, b = -b^2$ and $bx = -x$. Hence, $(e - a)x = -x$, and so $x = ex = -2x$, hence, $3x = 0$. As $a \in R$ is an idempotent, we see that $a = a^2$, hence, $a = -a$, and so $2a = 0$. It follows that $x = -ax = (2a)x - (3x)a = 0$.

Thus, $x = 0$ in any case. This implies that $eRf = 0$. Likewise, $fRe = 0$. Hence, $e \in R$ is central, an absurd. This completes the proof. □

Lemma 9.2.31. *Let R be any ring that consists entirely of units, weak idempotents, and nilpotents. Then eRe is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z}$ for any noncentral idempotent $e \in R$.*

Proof. Let $e \in R$ be a noncentral idempotent. In view of Lemma 9.2.30, eRe is a division ring. Set $f = 1 - e$. For any $u \in eRe$, we assume that $u \neq 0, e, -e$, then the matrix

$$X = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} eRe & eRf \\ fRe & fRf \end{pmatrix}$$

is not be a unit, a weak idempotent, a nilpotent. This gives a contradiction. Therefore $u = 0, e$ or $-e$, which completes the proof. □

A ring R is semiprime if it has no nonzero nilpotent ideals. We derive

Theorem 9.2.32. *Let R be any nonabelian ring that consists entirely of units, weak idempotents, and nilpotents. If R is semiprime, then it is isomorphic to $M_2(\mathbb{Z}_2)$ or $M_2(\mathbb{Z}_3)$.*

Proof. Suppose that R is semiprime. In view of Lemma 9.2.30, eRe is a division ring for any noncentral idempotent $e \in R$. It follows by [244, Lemma 21] that R is isomorphic to $M_2(D)$ for a division ring D. Choose $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(D)$. Then E_{11} is a noncentral idempotent. According to Lemma 9.2.31, $R \cong \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z}$, as desired. □

Recall that a ring R is a NJ-ring provided that for any $a \in R$, either $a \in R$ is regular or $1-a \in R$ is a unit (see [429]). Clearly, all rings in which every elements consists entirely of units, weak idempotents, and nilpotents are NJ-rings.

Theorem 9.2.33. *Let R be any nonabelian ring that consists entirely of units, weak idempotents, and nilpotents. If R is not semiprime, then it is isomorphic to the ring of a Morita context with zero pairings where the underlying rings are $\mathbb{Z}_2$ or $\mathbb{Z}_3$.*

Proof. Suppose that R is not semiprime. Clearly, R is an NJ-ring. In view of [429, Theorem 2], R is a regular ring, a local ring or isomorphic to the ring of a Morita context with zero pairings where the underlying rings are both division ring. If R is regular, it is semiprime, a contradiction. If R is local, it is abelian, a contradiction. Therefore, R is isomorphic to the ring of a Morita context $T = (A, B, M, N, \varphi, \psi)$ with zero pairings φ, ψ where the underlying rings are division rings A and B. Choose $E = \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix} \in T$. Then $E \in T$ is a noncentral idempotent. In light of Lemma 9.2.31, $A \cong ETE \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$. Likewise, $B \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$. This completes the proof. □

With these information we completely determine the structure of rings that consist entirely of units, weak idempotents, and nilpotents.

Theorem 9.2.34. *Let R be a ring. Then R consists entirely of units, weak idempotents, and nilpotents if and only if R is isomorphic to one of the following:*

(1) *a Boolean ring;*
(2) *$\mathbb{Z}_3 \times \mathbb{Z}_3$;*
(3) *$\mathbb{Z}_3 \times B$ where B is a Boolean ring;*
(4) *local ring with a nil Jacobson radical;*
(5) *$M_2(\mathbb{Z}_2)$ or $M_2(\mathbb{Z}_3)$; or*
(6) *the ring of a Morita context with zero pairings where the underlying rings are $\mathbb{Z}_2$ or $\mathbb{Z}_3$.*

Proof. $\Longrightarrow$ This is obvious by Theorem 9.2.29, Theorem 9.2.32 and Theorem 9.2.33.

$\Longleftarrow$ Cases (1)-(4) are easy. Case (5)-(6) are verified by checking all possible (generalized) matrices over $\mathbb{Z}_2$ and $\mathbb{Z}_3$. □

9.3 Strongly Weak Nil-clean Rings

A ring R is strongly weakly nil-clean provided that for any $a \in R$, a or $-a$ is strongly nil-clean. The aim of this section is to explore strongly weakly nil-clean rings. We begin with

Lemma 9.3.1. *Let R be a ring. Then the following are equivalent:*

(1) *R is strongly weakly nil-clean.*
(2) *For any $a \in R, a \pm a^2$ is nilpotent.*

Proof. This is proved as in Theorem 5.1.1. □

Proposition 9.3.2. *A ring R is strongly nil-clean if and only if*

(1) *$2 \in J(R)$;*
(2) *R is strongly weakly nil-clean.*

Proof. (1) $\Rightarrow$ (2) This is obvious.

(2) $\Rightarrow$ (1) Let $x \in J(R)$. Then $x \pm x^2 \in N(R)$ by Lemma 9.3.1. As $1 \pm x \in U(R)$, we see that $x \in N(R)$. Hence, $J(R)$ is nil. Let $a \in R$. Then $a \pm a^2 \in N(R)$. If $a + a^2 \in N(R)$, then $a - a^2 = (a + a^2) - 2a^2 \in N(R)$. Therefore R is strongly nil-clean, by Theorem 5.1.1. □

Every subring of strongly weakly nil-clean rings is strongly weakly nil-clean. Thus every center and every corner of a strongly weakly nil-clean ring are strongly weakly nil-clean. Next, we investigate various homomorphic images of strongly weakly nil-clean rings. Analogously to Theorem 5.1.16, we easily derive

Lemma 9.3.3. *Let I be a nil ideal of a ring R. Then R is strongly weakly nil-clean if and only if R/I is strongly weakly nil-clean.*

As a consequence, we prove that a ring R is strongly weakly nil-clean if and only if $R/P(R)$ is strongly weakly nil-clean. We are now ready to prove the following.

Theorem 9.3.4. *A ring R is strongly weakly nil-clean if and only if*

(1) *$J(R)$ is nil;*
(2) *$R/J(R)$ is weakly Boolean.*

Proof. $\Longrightarrow$ As in the proof of Proposition 9.3.2, $J(R)$ is nil. Let $a \in R$. In view of Lemma 9.3.1, $a \pm a^2 \in N(R)$. Hence, $(a-a^2)(a+a^2) \in N(R)$, i.e., $a^2-a^4 \in N(R)$. This shows that $a(a-a^3) \in N(R)$, and so $(1-a^2)a(a-a^3) = (a-a^3)^2 \in N(R)$. Therefore $a - a^3 \in N(R)$. In light of Lemma 8.4.19, $N(R)$ forms an ideal of R. This shows that $R/J(R)$ is weakly Boolean, as $N(R) \subseteq J(R)$.

$\Longleftarrow$ This is obvious, by Lemma 9.3.3, as every weakly Boolean ring is strongly weakly nil-clean. □

Corollary 9.3.5. *A ring R is strongly weakly nil-clean if and only if*

(1) *$J(R)$ is nil;*
(2) *$R/J(R)$ is isomorphic to a Boolean ring, or $\mathbb{Z}_3$ or the product of two such rings.*

Proof. This is obvious, by Theorem 9.3.4 and Theorem 9.2.23. □

Corollary 9.3.6. *A ring R is strongly weakly nil-clean if and only if*

(1) *R is periodic;*
(2) *$R/J(R)$ is weakly Boolean.*

Proof. $\Longrightarrow$ Let $a \in R$. By Lemma 9.3.1, $a - a^2 \in N(R)$ or $a + a^2 \in N(R)$. Then we have some $m \in \mathbb{N}$ such that $a^m = a^{m+1}f(a)$ for some $f(t) \in \mathbb{Z}[t]$. In light of Lemma 5.1.10, R is periodic.

$\Longleftarrow$ Since R is periodic, $J(R)$ is nil. By assumption, $R/J(R)$ is weakly Boolean; hence, it is strongly weakly nil-clean. Therefore we get the result by Lemma 9.3.3. □

Corollary 9.3.7. *A ring R is strongly weakly nil-clean if and only if*

(1) *R is right (left) quasi-duo;*
(2) *R is weakly nil-clean.*

Proof. $\Longrightarrow$ Obviously, R is weakly nil-clean. In light of Theorem 9.3.4 and Lemma 9.2.22, $R/J(R)$ is commutative. Similarly to Lemma 5.1.20, we prove that R is right (left) quasi-duo.

$\Longleftarrow$ Since R is right (left) quasi-duo, as in the proof of Lemma 5.1.20, $R/J(R)$ is abelian. Hence, $R/J(R)$ is strongly weakly nil-clean. Let $x \in J(R)$. Then we have an idempotent $e \in R$ and a nilpotent $w \in R$ such that $x = \pm e + w$. Write $w^m = 0$. Then $(x-w)^m \in J(R)$, and so $e \in J(R)$. This implies that $e = 0$. Hence, $x = w \in N(R)$. Thus $J(R)$ is nil. According to Lemma 9.3.3, R is strongly weakly nil-clean. □

Proposition 9.3.8. *A ring R is strongly weakly nil-clean if and only if*

(1) *R has no homomorphic image $\mathbb{Z}_3 \times \mathbb{Z}_3$;*
(2) *R is strongly 2-nil-clean.*

Proof. $\Longrightarrow$ If $\mathbb{Z}_3 \times \mathbb{Z}_3$ is a homomorphic image of R, then it is strongly weakly nil-clean. But $(1,-1) \in \mathbb{Z}_3 \times \mathbb{Z}_3$ is not strongly weakly nil-clean. This gives a contradiction. Thus proving (1). Let $a \in R$. In view of Lemma 9.3.1, $a \pm a^2 \in N(R)$. Hence, $a^2 - a^4 = (a - a^2)(a + a^2) \in N(R)$. Then $a - a^3 \in N(R)$, and thus proving (2) by Theorem 8.4.16.

$\Longleftarrow$ In view of Theorem 8.4.24, $R/J(R)$ is isomorphic to a Boolean ring, a Yaqub ring, or the product of such rings. By (1), $R/J(R)$ is isomorphic to a Boolean ring, $\mathbb{Z}_3$ or the product of such rings. In light of Corollary 9.3.5, R is strongly weakly nil-clean. □

Theorem 9.3.9. *Let R be a ring. Then the following are equivalent:*

(1) *R is strongly weakly nil-clean.*
(2) *$6 \in R$ is nilpotent and $R/6R$ is strongly weakly nil-clean.*
(3) *$N(R)$ forms an ideal of R and $R/N(R)$ is weakly Boolean.*

Proof. (1) $\Rightarrow$ (2) As R is strongly weakly nil-clean, $2 \pm 2^2 \in N(R)$; hence, $6 \in N(R)$. Also $R/6R$ is a homomorphic image of R, then it is strongly weakly nil-clean.

(2) $\Rightarrow$ (1) As 6 is nilpotent, then $6R$ is a nil ideal, so the implication follows from Lemma 9.3.3.

(1) $\Rightarrow$ (3) As in the proof of Theorem 9.3.4, $N(R)$ forms an ideal of R. Therefore $R/N(R)$ is weakly Boolean, as required.

(3) $\Rightarrow$ (1) Since every weakly Boolean ring is strongly weakly nil-clean, we obtain the result by Lemma 9.3.3. □

Corollary 9.3.10. *Let R be a ring. Then the following are equivalent:*

(1) *R is strongly weakly nil-clean.*
(2) *R is weakly nil-clean and $N(R)$ forms an ideal of R.*
(3) *R is weakly nil-clean and $R/J(R)$ is commutative.*

Proof. (1) $\Leftrightarrow$ (2) This is obvious, by Theorem 9.3.9.

(1) $\Rightarrow$ (3) Clearly, R is weakly nil-clean. As every weakly Boolean ring is commutative, $R/J(R)$ is commutative, by Theorem 9.3.4.

(3) $\Rightarrow$ (1) Since R is weakly nil-clean, we see that $R/J(R)$ is strongly weakly nil-clean and $J(R)$ is nil. Therefore we complete the proof, by Lemma 9.3.3. □

Theorem 9.3.11. *Let $\{R_i \mid i \in I\}$ be a family of rings. Then the direct product $R = \prod_{i \in I} R_i$ of rings R_i is strongly weakly nil-clean if and only if each R_i is strongly weakly nil-clean and at most one is not strongly nil-clean.*

Proof. $\Longrightarrow$ In view of Theorem 9.2.5, each R_i is weakly nil-clean and at most one is not nil-clean. By virtue of Corollary 9.3.7, R is right quasi-duo; and then so is each R_i. By Corollary 9.3.7 and Lemma 5.1.20, each R_i is strongly weakly nil-clean and at most one is not strongly nil-clean.

$\Longleftarrow$ By Corollary 9.3.7 and Lemma 5.1.20, each R_i is right quasi-duo, and then so is R. According to Theorem 9.2.5, R is weakly nil-clean. The result follows by Corollary 9.3.7. □

Corollary 9.3.12. *Let $L = \prod_{i \in I} R_i$ be the direct product of rings $R_i \cong R$ and $|I| \geq 2$. Then L is strongly weakly nil-clean if and only if L is strongly nil-clean if and only if R is strongly nil-clean.*

Corollary 9.3.13. *Let R be a ring. Then the following are equivalent:*

(1) *R is strongly nil-clean.*
(2) *$T_n(R)$ is strongly nil-clean for all $n \in \mathbb{N}$.*
(3) *$T_n(R)$ is strongly nil-clean for some $n \in \mathbb{N}$.*
(4) *$T_n(R)$ is strongly weakly nil-clean for some $n \geq 2$.*

Next, we exhibit a useful structure theorem of strongly weakly nil-clean rings.

Theorem 9.3.14. *Let R be a ring. Then the following are equivalent:*

(1) *R is strongly weakly nil-clean.*
(2) *R is a strongly nil-clean ring, an IU ring, or the direct product of such rings.*

Proof. $\Longrightarrow$ By virtue of Theorem 9.2.17, R is isomorphic to a nil-clean ring R_1, an IU ring R_2, or the product of such rings. It follows by Theorem 9.3.9 that $N(R)$ forms an ideal of R. Hence, $N(R_1)$ forms an ideal of R_1. By Corollary 5.1.6, R_1 is strongly nil-clean, as required.

$\Longleftarrow$ By hypothesis, R is a strongly nil-clean ring R_1, an IU ring R_2, or the direct product of such rings. By virtue of Theorem 9.2.15, $J(R_2)$ is nil and $R_2/J(R_2) \cong \mathbb{Z}_3$ is weakly Boolean. According to Theorem 9.3.4,

R_2 is strongly weakly nil-clean. Therefore R is strongly weakly nil-clean, in terms of Theorem 9.3.11. □

Corollary 9.3.15. *A ring R is strongly weakly nil-clean if and only if $R \cong R_1, R_2$ or $R_1 \times R_2$, where*

(1) *$R_1/J(R_1)$ is Boolean and $J(R_1)$ is nil;*
(2) *$R_2/J(R_2) \cong \mathbb{Z}_3$ and $J(R_2)$ is nil.*

Proof. $\Longrightarrow$ In view of Theorem 9.3.14, $R \cong R_1, R_2$ or $R_1 \times R_2$, where R_1 is strongly nil-clean, R_2 is an IU ring. In view of Theorem 5.1.5, $R_1/J(R_1)$ is Boolean and $J(R_1)$ is nil.

In light of Theorem 9.2.15, $R_2/J(R_2) \cong \mathbb{Z}_3$ and $J(R_2)$ is nil, as desired.

$\Longleftarrow$ In view of Theorem 5.1.5, R_1 is strongly nil-clean. It follows from Theorem 9.2.15 that R_2 is an IU ring. Therefore we complete the proof, by Theorem 9.3.14. □

A ring R is WUU if u or $-u$ is a unipotent for all $u \in U(R)$.

Lemma 9.3.16. *Let R be a weakly clean WUU ring. Then $6 \in R$ is nilpotent.*

Proof. Write $3 = \pm e + u$ where e is an idempotents and u is a unit. Then $u = \pm 1 + w$ for some $w \in N(R)$.

Case 1. $3 = e + u$. Then 2 or $4 = e + w$. It follows that $2 - 2^2$ or $4 - 4^2$ is nilpotent. Hence, $6 \in N(R)$.

Case 2. $3 = -e + u$. Then -2 or $-4 = e + w$. It follows that $(-2) - (-2)^2$ or $(-4) - (-4)^2$ is nilpotent. Hence, $30 = 2 \times 3 \times 5 \in N(R)$. Write $2^m \cdot 3^m \cdot 5^m = 0$. Then $R \cong R/2^m R \times R/3^m R \times R/5^m R$. Set $R_3 = R/5^m R$. Then $5 \in N(R_3)$, and so $4 = 5 - 1 \in U(R_3)$. This implies that $2 \in U(R_3)$. Since R is a WUU ring, then so is R_3. Thus, we see that $2 \in \pm 1 + N(R_3)$; hence, $3 \in N(R_3)$. As $(3, 5) = 1$, we have $5 \in U(R_3)$, a contradiction. Therefore $R \cong R/2^m R$, or $R/3^m R$, or the product of such rings. This implies that $6 \in N(R)$, as asserted. □

Lemma 9.3.17. *Let R be a weakly clean WUU ring. Then $J(R)$ is nil.*

Proof. By Lemma 9.3.16, $6 \in N(R)$. Say $6^n = 0$. Then $R \cong R_1 \times R_2$, where $R_1 \cong R/2^n R, R_2 \cong R/3^n R$.

Step 1. Let $x \in J(R_1)$. Since R_1 is a WUU ring with $2 \in N(R_1)$, $1 + x \in \pm 1 + N(R_1) = 1 + N(R_1)$, and so $x \in N(R_1)$. Hence $J(R_1)$ is nil.

Step 2. Let $x \in J(R_2)$. As R_2 is a WUU ring, $(1+x)^2 = 1+w$ for some $w \in N(R_2)$, i.e., $x(x+2) = w$. Since $3 \in N(R_2)$, we see that $2 = 3-1 \in U(R_2)$ and so $x+2 = 2(1+2^{-1}x) \in U(R_2)$. Thus, $x = w(x+2)^{-1} \in N(R_2)$, and so $J(R_2)$ is nil.

Therefore $J(R)$ is nil. □

Lemma 9.3.18. *Let R be an exchange WUU ring. Then $N(R) \subseteq J(R)$.*

Proof. Set $S = R/J(R)$. Suppose that $N(S) \neq 0$. Then we can find some $0 \neq a \in S$ such that $a^2 = 0$. Since S is an exchange ring and $J(S) = 0$, then by [389, Theorem 2.1], $eSe \cong M_2(T)$, for some idempotent $e \in S$ and some ring T.

Clearly, eSe is a WUU ring. Choose $U = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in M_2(T)$. Then $U \in U(M_2(R))$ and $U^2 - I_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, that is not nilpotent. Hence, U^2 is not unipotent. Thus $M_2(T)$ is not a WUU ring, a contradiction. This implies that $N(S) = 0$, i.e., S is reduced. Thus, $N(R) \subseteq J(R)$, hence the result. □

We have accumulated all the information necessary to prove the following.

Theorem 9.3.19. *A ring R is strongly weakly nil-clean if and only if*

(1) *R is weakly clean;*
(2) *R is WUU.*

Proof. $\Longrightarrow$ Clearly, R is weakly clean. Let $u \in U(R)$. Then $u \pm u^2 \in N(R)$; hence, $\pm u \in 1 + N(R)$, as desired.

$\Longleftarrow$ By Lemma 9.3.16, $6 \in N(R)$. Say $6^n = 0$. Then $R \cong R_1 \times R_2$, where $R_1 \cong R/2^nR, R_2 \cong R/3^nR$. Clearly, each R_i is a weakly clean WUU ring. In light of Lemma 9.3.17, $J(R)$ is nil, and then so is $J(R_i)$ for each i.

Step 1. Let $a \in R_1$. Since $2 \in N(R_1)$, there exists an idempotent $e \in R$ and a unit $u \in R$ such that $1 + a = e + u$. As R_1 is WUU, we have a nilpotent $w \in R$ such that $u = 1 + w$. Hence, $a = e + w$, and so R_1 is nil-clean. Thus R_1 is clean, and so it is exchange. In light of Lemma 9.3.18, $N(R_1) \subseteq J(R_1) \subseteq N(R_1)$. So $N(R_1)$ forms an ideal of R_1. Therefore R_1 is strongly nil-clean.

Step 2. Let $e \in R_2$. Then $1 - 2e \in U(R_2)$. Since R_2 is WUU, there exists a nilpotent $w \in R$ such that $1 - 2e = \pm 1 + w$. As $2 \in U(R_2)$, we see that $e = -\frac{w}{2}$ or $1 - e = \frac{w}{2}$. Thus, $e = 0$ or 1.

Let $a \in R_2$. Then we have a unit $u \in R$ such that $a = \pm 1 + u$. Furthermore, we have a nilpotent $v \in R_2$ such that $u = \pm 1 + v$. Hence, $a = v, -1+(v+3), 1+(v-3)$. As $3 \in N(R_3)$, we see that $v+3, v-3 \in N(R_3)$. Therefore R_2 is strongly weakly nil-clean.

Therefore R is strongly weakly nil-clean. □

A ring R is a UU ring if every unit in R is a unipotent. We now derive

Corollary 9.3.20. *A ring R is strongly nil-clean if and only if*

(1) *R is weakly clean;*
(2) *R is UU.*

Proof. $\Longrightarrow$ This is obvious.

$\Longleftarrow$ In view of Theorem 9.3.19, R is strongly weakly nil-clean. By hypothesis, $-1 \in R$ is a unipotent, whence, $2 \in N(R)$, the result follows by Proposition 9.3.2. □

A ring R is weakly exchange if for any $a \in R$ there exists an idempotent $e \in aR$ such that $1 - e \in (1 - a)R \cup (1 + a)R$. Every weakly clean ring is weakly exchange, the converse is true for an abelian ring (see [195, Theorem 2.1]).

Lemma 9.3.21. *Let R be a weakly exchange WUU ring. Then $6 \in R$ is nilpotent.*

Proof. Choose $f^2 = f \in (3)R$ such that $1-f \in (1-3)R$ or $1-f \in (1+3)R$.

Case I. $1 - f \in (1 - 3)R$. Write $f = 3a$, we may assume that $fa = a$. Write $1 - f = (-2)b$ where $(1 - f)b = b$. Then an easy calculation shows that $(3 - (1 - f))^{-1} = a - b$. Thus, $3 - (1 - f) \in U(R)$.

Case II. $1 - f \in (1 + 3)R$. Then $f^2 = f \in (-3)R, 1 - f \in (1 - (-3))R$. Similarly, we can find an idempotent $g \in R$ such that $(-3) - g \in U(R)$, and so $3 + g \in U(R)$.

Thus, $3 = \pm e + u$ with an idempotent $e \in R$ and a unit $u \in R$. As in the proof of Lemma 9.3.16, we prove that $6 \in N(R)$. □

Theorem 9.3.22. *A ring R is strongly weakly nil-clean if and only if*

(1) *R is weakly exchange;*
(2) *R is WUU.*

Proof. $\Longrightarrow$ One easily checks that every weakly clean ring is weakly exchange. Thus, this implication is obtained by Theorem 9.3.19.

$\Longleftarrow$ In view of Lemma 9.3.21, $6 \in N(R)$. Write $2^n 3^n = 0$. Then $R \cong R_1 \times R_2$ where $R_1 = R/2^n R$ and $R_2 = R/3^n R$. Obviously, each R_i is a weakly exchange WUU ring.

Step 1. Since $2 \in N(R_1)$, R_1 is an exchange UU ring. Let $x \in J(R_1)$. Then $1 + x \in 1 + N(R_1)$, and so $x \in N(R_1)$. Thus, $J(R_1)$ is nil. In view of Lemma 9.3.18, $N(R_1) \subseteq J(R_1)$. Hence, $N(R_1) = J(R_1)$. Let $S = R_1/J(R_1)$. Then S is an exchange UU ring. Moreover, S is reduced, and so it is abelian. In light of Corollary 1.1.9, S is clean. According to Corollary 9.3.20, S is strongly nil-clean. Therefore R_1 is strongly nil-clean, by Theorem 5.1.5.

Step 2. As in the proof of Theorem 9.3.19, we see that $Id(R_2) = \{0, 1\}$. Then R_2 is abelian. We note that every abelian weakly exchange ring is weakly clean. In light of Theorem 9.3.19, R_2 is strongly weakly nil-clean.

Therefore R is strongly weakly nil-clean. □

Corollary 9.3.23. *A ring R is strongly nil-clean if and only if*

(1) *R is weakly exchange;*
(2) *R is a UU ring.*

Proof. $\Longrightarrow$ This is obvious by Corollary 9.3.20, as every weakly clean ring is weakly exchange.

$\Longleftarrow$ As R is a UU ring, $2 \in N(R)$. This completes the proof by Theorem 9.3.22 and Proposition 9.3.2. □

9.4 Uniquely Weak Nil-clean Rings

A ring R is called uniquely weakly nil-clean provided that R is a weakly nil-clean ring in which every nil-clean element is uniquely nil-clean. We begin with

Theorem 9.4.1. *A ring R is uniquely weakly nil-clean if and only if*

(1) *R is abelian;*
(2) *R is weakly nil-clean.*

Proof. $\Longrightarrow$ Obviously, R is weakly nil-clean. Let $e^2 = e, x \in R$. Then $e+ex(1-e) \in R$ is an idempotent. Since $\big(e+ex(1-e)\big)+0 = e+ea(1-e)$,

by the uniqueness, $ex(1-e)=0$; hence, $ex=exe$. Likewise, $xe=exe$, and so $ex=xe$. Thus, R is abelian.

$\Longleftarrow$ Suppose that $a-e=a-f\in N(R)$ with idempotents $e,f\in R$. Then $e-f\in N(R)$. Since R is abelian, we see that $(e-f)^3=e-f$, and so $e=f$. Suppose that $a+e=a+f\in N(R)$ with idempotents $e,f\in R$. Similarly, we get $e=f$. This completes the proof. □

By Theorem 9.4.1 and Corollary 9.3.6, we prove that a ring R is uniquely weakly nil-clean if and only if R is abelian periodic and $R/J(R)$ is weakly Boolean. Furthermore, we derive

Corollary 9.4.2. *A ring R is uniquely weakly nil-clean if and only if*

(1) *R is abelian;*
(2) *$J(R)$ is nil;*
(3) *$R/J(R)$ is isomorphic to one of the following:*
   (a) $\mathbb{Z}_3$,
   (b) *a Boolean ring, or*
   (c) $\mathbb{Z}_3\times B$ *where B is Boolean.*

Proof. $\Longrightarrow$ In view of Theorem 9.4.1, R is abelian; hence, it is strongly weakly nil-clean. By using Corollary 9.3.5, we prove (2) and (3).

$\Longleftarrow$ This is obvious, by Corollary 9.3.5 and Theorem 9.4.1. □

Corollary 9.4.3. *A local ring R is uniquely weakly nil-clean if and only if*

(1) *$J(R)$ is nil;*
(2) *$R/J(R)$ is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$.*

Proof. $\Longrightarrow$ Since R is local, $R/J(R)$ is a division ring, and proving (2) by Corollary 9.4.2.

$\Longleftarrow$ This is clear from Corollary 9.4.2, as every local ring is abelian. □

Corollary 9.4.4. *A ring R is uniquely nil-clean if and only if*

(1) *$2\in R$ is nilpotent;*
(2) *R is uniquely weakly nil-clean.*

Proof. $\Longrightarrow$ Clearly, R is uniquely weakly nil-clean. In view of Theorem 5.2.3, $\overline{2}^2=\overline{2}$ in $R/J(R)$, and so $2\in J(R)$ is nilpotent, as desired.

$\Longleftarrow$ Since $2\in\mathbb{Z}_3$ is not nilpotent, by Corollary 9.4.2, R is abelian, $J(R)$ is nil, and that $R/J(R)$ is Boolean. The result follows from Theorem 5.2.3. □

Example 9.4.5. $\mathbb{Z}_3$ is uniquely weakly nil-clean, but $\mathbb{Z}_3 \times \mathbb{Z}_3$ is not uniquely weakly nil-clean.

Proof. Clearly, $\mathbb{Z}_3$ is uniquely weakly nil-clean. One easily checks that $(1,2) \in \mathbb{Z}_3 \times \mathbb{Z}_3$ can not be written as the sum of difference of a nilpotent and an idempotent, and so $\mathbb{Z}_3 \times \mathbb{Z}_3$ is not weakly nil-clean, and we are through. □

Theorem 9.4.6. *A ring R is uniquely weakly nil-clean if and only if*

(1) *R is abelian;*
(2) *$R/P(R)$ is uniquely weakly nil-clean.*

Proof. $\Longrightarrow$ Clearly, R is abelian. Since R is clean, it is an exchange ring. Thus, $R/P(R)$ is abelian. Obviously, $R/J(R)$ is weakly nil-clean. By Theorem 9.4.1, $R/P(R)$ is uniquely weakly nil-clean.

$\Longleftarrow$ For any $x \in J(R)$, $\overline{x} \in J\big(R/P(R)\big)$ is nilpotent. Since $P(R)$ is nil, $x \in R$ is an nilpotent; hence, $J(R)$ is nil. As every idempotent lifts modulo $P(R)$, we easily see that R is weakly nil-clean. According to Theorem 9.4.1, we obtain the result. □

Corollary 9.4.7. *A ring R is uniquely nil-clean if and only if*

(1) *R is abelian;*
(2) *$R/P(R)$ is uniquely nil-clean.*

Proof. One direction is obvious, by Theorem 5.3.3 and Corollary 5.3.4.

Conversely, assume that (1) and (2) hold. By virtue of Theorem 9.4.6, R is uniquely weakly nil-clean. As $R/P(R)$ is nil-clean, $\overline{2} \in R/P(R)$ is nilpotent. This implies that $2 \in R$ is nilpotent. Therefore R is nil-clean, hence the result. □

Theorem 9.4.8. *A ring R is strongly weakly nil-clean if and only if R is isomorphic to R_1, R_2 or $R_1 \times R_2$, where*

(1) *R_1 is unique nil-clean;*
(2) *R_2 is an IU ring.*

Proof. $\Longrightarrow$ In view of Theorem 9.4.1, R is abelian, and so it is strongly weakly nil-clean. By Theorem 9.3.14, R is isomorphic to R_1, R_2 or $R_1 \times R_2$, where R_1 is strongly nil-clean and R_2 is an IU ring. Since R_1 is abelian, it follows by Theorem 5.3.3 that R_1 is unique nil-clean, as desired.

$\Longleftarrow$ Since every uniquely nil-clean ring is an abelian strongly nil-clean ring, it follows from Theorem 9.3.14 that R is weakly nil-clean. As R_2

is an IU ring, as in the proof of Theorem 9.2.15, all idempotents in R_2 are $0, 1$, and so it is abelian. This implies that R is abelian. In light of Theorem 9.4.1, the proof is complete. □

We close this section with the discussion of uniquely weakly nil-clean property of group rings.

Lemma 9.4.9. *Let R be a ring, and let G be a group. If RG is uniquely weakly nil-clean, then so is R.*

Proof. Let $\omega : RG \to R$ be the augmentation of RG. Then $R \cong RG/Ker(\omega)$. Hence, R is weakly nil-clean. In view of Theorem 9.4.1, RG is abelian, and then so is R. By using Theorem 9.4.1 again, R is uniquely weakly nil-clean. □

Theorem 9.4.10. *Let R be a ring, and let G be a group. If $I(R, G)$ is nil, then RG is uniquely weakly nil-clean if and only if so is R.*

Proof. One direction is obvious by Lemma 9.4.9. Conversely, assume that R is uniquely weakly nil-clean. Let $a \in RG$. Then $a = \omega(a) + \big(a - \omega(a)\big)$. By hypothesis, there exists a weak idempotent $e \in R$ such that $w := \omega(a) - e \in N(R)$. Hence, $x = e + \big(w + (a - \omega(a))\big)$. Since $Ker(\omega)$ is nil, we see that $v := w + (a - \omega(a)) \in N(R)$. Thus, RG is weakly nil-clean.

As R is uniquely weakly nil-clean, it is abelian. Clearly, $Ker(\omega) \subseteq J(RG)$. It follows by Lemma 4.4.33 that every idempotent of RG is in R. Hence, RG is abelian. According to Theorem 9.4.1, RG is uniquely weakly nil-clean. □

Corollary 9.4.11. *Let R be a ring, and G be a group. Then RG is uniquely nil-clean if and only if R is uniquely nil-clean and $I(R, G)$ is nil.*

Proof. Suppose RG is uniquely nil-clean. Then RG is uniquely weakly nil-clean and $2 \in N(RG)$, by Corollary 9.4.4. Hence, R is uniquely weakly nil-clean and $2 \in N(R)$. By using Corollary 9.4.4 again, R is uniquely nil-clean. It follows from Lemma 5.3.7 that RG is uniquely clean. In view of Theorem 4.4.6, $RG/J(RG)$ is Boolean. For any $g \in G$, we see that $(1 - g) - (1 - g)^2 \in J(RG)$; hence, $1 - g \in J(RG)$. This implies that $Ker(\omega) \subseteq J(RG)$ is nil, as desired.

Conversely, assume that R is uniquely nil-clean and $Ker(\omega)$ is nil. In light of Corollary 9.4.4 and Theorem 9.4.10, $2 \in N(R)$ and RG is uniquely

weakly nil-clean. By using Corollary 9.4.4 again, RG is uniquely nil-clean, as asserted. □

Corollary 9.4.12. *Let R be a ring with a prime $p \in J(R)$, and let G be a locally finite p-group. Then RG is uniquely weakly nil-clean if and only if R is uniquely weakly nil-clean.*

Proof. $\Longrightarrow$ This is obvious, by Lemma 9.4.9.

$\Longleftarrow$ In view of Corollary 9.4.2, $J(R)$ is nil. We first suppose G is finite and prove the claim by induction on $|G|$. As the center of a nontrivial finite p-group contains more than one element, we may take $x \in G$ be an element in the center with the order p. Let (x) be the subgroup of G generated by x. Then $\overline{G} = G/(x)$ has smaller order. By induction hypothesis, $Ker(\overline{\omega})$ is nil, where $\overline{\omega} : R\overline{G} \to R$, $\sum\limits_{\overline{g} \in \overline{G}} r_{\overline{g}}\overline{g}$. Let $\varphi : RG \to R\overline{G}, \sum\limits_{g} r_g g \to \sum\limits_{g} r_g \overline{g}$. Then $Ker(\varphi) = (1-x)RG$. Since $x^p = 1$, we see that $(1-x)^p \in pRG$; hence, $1 - x \in RG$ is nilpotent. Thus, $Ker(\varphi)$ is nilpotent. But $\varphi\big(Ker(\omega)\big) = Ker(\overline{\omega})$ is nil. For any $z \in Ker(\omega)$, we have some $m \in \mathbb{N}$ such that $z^m \in Ker(\varphi)$ is nilpotent. Thus, $z \in RG$ is nilpotent.

Nor for the general case. Let $r \in I(R, G)$. If H is the subgroup of G generated by the support of r. Then $r \in I(R, H)$. As H is a finite p-group, by the preceding discussion, r is nilpotent. Thus, $I(R, G)$ is nil.

Therefore RG is uniquely weakly nil-clean, in terms of Theorem 9.4.10. □

Similarly, we now derive

Corollary 9.4.13. *Let R be a ring with a prime $p \in J(R)$, and let G be a locally finite p-group. Then RG is uniquely nil-clean if and only if R is uniquely nil-clean.*

Example 9.4.14. Let G be a locally finite 3-group. Then $\mathbb{Z}_3 G$ is not uniquely nil-clean, while it is uniquely weakly nil-clean.

Notes. Theorem 9.1.2, Theorem 9.1.5 and Theorem 9.1.11 are due to Ahn and Anderson [13]. Theorem 9.1.21 was proved by Arora and Kundu [26, Theorem 2.3]. Theorem 9.1.24, Theorem 9.1.26 and Theorem 9.1.28 are also due to [26, Theorem 3.5, Theorem 3.4 and Theorem 3.7]. Theorem 9.1.33 was proved by Kosan et al. [372, Theorem 8]. Theorem 9.2.5,

Theorem 9.2.8, Theorem 9.2.10 and Theorem 9.2.12 are essentially due to Breaz et al. [81]. Theorem 9.2.20 is due to Breaz et al. [81, Theorem 20]. Theorem 9.2.23 is due to Ahn and Anderson [13, Theorem 1.12]. Theorem 9.2.26 was proved in [81, Theorem 12]. Theorem 9.3.19 was established by Danchev [226] and [228]. Theorem 9.4.1 is due to Danchev [200, Theorem 2.12].

Chapter 10

Periodic and Weakly Periodic Rings

We investigate here various ring-theoretic properties of periodic rings and related ones. In Section 10.1, we discuss elementary properties of periodic rings. In Section 10.2, we are concerned with a subclass of abelian periodic rings. The commutativity of such rings is discussed. Following Lee and Zhou, a ring R is uniquely π-clean provided that for any $a \in R$ there exists some $m \in \mathbb{N}$ such that $a^m \in R$ is uniquely clean (see [386]). In Section 10.3, we establish various equivalences of such rings. The structure of uniquely π-clean rings is thereby determined in terms of periodicity. In the final section, we then study when a weakly periodic ring is periodic.

10.1 Periodic Rings

An element $p \in R$ is said to be potent provided that $p = p^n$ for some $n \geq 2$. A ring R is potent in case every element in R is potent. This section is devoted to find the conditions under which a ring R is periodic. The following elementary fact has significance for our subsequent investigations.

Theorem 10.1.1. *Let R be a ring. Then the following are equivalent:*

(1) *R is periodic.*
(2) *R is strongly π-regular and $U(R)$ is a torsion group.*
(3) *For any $a \in R$, there exists a potent $p \in comm(a)$ such that $a - p \in N(R)$.*

Proof. (1) $\Rightarrow$ (2) This is obvious.

(2) $\Rightarrow$ (1) Let $a \in R$. As in the proof of Theorem 1.1.2, there exists an idempotent $e \in R$ and an invertible $u \in R$ such that $a^n = eu = ue$ for some $n \in \mathbb{N}$. By hypothesis, we have some $m \in \mathbb{N}$ such that $u^m = 1$. Hence, $a^{mn} = e$, and so $a^{2mn} = a^{mn}$, as required.

(1) $\Rightarrow$ (3) For any $a \in R$, there exist $m, n (m > n)$ such that $a^m = a^n$. Thus, $a^n = a^m = a^{(m-n)+n} = \cdots = a^{n(m-n)+n}$, and so $a^n = \big(a^n\big)^{m-n+1}$. This implies that $\big(a^{n(m-n)}\big)^2 = \big(a^{n(m-n)+n}\big)\big(a^{n(m-n)-n}\big) = a^n\big(a^{n(m-n)-n}\big) = a^{n(m-n)}$. Choose $k = n(m-n)$. Then $\big(a^k\big)^2 = a^k$; hence, $a^k \in R$ is an idempotent. It is easy to verify that $\big(a^{k+1}\big)^{k+1} = \big(a^k\big)^{k+1}a^{k+1} = a^k a^{k+1} = a^{k+1}$ and $\big(a - a^{k+1}\big)^k = a^k\big(1 - a^k\big)^k = a^k\big(1 - a^k\big) = 0$. Set $p = a^{k+1}$ and $w = a - a^{k+1}$. Then $p \in comm(a)$ is potent and $w \in N(R)$. Furthermore, $a = p + w$, as desired.

(3) $\Rightarrow$ (1) For any $a \in R$, there exists a potent $p \in comm(a)$ and a nilpotent $w \in R$ such that $a = p + w$. Write $p^n = p$ for some $n \geq 2$. Then $a^n = p^n + wb$ for some $b \in comm(w)$. This shows that $a - a^n = w(1-b) \in N(R)$, thus completing the proof by Lemma 5.1.10. □

Corollary 10.1.2. *A ring R is periodic if and only if*

(1) *R is strongly π-regular;*
(2) *For any $u \in U(R)$, $u^{-1} \in \mathbb{Z}[u]$.*

Proof. $\Longrightarrow$ In view of Theorem 10.1.1, R is strongly π-regular. For any $u \in U(R)$, there exist distinct $m, n \in \mathbb{N}$ such that $u^m = u^{m+n}$. Hence, $u^n = 1$, and so $u^{-1} = u^{n-1} \in \mathbb{Z}[u]$.

$\Longleftarrow$ For any $a \in R$, we have $m \in \mathbb{N}$ and $b \in R$ such that $a^m = a^m b a^m, b = bab$ and $ab = ba$ from Lemma 1.1.1. Let $c = ba^m b$. Then $a^m = a^m c a^m, c = ca^m c$ and $ac = ca$. Set $u = 1 - a^m c + a^m$. Then $u^{-1} = 1 - a^m c + c$; hence, $u \in U(R)$. By hypothesis, there exists some $g(t) \in \mathbb{Z}[t]$ such that $ug(u) = 1$. Set $h(t) = tg^2(t) \in \mathbb{Z}[t]$. Then $uh(u) = 1$. Further, $a^m = a^m c\big(1 - a^m c + a^m\big) = a^m cu$. This implies that $a^m u^{-1} = a^m c$, and so $a^m = a^m c a^m = a^{2m} h(u) = a^m\big(a^m h(u)\big)$. Write $h(u) = b_1 u + \cdots + b_n u^n \in \mathbb{Z}[u]$. One easily checks that $a^m u = a^m\big(1 - a^m c + a^m\big) = a^{2m}, a^m u^2 = a^{2m} u = a^{3m}, \ldots, a^m u^n = a^{nm} u = a^{(n+1)m}$. This shows that $a^m = a^m\big(a^{2m} b_1 + a^{3m} b_2 + \cdots + a^{(n+1)m} b_n\big) = a^{m+1} k(a)$, where $k(a) = a^{2m-1} b_1 + a^{3m-1} b_2 + \cdots + a^{(n+1)m-1} b_n\big) \in \mathbb{Z}[a]$, hence the result by Lemma 5.1.10. □

Corollary 10.1.3. *A ring R is Boolean if and only if*

(1) *R is strongly π-regular;*
(2) $U(R) = \{1\}$.

Proof. $\Longrightarrow$ This is obvious.

$\Longleftarrow$ Let $a \in R$. In light of Theorem 10.1.1, there exists a potent $p \in comm(a)$ such that $w := a - p \in N(R)$. Write $p = p^n (n \geq 2)$. Then $\big(1 - p^{n-1} + p\big)^{-1} = 1 - p^{n-1} + p^{n-2}$. Hence, $p = p^{n-1}$, and so $p^2 = p^n = p$. Furthermore, $1 - w = 1$, and then $w = 0$. Therefore $a = p \in R$ is an idempotent, as desired. □

Corollary 10.1.4. *A ring R is potent if and only if R is a reduced periodic ring.*

Proof. One direction is obvious. Conversely, assume that R is a reduced periodic ring. Let $a \in R$. In view of Theorem 10.1.1, there exists a potent $p \in R$ such that $a - p \in N(R)$. As R is reduced, we get $a = p$, as desired. □

Let F_q be a finite field with q elements. Then $|U\big(F_q\big)| = q - 1$. Hence, every element of the group $U\big(F_q\big)$ is of finite order and its order is a divisor of $q-1$. Thus, $a^{q-1} = 1$ for all $a \in U\big(F_q\big)$, and so F_q is potent. Therefore we conclude that every finite field is potent. For instance, $\mathbb{Z}_2[x]/\big(x^2+x+1\big) = \{0, 1, \alpha, 1+\alpha\}$ is a potent field, where $\alpha^2 + \alpha + 1 = 0$. Further, the direct product of finitely many of potent rings is potent, e.g., $\mathbb{Z}_3 \times \mathbb{Z}_5$ is potent. Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \;\middle|\; a, b \in \mathbb{Z}_3 \right\}$. Then R is periodic as it is finite. But R is not reduced, and so it is not potent by Corollary 10.1.4.

Theorem 10.1.5. *A ring R is periodic if and only if for any $a \in R$ there exists a prime m such that $a - a^m \in N(R)$.*

Proof. One direction is trivial. Conversely, assume that R is periodic. Let $a \in R$. By virtue of Lemma 5.1.11, $a - a^n \in N(R)$ for some $n \geq 2$. According to the Dirichlet Theorem, there exists a prime m such that $m = s(n-1) + 1$ for some $s \in \mathbb{N}$. One easily checks that

$$\begin{aligned} a - aa^{n-1} &\in N(R), \\ aa^{n-1} - aa^{2(n-1)} &= (a - a^n)a^{n-1} \in N(R), \\ aa^{2(n-1)} - aa^{3(n-1)} &= (a - a^n)a^{2(n-1)} \in N(R), \\ &\vdots \\ aa^{(s-1)(n-1)} - aa^{s(n-1)} &= (a - a^n)a^{(s-1)(n-1)} \in N(R). \end{aligned}$$

Therefore $a - a^m = a - a^{s(n-1)+1} \in N(R)$, as asserted. □

Corollary 10.1.6. *A ring R is potent if and only if for any $a \in R$, there exists a prime m such that $a = a^m$.*

Proof. One direction is clear. Conversely, assume that R is potent. Then R is periodic. Let $a \in R$. It follows from Theorem 10.1.5 that there exists a prime m such that $a - a^m \in N(R)$. Clearly, R is reduced. Therefore $a = a^m$, as desired. □

It is an interesting problem to determine for which integers n, such that if R is a ring having the property that $a = a^n$ for each $a \in R$, then R is Boolean. We record that if $n = 2^{q(m+1)} + 2^m$, with $q, m \in \mathbb{N}$, then it is in the case that R is Boolean (see [472, Proposition 3]). For instance, if $a = a^6$ for every a in R, then R is Boolean. If R is a ring of characteristic 2, and there exist $s, t \in \mathbb{N}$ such that $(s, t) = 1$, $a = a^{2^s} = a^{2^t}$ for each $a \in R$, then R is Boolean (see [301, Theorem 11]).

Moreover, we have a result observed by Bell, Tominaga and Ohori (see [62] and [442]).

Proposition 10.1.7. *Let R be a ring. Then the following are equivalent:*

(1) *R is potent.*
(2) *For any $a \in R$, there exists a unique potent $p \in R$ such that $a - p \in N(R)$.*
(3) *For any $a \in R$, there exists a unique potent $p \in comm(a)$ such that $a - p \in N(R)$.*

Proof. (1) $\Rightarrow$ (2) This is clear, as every potent ring is reduced.

(2) $\Rightarrow$ (1) Write $2 = p + w$ where $p \in R$ is potent and $w \in N(R)$. Then $pw = wp$. Write $p^m = p$ for some $m > 1$. Then $2 - 2^m \in N(R)$. Thus, we can find $k \in \mathbb{N}$ such that $k \cdot 1_R = 0$. If $c^2 = 0$, then $1 + c = (1 + c)^{k+1}$ is potent. By the uniqueness, we get $c = 0$. Hence, R is reduced. Therefore every element of R is potent, as required.

(1) $\Leftrightarrow$ (3) This was proved as in the preceding argument. □

Following Abu-Khuzam and Yaqub, a ring R is a D^*-ring provided that for any zero-divisor $a \in R$ there exists a potent $p \in comm(a)$ such that $a - p \in N(R)$ (see [9]).

Lemma 10.1.8. *A ring R is a D^*-ring if and only if for any zero-divisor $a \in R$, there exist distinct $m, n \in \mathbb{N}$ such that $a^m = a^n$.*

Proof. It is obvious as in the proof of Theorem 10.1.1. □

Proposition 10.1.9. *Let R be an abelian D^*-ring. Then R is periodic or every zero-divisor in R is nilpotent.*

Proof. Suppose that there exists a zero-divisor $a \in R$ such that $a \notin N(R)$. In view of Lemma 10.1.8, there exist distinct $m, n \in \mathbb{N}$ such that $a^m = a^n (m > n)$. As in the proof of Theorem 10.1.1, $e := a^{n(m-n)} \in R$ is an idempotent. Clearly, $e \in R$ is a central nontrivial idempotent, and so $R = eRe \oplus (1-e)R(1-e)$. For any $x \in eRe$, $x \in R$ is a zero-divisor. In view of Lemma 10.1.8, $x \in R$ is periodic. Hence, eRe is periodic. Similarly, $(1-e)R(1-e)$ is periodic. Therefore R is periodic. □

Corollary 10.1.10. *Let R be not a domain. Then R is potent if and only if every zero-divisor in R is potent.*

Proof. One direction is obvious. Conversely, assume that every zero-divisor in R is potent. Let $e \in R$ be a nontrivial idempotent, and let $r \in R$. Then $er(1-e) = 0$; otherwise, $er(1-e) \in R$ is a zero-divisor, and so it is potent. Then $er(1-e) = 0$, a contradiction. Therefore $ere = er$. Likewise, $ere = re$. Hence, $er = re$, and so R is abelian. If every zero-divisor in R is nilpotent, by hypothesis, every zero-divisor in R is zero. Hence, R is a domain, a contradiction. It follows from Proposition 10.1.9 that R is periodic. Let $0 \neq w \in N(R)$. Then $w \in R$ is a zero-divisor, and so it is potent. Hence, $w = 0$, a contradiction. Thus, $N(R) = 0$. In light of Corollary 10.1.4, R is potent. □

Lemma 10.1.11. *Let R be a ring. If $char(R) = p_1^{r_1} \cdots p_s^{r_n}$, where $p_1, \ldots, p_n$ are primes and $r_1, \ldots, r_n \in \mathbb{N}$, then $R = R_1 \oplus \cdots \oplus R_n$ with each $char(R_i) = p_i^{r_i}$.*

Proof. Set $a_j = \prod_{i=1, i\neq j}^{n} p_i^{r_i}$. Then $(a_1, \ldots, a_n) = 1$. Hence, we can find some $k_1, \ldots, k_n \in \mathbb{Z}$ such that $k_1 a_1 + \cdots + k_n a_n = 1$. Let $e_j = k_j a_j \cdot 1_R$. Then $e_1 + \cdots + e_n = 1_R, e_i e_j = 0 (i \neq j)$ and $e_i = e_i^2$. Set $R_i = e_i R e_i$. Then $R = R_1 \oplus \cdots \oplus R_n$ with each $char(R_i) = p_i^{r_i}$, as asserted. □

Theorem 10.1.12. *A ring R is periodic if and only if*

(1) $char(R) \neq 0$;
(2) *For any $a \in R$ there exist $s, t, m, n \in \mathbb{N}$ such that $sa^m = ta^n$ with $(s,t) = 1$ and $m \neq n$.*

Proof. $\Longrightarrow$ This is clear.

$\Longleftarrow$ Since $char(R) \neq 0$, by Lemma 10.1.11, we may assume that $char(R) = p^s (p$ a prime, $s \in \mathbb{N})$. For any $a \in R$, there exist $s, t, m, n \in \mathbb{N}$ such that $sa^m = ta^n$ with $(s,t) = 1$ and $m \neq n$. If $(s,p) = (t,p) = 1$, then $(s,p^r) = (t,p^r) = 1$. By using Euler's Theorem, $s^{\varphi(p^r)} \equiv 1, t^{\varphi(p^r)} \equiv 1 \pmod{p^r}$, where $\varphi(p^r)$ is Euler's totient function. This implies that $a^{\varphi(p^r)m} = a^{\varphi(p^r)n}$, as $char(R) = p^s$. If $(s,p) \neq 1$ or $(t,p) \neq 1$, without loss of the generality, we write $s = pk$ for $k \in \mathbb{N}$. As $(s,t) = 1$, we have some $i, j \in \mathbb{Z}$ such that $is + jt = 1$. Hence, $a^n = (is + jt)a^n = isa^n + jsa^m = s\big(ia^n + ja^m\big) = p\big(pia^n + pja^m\big)$. Consequently, $a^{np^r} = 0 = a^{np^r+1}$, and therefore R is periodic. □

Corollary 10.1.13. *Let $s, t \in \mathbb{N}$ be such that $(s,t) = 1$. If for any $a \in R$ there exist distinct $m, n \in \mathbb{N}$ such that $sa^m = ta^n$, then R is periodic.*

Proof. If $s \neq t$, then $(s-t) \cdot 1_R = 0$. If $s = t$, then $s = 1$, and so $2^m = 2^n$ with $m \neq n$. Hence, $(2^m - 2^n) \cdot 1_R = 0$. Therefore $char(R) \neq 0$, and we are through by Theorem 10.1.12. □

Corollary 10.1.14. *Let R be a ring. If for any $a \in R$ there are $n \in \mathbb{N}$ and an odd k such that $a^n = a^{n+k}$, then R is potent.*

Proof. Since $(-1)^n = (-1)^{n+k}$, we see that $2 = 0$ in R. Let $w \in N(R)$. Then we have $m \in \mathbb{N}$ and an odd l such that $(1+w)^m = (1+w)^{m+l}$. Hence, $1 = (1+w)^l$, and so $w = w^2 f(w)$ for some $f(t) \in \mathbb{Z}[t]$. This shows that $w = 0$, and so R is reduced. In light of Theorem 10.1.12, R is periodic. This completes the proof, by Corollary 10.1.4. □

Lemma 10.1.15. *Let T be the ring of a Morita context $(A, B, M, N, \psi, \varphi)$. If A and B are periodic, then for any $a \in A, b \in B$, there exists $n \in \mathbb{N}$ such that $a - a^n \in N(A), b - b^n \in N(B)$.*

Proof. By hypothesis, we can find $p, q, s, t \in R (p < q, s < t)$ such that $a^p = a^q$ in A and $b^s = b^t$ in B. Hence, $a^{ps} = a^{qs}$ and $b^{ps} = b^{pt}$. This implies that

$$a^{ps} = a^{ps}a^{(q-p)s} = a^{ps}a^{2(q-p)s} = \cdots = a^{ps}a^{(t-s)p(q-p)s}.$$

Likewise, we get $b^{ps} = b^{ps}b^{(q-p)s(t-s)p}$. Choose $k = ps$ and $l = ps + (t - s)p(q - p)s$. Then $a^k = a^l, b^k = b^l (k < l)$. Choose $n = l - k + 1$. As in the proof of Lemma 5.1.11, we check that $a - a^n \in N(A), b - b^n \in N(B)$, this completes the proof. □

Theorem 10.1.16. *Let T be the ring of a Morita context $(A, B, M, N, \psi, \varphi)$. If $Im(\psi)$ and $Im(\varphi)$ are nilpotent, then A and B are periodic if and only if so is T.*

Proof. Suppose A and B are periodic. For any $\begin{pmatrix} a & n \\ m & b \end{pmatrix} \in T$, by Lemma 10.1.15, there exists $k \in \mathbb{N}$ such that $a - a^k \in N(A)$ and $b - b^k \in N(B)$. Hence,

$$\begin{pmatrix} a & n \\ m & b \end{pmatrix} - \begin{pmatrix} a & n \\ m & b \end{pmatrix}^k = \begin{pmatrix} a - a^k + c & * \\ * & b - b^k + d \end{pmatrix},$$

where $c \in Im(\psi)$ and $d \in Im(\varphi)$. Write $(a - a^k)^l = 0$ and $(b - b^k)^l = 0$. By hypothesis, $Im(\psi)$ and $Im(\varphi)$ are nilpotent ideals of A and B, respectively. Say $\big(Im(\psi)\big)^s = 0$ and $\big(Im(\varphi)\big)^t = 0$. Choose $p = \max(s, t)$ and $q = p(l + 1)$. Then

$$\big(a - a^k + c\big)^q = 0 \text{ and } \big(b - b^k + d\big)^q = 0.$$

Obviously,

$$\begin{pmatrix} a - a^k + c & * \\ * & b - b^k + d \end{pmatrix}^{q+1} \subseteq \begin{pmatrix} Im(\psi) & N \\ M & Im(\varphi) \end{pmatrix}.$$

Set $NM := Im(\psi)$ and $MN := Im(\varphi)$. As in the proof of Theorem 5.2.11, we see that

$$\begin{pmatrix} NM & N \\ M & MN \end{pmatrix}^{2l} = \begin{pmatrix} NM & (NM)N \\ (MN)M & MN \end{pmatrix}^{l} = \begin{pmatrix} (NM)^l & (NM)^l N \\ (MN)^l M & MN \end{pmatrix}$$

for any $l \in \mathbb{N}$. Choose $j = 2p(q + 1)$. As $(NM)^p = (MN)^p = 0$, we have

$$\begin{pmatrix} a - a^k + c & * \\ * & b - b^k + d \end{pmatrix}^j = \begin{pmatrix} 0 & t \\ s & 0 \end{pmatrix}$$

for some $s \in M, t \in N$. Hence,

$$\begin{pmatrix} 0 & t \\ s & 0 \end{pmatrix}^2 = \begin{pmatrix} \psi(t \otimes s) & 0 \\ 0 & \varphi(s \otimes t) \end{pmatrix},$$

and so

$$\left(\begin{pmatrix} a & n \\ m & b \end{pmatrix} - \begin{pmatrix} a & n \\ m & b \end{pmatrix}^k \right)^{2jp} = 0.$$

Accordingly, T is periodic, by Lemma 5.1.11. The converse is obvious. □

Corollary 10.1.17. *Let R be periodic, and let $s \in N(R) \cap C(R)$. Then $M_{(s)}(R)$ is periodic.*

Proof. Let $\psi : R \otimes R \to R, n \otimes m \mapsto snm$ and $\varphi : R \otimes R \to R, m \otimes n \mapsto smn$. Then $M_s(R) = (R, R, R, R, \psi, \varphi)$. As $s \in N(R) \cap C(R)$, we see that $Im(\varphi)$ and $Im(\psi) \subseteq J(R)$ are nilpotent, and we are through by Theorem 10.1.16. □

Example 10.1.18. Let R be periodic, let $A = B = \begin{pmatrix} R & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & R \end{pmatrix}, M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & R & 0 \end{pmatrix}$ and $N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ R & R & 0 \end{pmatrix}$, and let $\psi : N \bigotimes_B M \to A, \psi(n \otimes m) = nm$ and $\phi : M \bigotimes_A N \to B, \phi(m \otimes n) = mn$. Then $T = (A, B, M, N, \psi, \phi)$ is a Morita context with zero pairings. Hence, $Im(\psi)$ and $Im(\varphi)$ are nilpotent. Clearly, A and B are both periodic. In light of Theorem 10.1.16, T is periodic.

Let R be a ring, and let α be an endomorphism of R. Let $T_n(R, \alpha)$ be the set of all upper triangular matrices over the rings R. For any $(a_{ij}), (b_{ij}) \in T_n(R, \sigma)$, we define $(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$, and $(a_{ij})(b_{ij}) = (c_{ij})$ where $c_{ij} = \sum_{k=i}^{n} a_{ik}\alpha^{k-i}(b_{kj})$. Then $T_n(R, \alpha)$ is a ring under the preceding addition and multiplication. Clearly, $T_n(R, \alpha)$ will be $T_n(R)$ only when α is the identity morphism.

Proposition 10.1.19. *Let R be periodic, and let $\alpha : R \to R$ be an endomorphism. Then*

(1) *$R[[x, \alpha]]/(x^n)$ is periodic.*
(2) *$T_n(R, \alpha)$ is periodic for all $n \in \mathbb{N}$.*

Proof. (1) For any $f(x) \in R[[x]]/(x^n)$, there exists $m \in \mathbb{N}$ such that $f(0) - f^m(0) \in N(R)$. Hence, $f(x) - f^m(x) \in N\big(R[[x]]/(x^n)\big)$. According to Lemma 5.1.11, $R[[x]]/(x^n)$ is periodic.

(2) For any $(a_{ij}) \in T_n(R, \alpha)$, as in the proof of Lemma 10.1.15, we can find an $m \in \mathbb{N}$ such that each $a_{ii} - a_{ii}^m \in N(R)$. Thus, $(a_{ij}) - (a_{ij})^m \in N\big(T_n(R, \alpha)\big)$, as required. □

Corollary 10.1.20. *Let R be periodic. Then $M_{(x^m)}\big(R[[x]]/(x^n)\big)$ is periodic for all $1 \leq m \leq n$.*

Proof. Choose $\alpha = 1$. Then $R[[x]]/(x^n)$ is periodic, by Proposition 10.1.19. Choose $s = x^m (1 \leq m \leq n)$. Then $s \in N\big(R[[x]]/(x^n)\big) \cap C\big(R[[x]]/(x^n)\big)$. Applying Corollary 10.1.17 to $R[[x]]/(x^n)$, we obtain the result. □

10.2 Generalized n-Like Rings

The goal of this section is to consider a subclasses of abelain periodic rings. Let $n \geq 2$ be a fixed integer. Following Yaqub, a ring R is said to be a generalized n-like ring provided that for any $a, b \in R$, $(ab)^n - ab^n - a^n b + ab = 0$. We begin with the following elementary lemma which is due to Jacobson (see [341, Theorem 10.1.1]).

Lemma 10.2.1. *Every potent ring is commutative.*

Proof. Let R be potent. Then $R/J(R)$ is potent. We easily check that $R/J(R)$ is isomorphic to a subdirect product of some primitive rings R_i. Hence, we have some epimorphisms $\varphi_i : R/J(R) \to R_i$ such that $\bigcap_i Ker(\varphi_i) = 0$. By the structure theorem of primitive ring, there exists a division ring D_i such that either (1) there exists a subring S_i of R_i which admits an epimorphism $\phi_i : S_i \to M_2(D_i)$, (2) $R_i \cong M_{m_i}(D_i)$.

Case 1. Since the hypothesis is inherited by all subrings and all homomorphic images of R, we claim that $M_2(D_i)$ is potent. But $E_{12}^n \neq E_{12}$ for all $n \geq 2$. This gives a contradiction.

Case 2. By the preceding discussion, $m_i = 1$, and so $R_i \cong D_i$. Thus, D_i is a potent division ring. This implies that R_i is commutative (see [341, Theorem 7.12.3]).

For any $a, b \in R$, then $\overline{ab - ba} \in R/J(R)$. This implies that $\varphi_i\big(\overline{ab - ba}\big) = 0$, and so $\overline{ab - ba} \in \bigcap_i Ker(\varphi_i) = 0$. We infer that $ab - ba \in J(R)$. As R is potent, we can find some $m \geq 2$ such that $\big(ab - ba\big)^m = ab - ba$; hence, $\big(ab - ba\big)\big(1 - (ab - ba)^{m-1}\big) = 0$. Therefore $ab = ba$, as desired. □

Herstein extended Lemma 10.2.1 and proved that a ring R is commutative if and only if for any $a \in R$, there exists some $n(a) \geq 2$ such that $a - a^{n(a)} \in C(R)$ (see [53, Theorem H_3]).

Lemma 10.2.2. *Let R be a ring. Then the following are equivalent:*

(1) *R is abelian.*
(2) *For any idempotents $e, f \in R$, $e - f \in N(R) \Longrightarrow e = f$.*
(3) *For any idempotents $e, f \in R$, $(ef)^n = (fe)^n$ for some $n \in \mathbb{N}$.*

Proof. (1) $\Rightarrow$ (2) If $e - f \in N(R)$ for some idempotents $e, f \in R$, then $(e-f)^3 = e - f$, and so $e = f$.

(2) $\Rightarrow$ (1) Let $e \in R$ be an idempotent and $r \in R$. Choose $f = e + er(1-e)$. Then $e - f \in N(R)$; hence, $e = f$. This implies that $er = ere$. Likewise, $er = ere$. So $er = re$, and then R is abelian.

(1) $\Rightarrow$ (3) This is trivial.

(3) $\Rightarrow$ (1) Let $e \in R$ be an idempotent and $r \in R$. Choose $f = e + er(1-e)$. Then $(ef)^n = (fe)^n$ for some $n \in \mathbb{N}$; hence, $e + er(1-e) = e$, whence $er = ere$. Similarly, $er = ere$. Therefore $er = re$, the result follows. □

Theorem 10.2.3. *Let R be a generalized n-like ring, and let $n \geq 2$ be an integer. Then the following hold:*

(1) *R is an abelian periodic ring.*
(2) *$R/J(R)$ is commutative and $J(R) = N(R)$.*
(3) *$\alpha\beta = 0$ for all nilpotent $\alpha, \beta \in R$.*

Proof. (1) Let $a \in R$. Then $a^{2n} - 2a^{n+1} + a^2 = 0$, and so $(a - a^n)^2 = 0$. Thus, $a - a^n \in N(R)$. Accordingly, R is periodic by Lemma 5.1.11.

Let $e, f \in R$ be idempotents. Then we have

$$\big((1-e)f\big)^n e = \big((1-e)fe\big)^n - (1-e)fe + (1-e)fe = 0;$$
$$\big((1-e)f\big)^n = (1-e)f + (1-e)f - (1-e)f = (1-e)f.$$

Hence $(1-e)fe = 0$, and so $fe = efe$. Likewise, $ef = efe$. Therefore $ef = fe$. In light of Lemma 10.2.2, R is abelian.

(2) Clearly, $R/J(R)$ is isomorphic to a subdirect product of some primitive rings R_i. It is easy to check that $R/J(R)$ is a generalized n-like ring, and then so is R_i.

Case 1. There exists a subring S_i of R_i which admits an epimorphism $\phi_i : S_i \to M_2(D_i)$ where D_i is a division ring. Hence, S_i is a generalized n-like ring, and then so is $M_2(D_i)$. Choose $a = E_{11} + E_{12}, b = E_{22} \in M_2(D_i)$. Then $(ab)^n - ab^n - a^n b + ab = -\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0$, a contradiction.

Case 2. $R_i \cong M_{m_i}(D_i)$ for a division ring D_i. As in the preceding argument, $m_i = 1$. Thus, the division ring D_i is periodic, and so it is

potent. By virtue of Lemma 10.2.1, R_i is commutative. Similarly to the proof of Lemma 10.2.1, $R/J(R)$ is commutative.

Since R is periodic, it is strongly π-regular, and so $J(R)$ is nil. hence, $J(R) \subseteq N(R)$. If $a \in N(R)$, then $\overline{a} \in R/J(R)$ is nil. This implies that $\overline{a} \in J\big(R/J(R)\big)$; hence, $a \in J(R)$. Thus, $J(R) = N(R)$.

(3) If $a \in N(R)$, then $(a - a^n)^2 = 0$, and so $a^2 = a^3\big(2a^{n-2} - a^{2n-3}\big)$. This implies that $a^2 = 0$. For all nilpotent $\alpha, \beta \in R$, it follows from (2) that $\alpha\beta \in N(R)$, and then $\alpha^2 = \beta^2 = (\alpha\beta)^2 = 0$. As $(\alpha\beta)^n - \alpha\beta^n - \alpha^n\beta + \alpha\beta = 0$, we conclude that $\alpha\beta = 0$, as asserted. □

A ring R is an abelian periodic ring if and only if for any $a \in R$, there exists some $m \in \mathbb{N}$ such that $a^m \in R$ is uniquely nil-clean. As an immediate consequence of Theorem 10.2.3, we now derive

Corollary 10.2.4. *Let R be a generalized n-like ring. Then for any $a \in R$, there exists some $m \in \mathbb{N}$ such that $a^m \in R$ is uniquely nil clean.*

Proof. In view of Theorem 10.2.3, R is an abelian periodic ring, the result follows. □

Corollary 10.2.5. *Let R be a ring, and let $n \geq 2$ be an integer. Then the following are equivalent:*

(1) *R is a generalized n-like ring.*
(2) *For any $a, b \in R$, $(ab)^n = a^n b^n$ and $(a - a^n)(b - b^n) = 0$.*

Proof. (1) $\Rightarrow$ (2) Let $a, b \in R$. Then $a - a^n, b - b^n \in N(R)$. In view of Theorem 10.2.3, $(a - a^n)(b - b^n) = 0$. That is, $ab - ab^n - a^n b + a^n b^n = 0$. But $ab - ab^n - a^n b + (ab)^n = 0$. Therefore $(ab)^n = a^n b^n$.

(2) $\Rightarrow$ (1) Let $a, b \in R$. Then $(ab)^n = a^n b^n$ and $(a - a^n)(b - b^n) = 0$. Consequently, $(ab)^n - ab^n - a^n b + ab = a^n b^n - ab^n - a^n b + ab = (a - a^n)(b - b^n) = 0$, as required. □

Corollary 10.2.6. *A ring R is a generalized 2-like ring if and only if it is an abelian Boolean-like ring.*

Proof. One direction is obvious from Theorem 10.2.3 and Corollary 10.2.5. Conversely, assume that R is an abelian Boolean-like ring. By Theorem 5.2.16, R is strongly nil-clean. Let $a, b \in R$. Write $a = a_B + a_N$ and $b = b_B + b_N$. Here, $a_B, b_B \in R$ are idempotents and $a_N, b_N \in N(R)$.

As R is abelian, it follows by Theorem 5.2.16 that

$$\begin{aligned}(ab)^2 - ab^2 - a^2b + ab &= (a_Bb_B + a_Bb_N + a_Nb_B)^2 - ab^2 - a^2b + ab\\ &= \big(a_Bb_B + 2a_Bb_B(a_N + b_N)\big) - ab^2 - a^2b + ab\\ &= a^2b^2 - ab^2 - a^2b + ab\\ &= (a - a^2)(b - b^2)\\ &= 0,\end{aligned}$$

hence the result. □

Lemma 10.2.7. *Let R be a generalized n-like ring. Then $(n-1)[a,b] = 0$ for all $a \in N(R)$ and $b \in R$.*

Proof. Let $a \in N(R)$ and $b \in R$. Then $a^2 = 0$ by Theorem 10.2.3. In view of Corollary 10.2.5,

$$\begin{aligned}0 &= (ab)^n - ab^n - a^nb + ab = a^nb^n - ab^n - a^nb + ab = ab - ab^n;\\ 0 &= (ba)^n - ba^n - b^na + ba = b^na^n - ba^n - b^na + ba = ba - b^na.\end{aligned}$$

Hence, $[a, b^n] = ab^n - b^na = ab - ba = [a, b]$. By Theorem 10.2.3 again, we check that

$$\begin{aligned}(ab + b)^n &= ab^n + \big(bab^{n-1} + b^2ab^{n-2} + \cdots + b^{n-1}ab\big) + b^n;\\ (ba + b)^n &= \big(bab^{n-1} + b^2ab^{n-2} + \cdots + b^{n-1}ab\big) + b^na + b^n.\end{aligned}$$

Accordingly,

$$\begin{aligned}[a, b] = [a, b^n] &= (ab + b)^n - (ba + b)^n = (a + 1)^nb^n - b^n(a + 1)^n\\ &= [(a + 1)^n, b^n] = n[a, b^n] = n[a, b],\end{aligned}$$

as required. □

Lemma 10.2.8. *Every generalized n-like ring is a subdirect product of local rings.*

Proof. Let R be a generalized n-like ring. Then R is an abelian periodic ring by Theorem 10.2.3. In view of Lemma 4.4.19, $\bigcap\{P \mid P \in Pier(R)\} = 0$. Let $\varphi_P : R \to R/P$ be the natural epimorphism. Then $\bigcap_{P\in Pier(R)} Ker\varphi_P = \bigcap_{P\in Pier(R)} P = 0$. Hence, R is the subdirect product of all R/P, where $P \in Pier(R)$. Let $P \in Pier(R)$. Since R is clean, it follows by Corollary 7.1.7 that R/P is a local ring, as desired. □

Theorem 10.2.9. *Let n be an even positive integer. Then every generalized n-like ring is a commutative periodic ring.*

Proof. Let R be a generalized n-like ring. In view of Lemma 10.2.8, R is a subdirect product of local rings R_i. Thus, we have some epimorphisms

$\varphi_i : R \to R_i$ such that $\bigcap_i Ker(\varphi_i) = 0$. Clearly, R_i is a generalized n-like ring. For any $a \in N(R_i), b \in R_i$, we have $(n-1)[a,b] = 0$ by Lemma 10.2.7. On the other hand, $\left((-1) - (-1)^n\right)^2 = 0$, and then $4 = 0$. As n is even, $(4, n-1) = 1$. Thus, we can find some $k, l \in \mathbb{Z}$ such that $4k + (n-1)l = 1$. It follows that $[a,b] = 4k[a,b] + (n-1)l[a,b] = 0$, and therefore $ab = ba$. That is, $a \in N(R_i)$ is central.

In light of Theorem 10.2.3, $R_i/J(R_i)$ is commutative and $J(R_i) = N(R_i)$. Let $\overline{R_i} = R_i/N(R_i)$. Then $\overline{R_i}$ is a field. As R_i is periodic, we see that $\overline{R_i}$ is potent. In addition, $4 = 0$ in R_i. If $a, b \in R_i \setminus N(R_i)$, then the subring of $\overline{R_i}$ generated by $\overline{a} = a + N(R_i), \overline{b} = b + N(R_i)$ is a finite field. As is well known, the multiplicative group of all nonzero elements of any finite field is cyclic (see [457, Corollary 64]). Thus, there exist $g \in R$ and $s, t \geq 0$ such that $a = g^s + c$ and $b = g^t + d$, where $c, d \in N(R_i)$. Therefore $ab = ba$, i.e., R_i is commutative.

For any $x, y \in R$, we have $\varphi_i(xy - yx) = 0$, by the preceding argument. Hence, $xy - yx \in \bigcap_i Ker(\varphi_i) = 0$, and so $xy = yx$. Therefore R is commutative. □

For any generalized n-like ring with odd number n, the situation is not the same. But we have

Proposition 10.2.10. *Every generalized 3-like ring is a commutative periodic ring.*

Proof. Let R be a generalized 3-like ring. In view of Lemma 10.2.8, we may assume that R is local. It will suffice to show that if $a, b \in U(R)$ then $ab = ba$. By Corollary 10.2.5, $(a - a^3)(b - b^3) = 0$, and so $(1-a^2)(1-b^2) = 0$. Hence, $1 - b^2 - a^2 + a^2b^2 = 0$. Likewise, $1 - a^2 - b^2 + b^2a^2 = 0$. We infer that $a^2b^2 = b^2a^2$. Therefore $(ab)^3 = a^3b^3 = a(a^2b^2)b = a(b^2a^2)b = (ab)(ba)(ab)$. As $a, b \in U(R)$, we get $ab = ba$, as required. □

Example 10.2.11. Let $R = \left\{ \begin{pmatrix} x & y & z \\ 0 & x^2 & 0 \\ 0 & 0 & x \end{pmatrix} \;\middle|\; x, y, z \in GF(4) \right\}$. It is easy to check that for each $a \in R$, $a^7 = a$ or $a^7 = a^2 = 0$. Therefore R is a generalized 7-like ring. In this case, R is a noncommutative periodic ring.

Let $n \geq 2$ be a fixed integer. Following Yaqub, a ring R is said to be n-like provided that R is a generalized n-like ring with $n \cdot 1_R = 0$. For instance, $\mathbb{Z}_3 \times \mathbb{Z}_5$ is 45-like.

Theorem 10.2.12. *A ring R is n-like if and only if*

(1) *R is commutative;*
(2) *$n \cdot 1_R = 0$;*
(3) *$\alpha\beta = 0$ for all nilpotent $\alpha, \beta \in R$;*
(4) *For any $a \in R$, there exists some $p \in R$ such that $a - p \in N(R)$ and $p^n = p$.*

Proof. Suppose R is n-like. Then $n \cdot 1_R = 0$. (3) follows from Theorem 10.2.3. Let $a \in R$. Then $(a-a^n)^2 = 0$, i.e., $a^{2n} = 2a^{n+1} - a^2$. Further, $a^{2n}a^{n-1} = 2a^{2n} - a^{n+1}$, and so $a^{3n-1} = 3a^{n+1} - 2a^2$. Multiplying by a^{n-1}, we have $a^{4n-2} = 4a^{n+1} - 3a^2$. By iteration of this process, we get $a^{nn-(n-2)} = na^{n+1} - (n-1)a^2$. As $n \cdot 1_R = 0$, we see that $a^{n^2-n+2} = a^2$, and thus $(a^n)^n = a^{n^2-n+2}a^{n-2} = a^2a^{n-2} = a^n$. Choose $p = a^n$. Then $a - p \in N(R)$ and $p^n = p$.

In view of Lemma 10.2.8, R is a subdirect product of local rings R_i. Let $c \in N(R_i)$ and $d \in R_i$. By Lemma 10.2.7, $[c, d] = n[c, d] = 0$ as $n \cdot 1_R = 0$. Thus, $c \in R_i$ is central. Let $a, b \in R_i \setminus N(R_i)$. As in the proof of Theorem 10.2.9, R_i is commutative. Further, R is commutative.

Conversely, we let $a, b \in R$. Then $a - a^n, b - b^n \in N(R)$, by (1) and (4). Hence, it follows from (3) that $(a-a^n)(b-b^n) = 0$, and so $ab - ab^n - a^nb + a^nb^n = 0$. Since R is commutative, we get $(ab)^n - ab^n - a^nb + ab = 0$, as required. □

Corollary 10.2.13. *A ring R is 2-like if and only if*

(1) *R is commutative;*
(2) *$char(R) = 2$;*
(3) *R is Boolean-like.*

Proof. Suppose R is 2-like. Then (1) and (2) hold by Theorem 10.2.12. In addition, R is a generalized 2-like ring. By Corollary 10.2.6, R is Boolean-like.

Conversely, assume that (1), (2) and (3) hold. For any $a \in R$, $a - a^2 \in N(R)$. In light of Theorem 5.1.1, R is strongly nil-clean. Thus, for any $a \in R$, there exists $p \in R$ such that $a - p \in N(R)$ and $p^2 = p$. In light of Theorem 5.2.16, the product of any two nilpotents is zero. Therefore R is 2-like by Theorem 10.2.12. □

Corollary 10.2.14. *A ring R is 2-like if and only if*

(1) *R is commutative;*

(2) $char(R) = 2$;
(3) $R/J(R)$ *is Boolean and* $J^2(R) = 0$.

Proof. Suppose R is 2-like. Then (1) and (2) hold by Corollary 10.2.13. Further, R is Boolean-like. In view of Theorem 5.2.16, R is strongly nil clean. It follows from Theorem 5.1.5 that $R/J(R)$ is Boolean and $J(R)$ is nil. Thus, $J^2(R) = 0$ from Theorem 10.2.12.

Conversely, assume that (1), (2) and (3) hold. Then for any $a, b \in R$, $a - a^2, b - b^2 \in J(R)$, and so $(a - a^2)(b - b^2) = 0$. Therefore we conclude that R is Boolean-like, hence the result by Corollary 10.2.13. □

Proposition 10.2.15. *Let R be 2-like, and let S be a multiplicatively closed subset of R. Then $S^{-1}R$ is 2-like.*

Proof. Clearly, $char\big(S^{-1}R\big) = 2$. For any $\frac{r_1}{s_1}, \frac{r_2}{s_2} \in S^{-1}R$, we check that

$$\begin{aligned}
&\Big(\tfrac{r_1}{s_1} - (\tfrac{r_1}{s_1})^2\Big)\Big(\tfrac{r_2}{s_2} - (\tfrac{r_2}{s_2})^2\Big)\\
&= \tfrac{r_1 s_1^2 - r_1^2 s_1}{s_1^3} \cdot \tfrac{r_2 s_2^2 - r_2^2 s_2}{s_2^3}\\
&= \tfrac{(r_1 s_1)^2 - r_1 s_1}{s_1^3} \cdot \tfrac{(r_2 s_2)^2 - r_2 s_2}{s_2^3}\\
&= \tfrac{\big((r_1 s_1)^2 - r_1 s_1\big)\big((r_2 s_2)^2 - r_2 s_2\big)}{s_1^3 s_2^3}\\
&= \tfrac{0}{1}.
\end{aligned}$$

Therefore $S^{-1}R$ is 2-like, by Corollary 10.2.13. □

10.3 Uniquely π-Clean Rings

A ring R is uniquely π-clean provided that for any $a \in R$ there exists some $n \in \mathbb{N}$ such that $a^n \in R$ is uniquely clean. Our main concern in this section is to find necessary and sufficient conditions under which a ring R is uniquely π-clean in terms of its potent property.

Lemma 10.3.1. *Every uniquely π-clean ring is an abelian exchange ring.*

Proof. Let R be uniquely π-clean, let $e \in R$ be an idempotent, and let $r \in R$. Choose $a = 1 - \big(e + er(1 - e)\big)$. Then there exists some $n \in \mathbb{N}$ such that $a^n \in R$ is uniquely clean. As in the proof of Lemma 4.4.1, we get $e = e + er(1 - e)$, and then $er = ere$. Likewise, $re = ere$. Thus, $er = re$, and therefore R is abelian.

For any $a \in R$, we have $m \in \mathbb{N}$ such that $a^m \in R$ is clean. Write $a^m = f + v$, where $f = f^2, v \in U(R)$. Then $a^m - f^m = v$, and so $a - f \in U(R)$. This implies that R is clean. According to Lemma 7.1.9, R is an exchange ring. □

Since every abelian exchange ring has stable range 1, it follows from Lemma 10.3.1 that the class of uniquely clean rings is a subclass of strongly clean rings having stable range 1. It is easy to verify that $\mathbb{Z}/3\mathbb{Z}$ is not uniquely clean as $2 = 0 + 2 = 1 + 1$, while $\mathbb{Z}/3\mathbb{Z}$ is uniquely π-clean. Let $R = \bigoplus_{p \text{ is prime}} \mathbb{Z}/(p+1)\mathbb{Z}$. Then R is strongly clean, but it is not uniquely π-clean. In fact, we have {uniquely clean rings} $\subsetneq$ {uniquely π-clean rings} $\subsetneq$ {strongly clean rings}.

Theorem 10.3.2. *A ring R is uniquely π-clean if and only if*

(1) *R is abelian;*
(2) *Every idempotent lifts modulo $J(R)$;*
(2) *$R/J(R)$ is uniquely π-clean.*

Proof. $\Longrightarrow$ By virtue of Lemma 10.3.1, R is an abelian exchange ring, and so every idempotent lifts modulo $J(R)$. For any $\overline{a} \in R/J(R)$, we have some $n \in \mathbb{N}$ such that $a^n \in R$ is uniquely clean. Then $a^n = e + u, e = e^2 \in R, u \in U(R)$. Hence, $\overline{a}^n = \overline{e} + \overline{u}$ is clean. Write $\overline{a}^n = \overline{f} + \overline{v}, \overline{f} = \overline{f}^2 \in R/J(R), \overline{v} \in U\big(R/J(R)\big)$. Since idempotents and units lift modulo $J(R)$, we may assume that $f = f^2 \in R, v \in U(R)$. Thus there exists some $r \in J(R)$ such that $a^n = e + u = f + (v + r)$. By the uniqueness, we get $e = f$. Therefore $R/J(R)$ is uniquely π-clean, as required.

$\Longleftarrow$ For any $a \in R$, we have $\overline{a} \in R/J(R)$, and so there exists some $n \in \mathbb{N}$ such that $\overline{a}^n \in R$ is uniquely clean. Thus $a^n = e+u, e = e^2 \in R, u \in U(R)$. That is, a^n is clean. Write $a^n = f+v, f = f^2, v \in U(R)$. Then $\overline{a}^n = \overline{f}+\overline{v}$. By the uniqueness, we get $\overline{e} = \overline{f}$, i.e., $e - f \in J(R)$. This infers that $f(1-e) = (e-f)(e-1) \in J(R)$. As R is abelian, $f(1-e) \in R$ is an idempotent, and then $f(1-e) = 0$. It follows that $f = fe$. Likewise, $e = ef$. Consequently, $e = f$. Therefore R is uniquely π-clean. □

Corollary 10.3.3. *Every corner of a uniquely π-clean ring is uniquely π-clean.*

Proof. Let R be uniquely π-clean, and let $e = e^2 \in R$. In light of Theorem 10.3.2, $e \in R$ central. For any $eae \in eRe$, then $eae + 1 - e \in R$

is uniquely π-clean. So we have some $n \in \mathbb{N}$ such that $(eae + 1 - e)^n \in R$ is uniquely clean. Thus, $(eae + 1 - e)^n = f + u, f = f^2 \in R, u \in U(R)$, and so $(eae)^n = efe + eue$ is clean in eRe. Write $(eae)^n = g + v, g = g^2 \in eRe, v \in U(eRe)$. Then $(eae + 1 - e)^n = (eae)^n + 1 - e = g + (v + 1 - e)$, where $g = g^2 \in R, v + 1 - e \in U(R)$. Accordingly $g = f = efe$, hence the result. □

Theorem 10.3.4. *A ring R is uniquely π-clean if and only if*

(1) *R is abelian;*
(2) *For any $a \in R$ there exists $n \in \mathbb{N}$ and a unique idempotent $e \in a^n R$ such that $1 - e \in (1 - a^n)R$.*

Proof. $\Longrightarrow$ In view of Theorem 10.3.2, R is abelian. For any $a \in R$, there exists some $n \in \mathbb{N}$ such that $a^n \in R$ is uniquely clean. Write $a^n = f + v$, where $f = f^2, v \in U(R)$. Set $g = 1 - f$. Then $g = g^2 \in R$. Obviously, we get

$$\begin{aligned}(a^n - g)v &= \big(f + v - (1 - f)\big)v \\ &= v^2 + fv - v + vf \\ &= a^{2n} - a^n.\end{aligned}$$

Thus $g - a^n \in (a^n - a^{2n})R$, and so $g \in a^n R$ and $1 - g \in (1 - a^n)R$.

If there exists an idempotent $h \in a^n R$ such that $1 - h \in (1 - a^n)R$. Write $h = a^n x, xh = x$. Then $xa^n x = x$. It is easy to verify that $xa^n = x(a^n x)a^n = a^n x(xa^n) = a^n(xa^n)x = a^n x$. Write $1 - h = (1 - a^n)y, y(1 - h) = y$. Likewise, $y(1 - a^n) = (1 - a^n)y$. One directly checks that $\big(a^n - (1 - h)\big)^{-1} = x - y$, i.e., $a^n - (1 - h) \in U(R)$, By the uniqueness, we get $1 - h = f$. Hence, $g = 1 - f = h$, as desired.

$\Longleftarrow$ For any $a \in R$, there exists $n \in \mathbb{N}$ and a unique idempotent $e \in a^n R$ such that $1 - e \in (1 - a^n)R$. As in the preceding discussion, we have $a^n - (1 - e) \in U(R)$. Write $a^n = f + v$, where $f = f^2, v \in U(R)$. Set $g = 1 - f$. Then $g = g^2 \in R$. Similarly, $g \in a^n R$ and $1 - g \in (1 - a^n)R$. By the uniqueness, we get $g = e$, and so $f = 1 - e$, hence the result. □

Corollary 10.3.5. *A ring R is uniquely π-clean if and only if*

(1) *R is abelian;*
(2) *For any $a \in R$ there exists $n \in \mathbb{N}$ and a unique idempotent $e \in Ra^n$ such that $1 - e \in R(1 - a^n)$.*

Proof. Obviously, a ring R is uniquely π-clean if and only if so is the opposite ring R^{op}. Applying Theorem 10.3.4 to R^{op}, we complete the proof. □

Lemma 10.3.6. *Let I be an ideal of an exchange ring R. If $e \in R$ is an idempotent such that $\overline{e} = \overline{xy}$ in R/I, then there exists an idempotent $q \in I$ and an element $r \in eRe$ such that*

$$a = (e-q)x, b = yr, a = aba, b = bab, \overline{e} = \overline{r}, \overline{a} = \overline{ex}, \overline{b} = \overline{ye}.$$

Proof. Since $\overline{e} = \overline{exye}$, $e - exye \in I$. As R is an exchange ring, we have some idempotent $q \in (e-exye)R \subseteq I$ such that $1-q \in \big(1-(e-exye)\big)R$, and so $e - q \in (exye)(eRe)$ is an idempotent. Write $e - q = exyr$ for a $r \in eR(e-q) \subseteq eRe$. Set $a = (e-q)x$ and $b = yr$. Then $\overline{e} = \overline{exyr} = \overline{r}$, $\overline{a} = \overline{(e-q)x} = \overline{ex}, \overline{b} = \overline{yr} = \overline{ye}$. Moreover, $e-q = exyr = (e-q)xyr(e-q)$, and then $a = aba$. Also we have $bab = yr(e-q)xyr = yr(e-q)(exyr) = yr(e-q)(e-q) = yr(e-q) = b$, as required. □

Lemma 10.3.7. *Let I be an ideal of an exchange ring R, and let x, y and z be elements in R/I such that $p = xy, y = yp, 1-p = (1-x)(1-z), (1-z)p = 0$. If $\overline{a} = x$ with $a \in R$, then we can find b and c in R with $\overline{b} = y$ and $\overline{c} = z$ such that*

$$e = ab, b = be, 1-e = (1-a)(1-c), (1-c)e = 0.$$

Proof. Clearly, $(1-p)p = (1-x)(1-z)p = 0$, and so $p = xy \in R/I$ is an idempotent. Thus, $yx \in R/I$ is an idempotent. In view of Lemma 4.4.5, yx can be lifted by an idempotent $p_1 \in R$. Write $\overline{b'} = y$ for a $b' \in R$. Then $\overline{p_1} = \overline{b'a}$. In light of Lemma 10.3.6, there exists an idempotent $q_1 \in I$ and an element $r_1 \in p_1Rp_1$ such that $a_1 = (p_1 - q_1)b', b_1 = ar_1, a_1 = a_1b_1a_1, b_1 = b_1a_1b_1, \overline{p_1} = \overline{r_1}, \overline{a_1} = \overline{p_1b'}$ and $\overline{b_1} = \overline{ap_1}$. Thus, $e_1 = b_1a_1 \in R$ is an idempotent, and that $\overline{e_1} = \overline{b_1a_1} = \overline{ap_1b'} = \overline{ab'ab'} = \overline{ab'} = xy = p$.

Write $\overline{c'} = z$ for a $c' \in R$. Set $z_1 = 1-(1-c')(1-e_1)$. Then $\overline{z_1} = 1-(1-z)(1-p) = z = \overline{c'}$. Clearly, $(1-z)(1-x) \in R/I$ can be lifted by an idempotent $p_2 \in R$. Hence, $\overline{p_2} = \overline{(1-z_1)(1-a)}$. By using Lemma 10.3.6 again, there exists an idempotent $q_2 \in I$ and an element $r_2 \in p_2Rp_2$ such that $a_2 = (p_2-q_2)(1-z_1), b_2 = (1-a)r_2, a_2 = a_2b_2a_2, b_2 = b_2a_2b_2, \overline{p_2} = \overline{r_2}, \overline{a_2} = \overline{p_2(1-z_1)}$ and $\overline{b_2} = \overline{(1-a)p_2}$.

Thus, $f = b_2a_2 \in R$ is an idempotent, and that

$$\overline{f} = \overline{(1-a)p_2(1-c')} = (1-x)(1-z)(1-x)(1-z) = 1-p = 1-\overline{e_1}.$$

As $f = f(1-e_1)$ by the construction of z_1, we see that $q = (1-e_1)(1-f) \in I$ is an idempotent.

As R is an exchange ring, so is qRq by Lemma 7.2.1. Thus, we can find some $t \in qaq(qRq)$ such that $q - t \in (q-qaq)(qRq)$. Write $t = qaqkq$ and $q - t = q(1-a)qlq$. Set $s = aqkt$. Then $s^2 = s \in I$. Set $e_2 = e_1 + (1-e_1)s$.

Then $e_2 \in R$ is an idempotent, as

$$\begin{aligned}(1-e_1)se_1 = (1-e_1)aqkte_1 &= (1-e_1)aqkqaqkqe_1\\ &= (1-e_1)aqkqaqk(1-e_1)(1-f)e_1\\ &= (1-e_1)aqkqaqk(1-e_1)e_1\\ &= 0.\end{aligned}$$

Moreover, $\overline{e_2} = \overline{e_1} = p$. Let $y_2 = r_1a_1(1-s) + qkt$. Then $\overline{y_2} = \overline{r_1a_1} = yxy = yp = y$. It is easy to check that

$$ay_2 = ar_1a_1(1-s)+s = b_1a_1(1-s)+s = e_1(1-s)+s = e_1+(1-e_1)s = e_2.$$

Let $A = e_2R + (1-a)R$. Then $e_2e_1, e_2(1-e_1) \in A$. Thus,

$$\begin{aligned}e_2e_1 &= e_1 + (1-e_1)se_1\\ &= e_1 + (1-e_1)aqkqaqkqe_1\\ &= e_1 + (1-e_1)aqkqaqk(1-e_1)(1-f)e_1\\ &= e_1 + (1-e_1)aqkqaqk(1-e_1)e_1\\ &= e_1 \in A,\end{aligned}$$

and so $e_2(1-e_1) = e_2 - e_1 = (1-e_1)s = s - e_1s \in A$. Thus $s \in A$. Clearly, $f = (1-a)r_2a_2 \in A$, and so $(1-e_1)f = f - e_1f \in A$. Hence, $1-q = e_1+(1-e_1)f \in A$. Now $s \in A$ and $t = qs$, we get $t = s-(1-q)s \in A$. Since $q - t = q(1-a)qlq = (1-a)qlq - (1-q)(1-a)qlq \in A$, we see that $q = (q-t) + t \in A$. Write $q = e_2u + (1-a)v$ for some $u, v \in Rq$. Then $u, v \in I$ as $q \in I$. Put $z_2 = 1 - (r_2a_2 + v)$ and $e_3 = e_1(1-f) + u$. Then $\overline{1-z_2} = \overline{r_2a_2} = \overline{p_2}\overline{(p_2-q_2)(1-z_1)} = (1-z)(1-x)(1-z) = (1-z)(1-p) = 1-z$. In addition, $\overline{e_3} = \overline{e_1}(1-\overline{f}) = p$. Moreover, we get

$$\begin{aligned}(1-a)(1-z_2) + e_2e_3 &= b_2a_2 + (1-a)v + e_2e_1(1-f) + e_2u\\ &= f + q + e_1(1-f)\\ &= f + e_1(1-f) + (1-e_1)(1-f)\\ &= 1.\end{aligned}$$

Set

$$e = e_2 + e_2e_3(1-e_2), b = y_2e, c = 1-(1-z_2)(1-e_2)(1-e).$$

Then $\overline{e} = \overline{e_2 + e_2e_3(1-e_2)} = p + pp(1-p) = p, \overline{b} = \overline{y_2e} = yp = y$ and $\overline{c} = 1-(1-z)(1-p)(1-p) = z$. Therefore, we get $ab = ay_2e = e_2e = e, be = y_2ee = b, (1-c)e = (1-z_2)(1-e_2)(1-e)e = 0$, and

$$\begin{aligned}(1-a)(1-c) &= (1-a)(1-z_2)(1-e_2)(1-e)\\ &= (1-e_2e_3)(1-e_2)(1-e)\\ &= \big(1-e_2-e_2e_3(1-e_2)\big)(1-e)\\ &= 1-e,\end{aligned}$$

and the result follows. □

Theorem 10.3.8. *Every homomorphic image of a uniquely π-clean ring is uniquely π-clean.*

Proof. Let I be an ideal of a uniquely π-clean ring R, and let $\varphi : R \to R/I$. In light of Lemma 10.3.1, R is an abelian exchange ring. By virtue of Lemma 4.4.5, R/I is also an abelian exchange ring. For any $\overline{a} \in R$, it follows from Theorem 10.3.4 that there are $n \in \mathbb{N}$ and a unique idempotent $e \in a^n R$ such that $1 - e \in (1 - a^n)R$. Hence, $\overline{e} \in \overline{a}^n \overline{R}$ such that $1 - \overline{e} \in (1 - \overline{a}^n)\overline{R}$.

Suppose that there exists an idempotent $\overline{f} \in \overline{a}^n \overline{R}$ such that $1 - \overline{f} \in (1 - \overline{a}^n)R$. Write $\overline{f} = \overline{a^n}\overline{y}, \overline{y} = \overline{y}\overline{f}, 1 - \overline{f} = (1 - \overline{a^n})(1 - \overline{z}), (1 - \overline{z})\overline{f} = 0$. In view of Lemma 10.3.7, we have some b, c in R with $\varphi(b) = \overline{y}$ and $\varphi(c) = \overline{z}$, such that $e = a^n b$, $b = be, 1 - e = (1 - a^n)(1 - c), (1 - c)e = 0$. Thus, $e^2 = a^n be = e \in R$. Further, $e \in a^n R$ and $1 - e \in (1 - a^n)R$. By the uniqueness, we get $e = f$. Hence, $\overline{e} = \overline{f}$. According to Theorem 10.3.4, R/I is uniquely π-clean, as required. □

Let $\mathbb{Z}_9 = \{\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{8}\}$. Then $\mathbb{Z}_9$ is a local ring with $J(\mathbb{Z}_9) = \{\overline{0}, \overline{3}, \overline{6}\}$. Since $\mathbb{Z}_9/J(\mathbb{Z}_9) \cong \mathbb{Z}_3$ is potent, $\mathbb{Z}_9$ is uniquely π-clean Theorem 10.3.8. But $\overline{8} = \overline{0} + \overline{8} = \overline{1} + \overline{7}$. Therefore $\mathbb{Z}_9$ is not uniquely clean.

Lemma 10.3.9. *Every abelian exchange primitive ring is local.*

Proof. Let R be an abelian exchange primitive ring. If $x + y = 1$ in R, there exists an idempotent $e \in R$ such that $e \in xR$ and $1 - e \in yR$. As R is prime, it follows from $eR(1 - e) = 0$ that $e = 0$ or $1 - e = 0$. This shows that x or y is right invertible. As R is abelian, every right invertible element in R is invertible. Therefore either x or $1 - x$ is invertible, hence the result. □

We are now ready to prove the following.

Theorem 10.3.10. *A ring R is uniquely π-clean if and only if*

(1) *R is abelian;*
(2) *Every idempotent lifts modulo $J(R)$;*
(2) *$R/J(R)$ is potent.*

Proof. $\Longrightarrow$ (1) and (2) are obvious. In view of Theorem 10.3.8, $R/J(R)$ is uniquely π-clean. Clearly, $R/J(R)$ is isomorphic to a subdirect product of some primitive rings R_i. That is, there exist epimorphisms $\varphi_i : R/J(R) \to R_i$ such that $\bigcap Ker(\varphi_i) = 0$. By using Theorem 10.3.8 again, R_i is uniquely

π-clean. In light of Lemma 10.3.9, R_i is local. Let $a \in R$. Then there exists $n \in \mathbb{N}$ such that $a^n \in R$ is uniquely clean. By checking the proof of Theorem 10.3.8, we see that $\overline{a}^n \in R_i$ is uniquely clean.

If $\overline{a}, \overline{1-a^n}$ are both invertible, then $\overline{a^n}$ is not uniquely clean in R_i, a contradiction. Hence $\overline{a(1-a^n)} \in J(R_i)$. This shows that $\overline{a} = \overline{a}^{n+1}$, and so $a - a^{n+1} \in Ker(\varphi_i)$. Therefore $a - a^{n+1} \in \bigcap Ker(\varphi_i) = 0$, whence $a = a^{n+1}$ in $R/J(R)$. Thus, $R/J(R)$ is potent.

$\Longleftarrow$ By hypothesis, $S := R/J(R)$ is potent. Let $a \in S$. Then $a^m = a$ for some $m \geq 2$. Thus, $a^{m-1} \in S$ is an idempotent. Hence, $a^{m-1} = \big(1 - a^{m-1}\big) + \big(2a^{m-1} - 1\big)$, where $1 - a^{m-1} \in S$ is an idempotent and $2a^{m-1} - 1 = \big(2a^{m-1} - 1\big)^{-1} \in U(S)$. If there exists an idempotent $f \in S$ and a unit $u \in R$ such that $a^{m-1} = f + u$. As R is abelian, it is easy to verify that $\big(a^{m-1} + f - 1\big)\big(a^{m-1} - f\big)^2 = 0$, and so $f = 1 - a^{m-1}$; hence, $R/J(R)$ is uniquely π-clean. Therefore we complete the proof by Theorem 10.3.2. □

Corollary 10.3.11. *Every abelian periodic ring is strongly π-clean.*

Proof. This is obvious by Theorem 5.3.2, Lemma 5.1.11 and Theorem 10.3.10. □

Corollary 10.3.12. *A ring R is uniquely clean if and only if*

(1) *R is uniquely π-clean;*
(2) $J(R) = \{x \in R \mid x - 1 \in U(R)\}$.

Proof. One direction is obvious from Theorem 4.4.9.

Conversely, assume that (1) and (2) hold. In view of Theorem 10.3.10, $R/J(R)$ is potent. It follows from $J(R) = \{x \in R \mid x - 1 \in U(R)\}$ that $U\big(R/J(R)\big) = \{\overline{1}\}$. By virtue of Corollary 10.1.3, $R/J(R)$ is Boolean. Therefore the result follows by Theorem 10.3.2 and Theorem 4.4.6. □

Theorem 10.3.13. *A ring R is uniquely π-clean if and only if*

(1) *For any $a \in R$, there are $m \in \mathbb{N}$ and a unique $e \in R$ such that $a^m - e \in J(R)$;*
(2) $J(R) = \{x \in R \mid x^m - 1 \in U(R)$ *for all* $m \in \mathbb{N}\}$.

Proof. Suppose that R is uniquely π-clean. Let $a \in R$. In view of Theorem 10.3.10, there are $m \in \mathbb{N}$ and $g \in B(R)$ such that $a^m - g \in J(R)$. If there exists an idempotent $f \in R$ such that $a^m - f \in J(R)$, then

$g-f=(a^m-f)-(a^m-g)\in J(R)$. Clearly, $(g-f)^3=g-f$, and so $(g-f)\big(1-(g-f)^2\big)=0$. Thus, $g=f$, i.e., the uniqueness is verified.

Clearly, $J(R)\subseteq\{x\in R\mid x^m-1\in U(R) \text{ for all } m\in\mathbb{N}\}$. If $x\notin J(R)$, then $0\neq xR\nsubseteq J(R)$. In view of Lemma 4.4.2, there exists an idempotent $0\neq e\in xR$. Write $e=xr$ for $r\in R$. Choose $a=exe+1-e$. Then we can find some $m\in\mathbb{N}$ such that $a^m\in R$ is uniquely clean. In addition, R is abelian by Lemma 10.3.1. Obviously, $a^m=0+\big(ex^me+1-e\big)=e+\big(e(x^m-1)e+1-e\big)$. If $x^m-1\in U(R)$, then $0=e$, a contradiction. This implies that $x^m-1\notin U(R)$. That is, $x\notin\{x\in R\mid x^m-1\in U(R) \text{ for all } m\in\mathbb{N}\}$. Therefore $\{x\in R\mid x^m-1\in U(R) \text{ for all } m\in\mathbb{N}\}\subseteq J(R)$, as required. □

Conversely, assume that (1) and (2) hold. Let $x\in N(R)$. Then $x^m-1\in U(R)$ for all $m\in\mathbb{N}$. By hypothesis, we get $x\in J(R)$. Therefore, every nilpotent in R is contained in $J(R)$. Let $e\in R$ be an idempotent, and let $r\in R$. Then $e+er(1-e)\in R$ is an idempotent. Hence, there exists a unique $f\in R$ such that $\big(e+er(1-e)\big)-f\in J(R)$. By the preceding discussion, $\big(e+er(1-e)\big)-e=er(1-e)\in J(R)$. The uniqueness forces $e=f$. But $\big(e+er(1-e)\big)-\big(e+er(1-e)\big)\in J(R)$, and so $e+er(1-e)=f=e$. This shows that $er=ere$. Likewise, $re=ere$. That is, $er=re$, and so R is abelian. For any $a\in R$, there are $m\in\mathbb{N}$ and a unique $e\in R$ such that $w:=a^m-e\in J(R)$. Then $a^m=(1-e)+(2e-1+w)$. As $(2e-1)^2=1$, we see that $2e-1+w\in U(R)$. If there exists an idempotent $f\in R$ such that $a^m-f\in U(R)$, then $e-f=(a^m-f)-(a^m-e)\in U(R)$. One easily checks that $(e+f-1)(e-f)^2=0$, and therefore $e+f-1=0$. Thus, $f=1-e$, hence the result. □

Let R be uniquely π-clean. It follows from Theorem 10.3.13 that every nilpotent in R is contained in $J(R)$.

Corollary 10.3.14. *Let R be a local ring. Then the following are equivalent:*

(1) *R is uniquely π-clean.*
(2) *$U(R)=\{x\in R\mid \exists\ m\in\mathbb{N}$ such that $x^m-1\in J(R)\}$.*
(3) *$J(R)=\{x\in R\mid x^m-1\in U(R)$ for all $m\in\mathbb{N}\}$.*

Proof. (1) ⇒ (3) This is clear from Theorem 10.3.13.

(3) ⇒ (2) Obviously, $\{x\in R\mid \exists\ m\in\mathbb{N}$ such that $x^m-1\in J(R)\}\subseteq U(R)$. For any $x\in U(R)$, $x\notin J(R)$. By hypothesis, there exists some $m\in\mathbb{N}$ such that $x^m-1\notin U(R)$. As R is local, $x^m-1\in J(R)$. This

implies that $U(R) \subseteq \{x \in R \mid \exists\ m \in \mathbb{N}$ such that $x^m - 1 \in J(R)\}$, as required.

(2) $\Rightarrow$ (1) For any $x \in R$, we see that either $x \in J(R)$ or $x \in U(R)$. This implies that $\overline{x} = \overline{0}$ or $\overline{x}^m = \overline{1}$ in $R/J(R)$. Thus $R/J(R)$ is potent. In light of Theorem 10.3.10, R is uniquely π-clean. □

Lemma 10.3.15. *Let R be a ring. Then the following are equivalent:*

(1) *R is strongly π-regular.*
(2) *R/P is strongly π-regular for all $P \in Spec(R)$.*

Proof. (1) $\Rightarrow$ (2) This is obvious.

(2) $\Rightarrow$ (1) Suppose that R is not strongly π-regular. Then there exists some $a \in R$ such that $a^m \notin a^{m+1}R$ for any $m \in \mathbb{N}$. Let $\Omega = \{I \lhd R \mid \overline{a}^m \notin \overline{a}^{m+1}\overline{R}$ for any $m \in \mathbb{N}\}$. Then Ω is a non-empty inductive set. By using Zorn's Lemma, there exists an ideal Q of R such that Q is maximal in Ω. If Q is not prime, there exist two ideals K and L such that $K, L \nsubseteq Q$, while $KL \subseteq Q$. By the maximality, we can find some $s, t \in \mathbb{N}$ and $c, d \in R$ such that $a^s - a^{s+t+1}c \in K + Q$ and $a^t - a^{s+t+1}d \in L + Q$. Thus $\big(a^s - a^{s+t+1}c\big)\big(a^t - a^{s+t+1}d\big) \in \big(K + Q\big)\big(L + Q\big) \subseteq Q$. Hence, we have that $\overline{a}^{s+t} \in \overline{a}^{s+t+1}\big(R/Q\big)$. This contradicts to the choice of Q. Therefore $Q \in Spec(R)$, and then R/Q is strongly π-regular, by hypothesis. This gives a contradiction, and we therefore complete the proof. □

Lemma 10.3.16. *Let R be a commutative ring. Then the following are equivalent:*

(1) *R is strongly π-regular.*
(2) *Every prime ideal of R is maximal.*

Proof. (1) $\Rightarrow$ (2) Assume that $P \in Spec(R)$ is not an maximal ideal of R. Then there exists a maximal ideal M such that $P \subsetneq M$. Choose $a \in M$, while $a \notin P$. Write $a^n = a^n x a^n$ for some $n \in \mathbb{N}$. Set $e = a^n x$. Then $e \notin P$; otherwise, $a \in P$, a contradiction. This implies that $1 - e \in P \subseteq M$. But $e = a^n x \in M$, and so $1 = (1 - e) + e \in M$, a contradiction. Therefore P is maximal, as desired.

(2) $\Rightarrow$ (1) By hypothesis, R/P is a field for all $P \in Spec(R)$. In light of Lemma 10.3.15, R is strongly π-regular. □

A ring R is said to be torsion provided that for any $0 \neq a \in R$, there exists some $m \in \mathbb{N}$ such that $a^m = 1$.

Theorem 10.3.17. *A ring R is uniquely π-clean if and only if*

(1) *R is an abelian exchange ring;*
(2) *R/P is torsion for all $P \in J\text{-}spec(R)$.*

Proof. Suppose R is uniquely π-clean. In view of Theorem 10.3.10, R is an abelian exchange ring and $R/J(R)$ is potent. Let $P \in J\text{-}spec(R)$. Then $R/J(R)/P/J(R) \cong R/P$ is prime; hence, $P/J(R)$ is a prime ideal of $R/J(R)$. In view of Lemma 10.2.1, $R/J(R)$ is a commutative π-regular ring. It follows from Lemma 10.3.16 that $P/J(R)$ is a maximal ideal of $R/J(R)$. Hence P is a maximal ideal of R.

In light of Theorem 10.3.8, R/P is uniquely π-clean. Then it is an abelian exchange ring by Lemma 10.3.1. Since P is maximal, R/P is simple. For any $0 \neq x \in R/P$, we have $\overline{R}x\overline{R} = \overline{R}$. As in the proof of [120, Proposition 7.1.12], we can find $e^2 = e \in x\overline{R}$ such that $\overline{R}e\overline{R} = \overline{R}$. Hence, $e = \overline{1}$. This implies that $x \in U(R/P)$. Hence, R/P is a division ring. On the other hand, $R/P \cong R/J(R)/P/J(R)$ is potent. Then we have some $m \in \mathbb{N}$ such that $x^{m+1} = x$, and so $x^m = 1$. This implies that R/P is torsion, as required.

Conversely, assume that (1) and (2) hold. Assume that R is not uniquely π-clean. Set $S = R/J(R)$. By virtue of Corollary 10.3.11 and Theorem 10.3.2, we claim that S is not periodic. In view of Lemma 5.1.10, there exists some $a \in S$ such that $a^m \neq a^{m+1}f(a)$ for any $m \in \mathbb{N}$ and any $f(x) \in \mathbb{Z}[x]$. Let $\Omega = \{I \lhd S \mid \overline{a}^m \neq \overline{a}^{m+1}f(\overline{a})$ in S/I for any $m \in \mathbb{N}$ and any $f(x) \in \mathbb{Z}[x]\}$. Then Ω is an nonempty inductive. By using Zorn's Lemma, there exists an ideal Q of S which is maximal in Ω. If Q is not prime, then there exist two ideals K and L of S such that $K, L \nsubseteq Q$ and $KL \subseteq Q$. By the maximality of Q, we can find some $s, t \in \mathbb{N}$ and some $f(x), g(x) \in \mathbb{Z}[x]$ such that $\overline{a}^s = \overline{a}^{s+1}f(\overline{a})$ in $R/\big(K+Q\big)$ and $\overline{a}^t = \overline{a}^{t+1}g(\overline{a})$ in $R/\big(L+Q\big)$. Thus, $a^s - a^{s+1}f(a) \in K + Q$ and $a^t - a^{t+1}g(a) \in L + Q$, and so $\big(a^s - a^{s+1}f(a)\big)\big(a^t - a^{t+1}g(a)\big) \in (K+Q)(L+Q) \subseteq KL + Q \subseteq Q$. In S/Q, we have $\overline{a}^{s+t} = \overline{a}^{s+t+1}h(\overline{a})$ for some $h(x) \in \mathbb{Z}[x]$. This contradicts the choice of Q. Therefore $Q \in Spec(S)$. Write $Q = M.J(R)$. Then $K \in J\text{-}spec(R)$. By hypothesis, $S/Q \cong R/K$ is torsion, and so it is periodic, which is imposable. Therefore R is uniquely π-clean. □

Theorem 10.3.18. *The ideal-extension $I(R;S)$ is uniquely π-clean and S is idempotent-free if and only if the following conditions are satisfied:*

(1) *R is uniquely π-clean;*
(2) *If $e = e^2 \in R$, then $es = se$ for all $s \in S$;*

(3) *If $s \in S$, then there exists some $s' \in S$ such that $ss' = s's$ and $s + s' + ss' = 0$.*

Proof. Assume that (1) – (3) hold. Then S is idempotent-free by (3). Let $(a, s) \in I(R; S)$. Then $a \in R$ is uniquely π-clean. Thus, we have some $n \in \mathbb{N}$ such that $a^n \in R$ is uniquely clean. Write $a^n = e+u, e = e^2 \in R, u \in U(R)$. Hence, $(a, s)^n = (a^n, x) = (e, 0) + (u, x)$ for some $x \in S$. Clearly, $(e, 0)^2 = (e, 0)$. As in the proof of Proposition 4.4.22, $(u, x)^{-1} = (u^{-1}, zu^{-1})$ for $z \in S$. Write $(a, s)^n = (f, y) + (v, w), (f, y)^2 = (f, y)$ and $(v, w) \in U\big(I(R; S)\big)$. Then $f = f^2 \in R, y = 0$ and $v \in U(R)$. Further, $a^n = f + v$ and $x = w$. This implies that $f = e, v = u$, and so $(f, y) = (e, 0), (v, w) = (u, x)$. As a result, $(a, s) \in I(R; S)$ is uniquely π-clean, as desired.

Assume that $I(R; S)$ is uniquely π-clean and S is idempotent-free. Then R is uniquely π-clean. Let $e = e^2 \in R$ and $s \in S$. Then $(e, 0) = (e, 0)^2 \in I(R; S)$ is central. Hence, $(e, 0)(0, s) = (0, s)(e, 0)$, and so $es = se$. For any $s \in S$, there exists some $n \in \mathbb{N}$ such that $(1, s)^n \in I(R; S)$ is uniquely clean. Write $(1, s)^n = (1, x) = (f, y) + (u, v)$ where $x \in S, (f, y) \in I(R; S)$ is an idempotent and $(u, v) \in I(R; S)$ is a unit. Clearly, $f = 0$, and so $y = 0$. This implies that $x = y + v = v$; hence, $(1, x) \in I(R; S)$ is a unit. Further, $(1, s) \in I(R; S)$ is a unit. Write $(1, s)^{-1} = (1, s')$ for a $s' \in S$. Then $ss' = s's$ and $s + s' + ss' = 0$, as asserted. □

Example 10.3.19. Let R be uniquely π-clean. Then $S = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn}\}$ is uniquely π-clean.

Example 10.3.20. Let R be a ring, and let $\alpha : R \to R$ be a ring endomorphism. Then $R[[x, \alpha]]$ is uniquely π-clean if and only if R is uniquely π-clean and $\alpha(e) = e$ for all $e = e^2 \in R$.

Let $R = \mathbb{Z}_4 \oplus \mathbb{Z}_4$ and $\alpha : R \to R$ given by $\alpha(a, b) = (b, a)$. Then $R[[x, \alpha]]$ is not uniquely π-clean even though R is uniquely clean.

Example 10.3.21. Let R be a ring. Then the following are equivalent:

(1) R is uniquely π-clean.
(2) $R[[x]]/(x^2)$ is uniquely π-clean.
(3) $R[[x]]/(x^n)$ is uniquely π-clean for some $n \geq 2$.
(4) $R[[x]]/(x^n)$ is uniquely π-clean for all $n \geq 2$.

Finally, we close this section by the relations between abelian periodic rings and unique π-cleanness.

Theorem 10.3.22. *A ring R is an abelian periodic ring if and only if*

(1) *$J(R)$ is nil;*
(2) *R is uniquely π-clean.*

Proof. $\Longrightarrow$ This is obvious, by Corollary 10.3.11.

$\Longleftarrow$ In light of Theorem 10.3.10, R is abelian and $R/J(R)$ is potent. Let $a \in R$. Then $a - a^n \in J(R)$ for some $n \geq 2$. Hence, $a - a^n \in N(R)$. This completes the proof, by Lemma 5.1.11. □

As a consequence, we deduce that every commutative periodic ring is uniquely π-clean.

Corollary 10.3.23. *Every generalized n-like ring is uniquely π-clean.*

Proof. It is immediate from Theorem 10.2.3 and Theorem 10.3.22. □

10.4 Weakly Periodic Rings

A ring R is said to be weakly periodic in case for any $a \in R$, there exists a potent $p \in R$ such that $a - p \in N(R)$. As is well known, every periodic ring is weakly periodic; whether the converse holds is an open question. The goal of this section is to investigate when a weakly periodic ring is periodic.

Lemma 10.4.1. *Let R be weakly periodic. Then $J(R)$ is nil.*

Proof. Let $x \in J(R)$. Then we can find a potent $p \in R$ such that $w := x - p \in N(R)$. Write $w^n = 0$ for some $n \in \mathbb{N}$. Then $(x-p)^n = 0$, and then $p^n \in J(R)$. Write $p = p^m (m \geq 2)$. Then $p = p^{m^n} = p^n p^{m^n - n} \in J(R)$. It follows from $p\big(1 - p^{m-1}\big) = 0$ that $p = 0$, and so $x = w \in N(R)$. Accordingly, $J(R)$ is nil. □

For any $x, y \in R$, $[x, y]_1 = [x, y]$, and $[x, y]_k = \big[[x, y]_{k-1}, y\big]$ by induction. We are now ready to prove the following which is due to Yaqub (see [553, Theorem 2.4]).

Theorem 10.4.2. *Let R be weakly periodic. If for any $x, y \in R \backslash N(R)$ there exists some $k \in \mathbb{N}$ such that $[xy, yx]_k \in R$ is potent, then R is periodic.*

Proof. Obviously, $R/J(R)$ is isomorphic to a subdirect product of some primitive rings R_i.

Case 1. There exists a subring S_i of R_i which admits an epimorphism $\phi_i : S_i \to M_2(D_i)$ where D_i is a division ring. Since the hypothesis is inherited by all subrings and all homomorphic images of R, we claim that, for any $x, y \in M_2(D_i)\backslash N\big(M_2(D_i)\big)$, there exists some $k \in \mathbb{N}$ such that $[xy, yx]_k \in M_2(D_i)$ is potent. Choose $x = E_{11}, y = E_{11} + E_{12} \in M_2(D_i)$. Then $x, y \in M_2(D_i)\backslash N\big(M_2(D_i)\big)$. Further, $xy = E_{11} + E_{12}$ and $yx = E_{11}$, and so $[xy, yx]_1 = -E_{12}$. Hence, $[xy, yx]_2 = \big[[xy, yx]_1, yx\big] = E_{12}$. By induction, $[xy, yx]_k = (-1)^k E_{12}$ is not potent, a contradiction.

Case 2. $R_i \cong M_{m_i}(D_i)$ for a division ring D_i. By the preceding discussion, $m_i = 1$, and so $R_i \cong D_i$. Clearly, every element in R_i is the sum of a potent and a nilpotent. Hence, R_i is potent, an so R_i is commutative by Lemma 10.2.1. Therefore R_i is commutative. As in the proof of Theorem 5.1.5, $R/J(R)$ is commutative, and so $N(R) \subseteq J(R)$. By Lemma 10.4.1, $J(R) \subseteq N(R)$; hence, $N(R) = J(R)$ is an ideal of R. This implies that $a - a^n \in N(R)$. In light of Lemma 5.1.11, R is periodic. □

Proposition 10.4.3. *Let R be weakly periodic. If $xR \subseteq N(R)$ for any $x \in N(R)$, then R is periodic.*

Proof. Let $x \in N(R)$. Then for any $r \in R$, we have $xr \in N(R)$, and so $1 - xr \in U(R)$. This implies that $x \in J(R)$; hence, $N(R) \subseteq J(R)$. By virtue of Lemma 10.4.1, $N(R) = J(R)$. For any $a \in R$, we can find a potent $p \in R$ such that $w := a - p \in N(R)$. Write $p = p^n (n \geq 2)$. Then $a - a^n = (p + w) - (p + w)^n \in N(R)$. According to Lemma 5.1.11, R is periodic. □

Corollary 10.4.4. *Let R be a weakly periodic ring in which every right zero-divisor is nilpotent. Then R is periodic.*

Proof. Let $x \in N(R)$ and $r \in R$. If $x = 0$, then $xr \in N(R)$. If $x \neq 0$, then $x^n = 0$ and $x^{n-1} \neq 0$ for some $n \in \mathbb{N}$. As $x^{n-1}(xr) = 0$, we see that $xr \in N(R)$. Thus, $xR \subseteq N(R)$ and completes the proof. □

We say that $N(R)$ is commutative in case $xy = yx$ for all $x, y \in N(R)$. Let R be weakly periodic. If $N(R)$ forms an ideal, then R is periodic. Furthermore, we now derive

Theorem 10.4.5. *Let R be weakly periodic. If $N(R)$ is commutative, then R is periodic.*

Proof. Let $e \in E(R)$ and $r \in N(R)$. Write $r^{2^n} = 0$ for some $n \in \mathbb{N}$. Since $\big(er(1-e)\big)^2 = \big((1-e)re\big)^2 = 0$, we see that $er(1-e), (1-e)re \in N(R)$. Hence, $er^2e - (ere)^2 = er(1-e)re = e\big(er(1-e)\big)\big(1-e)re\big) = e\big(1-e)re\big)\big(er(1-e)\big) = 0$. Thus, $er^4e = e(r^2)^2e = \big(er^2e\big)^2 = (ere)^4$. Repeating this procedure, $(ere)^{2^n} = er^{2^n}e = 0$, and so $ere \in N(R)$. Consequently, $E(R)N(R) \subseteq N(R)$.

Let $x \in R$ be potent, and let $r \in N(R)$. Then we have some $m \in \mathbb{N}$ such that $x^m \in E(R)$, and suppose that $m \geq 2$. By the previous claim, $x^m r \in N(R)$. Write $\big(x^m r\big)^k = 0 (k \in \mathbb{N})$. Then $\big(x^i r x^{m-i}\big)^{k+1} = x^i r\big(x^m r\big)^k x^{m-i} = 0$; hence, $x^i r x^{m-i} \in N(R)$ for any $0 \leq i \leq m$. If $m = 2n$, then $x^n r x^n \in N(R)$, and so $x^n r \in N(R)$. If $m = 2n+1$, then

$$b := \big(xrx^{2n}\big)\big(x^2 r x^{2n-1}\big) \cdots \big(x^{2n+1} r\big) \in N(R).$$

This shows that $x^m b = \big(x^{m+1} r\big)^m$. As $b \in N(R)$, we get $x^m b \in N(R)$, and then $\big(x^{m+1} r\big)^m \in N(R)$. Hence, $x^{m+1} r \in N(R)$. In any case, we can find some $m' < m$ such that $x^{m'} N(R) \subseteq N(R)$. By iteration of this process, we get $xN(R) \subseteq N(R)$. That is, $xN(R) \subseteq N(R)$ for any potent $x \in R$. Analogously, we deduce that $N(R)x \subseteq N(R)$ for any potent $x \in R$.

For any $a \in R$, there exists a potent $p \in R$ such that $w := a - p \in N(R)$. Write $p = p^k$ for some $k \geq 2$. Then $a - a^k = (p+w) - (p+w)^k \in N(R)$. Accordingly, R is periodic, by Lemma 5.1.11. □

An ideal I of a ring R is weakly periodic provided that for any $x \in I$, there exists a potent $p \in I$ such that $x - p \in N(R)$.

Lemma 10.4.6. *Every ideal of a weakly periodic ring is weakly periodic.*

Proof. Let I be an ideal of a periodic ring R. For any $x \in I$, there exists a potent $p \in R$ such that $w := x - p \in N(R)$. Write $w^n = 0 (n \in \mathbb{N})$. Then $0 = (p-x)^n = p^n + v$ for some $v \in I$. So $p \in I$, as p is potent. Hence, I is weakly periodic, as asserted. □

Lemma 10.4.7. *Let I be a periodic ideal of a ring. Then any nilpotent of R/I is of the form $w + I$ for some nilpotent $w \in R$.*

Proof. Suppose $(x+I)^n = I$ in $R/I (n \in \mathbb{N})$. Then $x^n \in I$. As I is periodic, we have distinct $s, t \in \mathbb{N}$ with $s < t$ such that $x^{sn} = x^{tn}$. Set $w = x - x^{1+n^2 s(t-s)} \in R$. As in the proof of Theorem 10.1.1, $w^{n^2 s(t-s)} = 0$, and so $w \in N(R)$. Clearly, $x - w = x^{1+n^2 s(t-s)} \in I$. Hence, $x + I = w + I$, the result follows. □

Lemma 10.4.8. *Let S be a subring of a ring R. If $[R : S] < \infty$, then there exists an ideal I of R such that $I \subseteq S$ and $[R : I] < \infty$.*

Proof. Since the sum of two ideals is again an ideal, there exists a largest ideal J of R contained in S. Let $S' = S/I$ and $R' = R/I$. Then S is a subring of R', and S' contains no nonzero ideal of R'. As $[R : S] < \infty$, $m := [R' : S'] < \infty$. Assume that $|S'| > m$. Write the abelian group $R'/S' = \{u_1 + S', \ldots, u_m + S'\}$, where $u_1, \ldots, u_m \in R'$. For any $s \in S'$, we define a map $f_s : \{1, \ldots, m\} \to \{1, \ldots, m\}$ given by $u_i s u_j - u_{f_s(i,j)} \in S'$. Then there are only m^{m^2} such maps. If $|S'| > m^{m^2}$, then we can find two distinct $s_1, s_2 \in S$ such that $f_{s_1} = f_{s_2}$. Choose $I = (s_1 - s_2)$, i.e., the principal ideal generated by $s_1 - s_2$ in R'. Clearly, $u_i(s_1 - s_2)u_j \in S'$. For any $r \in R'$, there exists some u_i such that $r - u_i \in S'$. We infer that $0 \neq I \subseteq S'$, a contradiction. Therefore $|S'| < \infty$, i.e., $[S : J] < \infty$. We easily check that $[R : J] < \infty$ as $[R : S] < \infty$. This completes the proof. □

We note that most of previous results also hold for general rings which may be without units. A zero ring is a general ring (without unity) in which the product of any two elements is 0 (the additive identity element).

Lemma 10.4.9. *An infinite prime ring with a nonzero nilpotent contains an infinite zero ring.*

Proof. Let R be an infinite prime ring with a nonzero nilpotent. Say $a^2 = 0$ for a nonzero $a \in R$. Assume that every zero subring of R is finite. Then aRa is finite. Let $\varphi : R \to aRa$ be a group morphism given by $r \mapsto ara$ for any $r \in R$. Set $K = Ker(\varphi)$. Then $[R : K] < \infty$. Further, we have some $b_1, \ldots, b_n \in R$ such that $R = \bigcup_{i=1}^{n} (K + b_i)$. As $aKa = 0$, the subring $aRaK$ is a zero subring, hence is finite. Furthermore, $aRab_i$ is finite for $i = 1, \ldots, n$. Now $aRaR \subseteq aRaK + \sum_{i=1}^{n} aRab_i$, so the subring $aRaR$ is finite. Since R is prime, $aRa \neq 0$. Choose $0 \neq c \in aRa$. Then cR is finite.

Construct a group morphism $\varphi : R \to cR$ given by $r \mapsto cr$ for any $r \in R$. Set $L = Ker(\varphi)$. Then $[R : L] < \infty$. Additionally, L is a subring of R. By Lemma 10.4.8, there exists an ideal I of R such that $I \subseteq L$ and $[R : I] < \infty$. This implies that $I \neq 0$, as R is infinite. But $RcRI = 0$, and then $c = 0$ or $I = 0$, a contradiction. This completes the proof. □

A subdirectly irreducible ring is a ring with a unique, nonzero minimum two-sided ideal. The following conditions are equivalent for a ring

R: (1) R is subdirectly irreducible; (2) R cannot be written as a subdirect product of two or more rings; (3) The intersection of all nonzero ideals of R is nonzero. We have at our disposal all the information necessary to prove the result.

Theorem 10.4.10. *Let R be a weakly periodic ring with only finitely many noncentral nilpotents. Then R is periodic.*

Proof. We shall prove this result holds for any general ring which may be without units. Assume that there is a counterexample R. In view of Theorem 10.4.5, $N(R)\backslash C(R) \neq \emptyset$. For $x \in N(R)\backslash C(R)$ and $y \in N(R) \cap C(R)$, $x+y \in N(R)\backslash C(R)$; hence $N(R) \cap C(R)$ is finite, and then $N(R)$ is finite.

We may suppose that R is a counterexample with a minimal number of nilpotents. Let $\Omega = \{I \lhd R \mid I \text{ is a periodic ideal of } R\}$. Then $\Omega \neq \emptyset$. Given any $I_1 \subseteq I_2 \subseteq \cdots$ in Ω, we set $M = \bigcup_{i=1}^{\infty} I_i$. Then M is a periodic ideal of R. By using Zorn's Lemma, we have a periodic ideal Q of R, which is maximal in Ω. Let $S = R/Q$. Then S is weakly periodic. It follows from Lemma 10.4.7 that S has no more nilpotents than R has. Furthermore, S is not periodic, by Lemma 5.1.10. If K/Q is a nonzero periodic ideal of S, then $Q \subsetneq K$ and K is periodic in terms of Lemma 5.1.10. This contradicts to the choice of Q. Hence, S has no nonzero periodic ideal.

Let J be any nonzero ideal of S. Then it must contains nonzero nilpotents; otherwise, J is periodic, a contradiction. If J contains some but not all of nonzero nilpotents of S, the minimality would force J to be periodic. Therefore J contains all nonzero nilpotents of S, and thus S is subdirectly irreducible.

Hence, we have a nonzero minimal ideal I of S. In addition, I is not periodic. Clearly, I contains only finitely many nonzero nilpotents. We infer that I do not contain any infinite zero rings. In light of Lemma 10.4.9, I is finite; hence, it is periodic. This contradicts the choice of I, and the proof is complete. □

Corollary 10.4.11. *Let R be weakly periodic with only finitely many noncentral zero-divisors. Then R is periodic.*

Proof. As every nilpotent is zero-divisor, we complete the proof by Theorem 10.4.10. □

By Theorem 10.4.10, every weakly periodic ring with finitely many nilpotents is periodic. This should be contrast to the following: every

ring with a finite set of non-nilpotents is finite, and then periodic (see [60, Theorem 2.1]).

Notes. Theorem 10.1.5 is a generalization of [402, Theorem 5]. Theorem 10.1.12 is a modification of [65, Theorem 2.2]. Theorem 10.2.3 is due to Tominaga and Yaqub [510]. Theorem 10.2.9 was proved by Moore [422, Theorem 2]. Theorem 10.2.12 is due to Yaqub [551, Theorem 3]. Theorem 10.3.8 and Theorem 10.3.10 were first proved by Lee and Zhou [386, Corollary 3.4 and Theorem 3.1]. Theorem 10.4.5 is due to Bell and Tominaga [62, Corollary]. Theorem 10.4.10 was proved by Bell and Klein [60, Theorem 4.7].

This page intentionally left blank

Chapter 11

Conditions on Zero-divisors

Let R be a ring with an identity. An element $a \in R$ is a left (right) zero-divisor if there exists a nonzero $b \in R$ such that $ab = 0 (ba = 0)$. An element that is a left and right zero-divisor is simply called a zero-divisor. Zero-divisors occur in many classes of rings. For instance, every non-trivial idempotent is a zero-divisor. A zero-divisor that is nonzero is called a nontrivial zero-divisor. A commutative ring without nontrivial zero-divisors is called an integral domain. In the ring of $n \times n$ matrices over some integral domain, the zero-divisors are precisely the matrices with determinant zero. In this chapter we establish a number of results concerning the structures of rings with certain conditions on zero-divisors.

11.1 Zero-divisors, Idempotents and Nilpotents

The aim of this section is to collect some elementary results on zero-divisors which will also be used in our subsequent investigations.

Proposition 11.1.1. *Let R be a ring with nontrivial zero-divisors. Then the following are equivalent:*

(1) *R is Boolean.*
(2) *Every zero-divisor is an idempotent.*

Proof. (1) $\Rightarrow$ (2) This is trivial.

(2) $\Rightarrow$ (1) Let $x \in R$ be a nontrivial zero-divisor. Then $x^2 = x$. Let $a \in R$. We have $\big(xa(1-x)\big)^2 = 0$, and so $xa(1-x) = 0$, i.e., $xa = xax$. Likewise, $ax = xax$. Thus $xa = ax$.

Case 1. $xa = 0$. Then $a \in R$ is a zero-divisor, and so it is an idempotent.

Case 2. $xa \neq 0$. Since $xa(1-x) = 0 = (1-x)xa$, $xa \in R$ is a zero-divisor, and so $xa = (xa)^2$. Hence, $xa(1-a) = 0$. This implies that $1-a \in R$ is a zero-divisor, and then $1-a = (1-a)^2$. Thus, $a = a^2$.

Therefore $a \in R$ is an idempotent, as required. □

Corollary 11.1.2. *Let R be a ring in which every zero-divisor is an idempotent. Then R is Boolean or a domain.*

Proof. This is obvious from Proposition 11.1.1. □

Lemma 11.1.3. *Let R be a ring in which every zero-divisor is potent. Then R is reduced.*

Proof. Let $a \in R$ be such that $a^2 = 0$. Then $a \in R$ is a zero-divisor; hence, it is potent. Write $a = a^n (n \geq 2)$. Then $a = 0$. This completes the proof. □

As every reduced ring is abelian, we see that every ring in which every zero-divisor is potent is abelian.

Theorem 11.1.4. *A ring R is potent if and only if*

(1) *R is periodic;*
(2) *Every zero-divisor is potent.*

Proof. One direction is obvious.

Assume that (1) and (2) hold. In view of Lemma 11.1.3, R is reduced. Therefore we complete the proof, by Corollary 10.1.4. □

Corollary 11.1.5. *Let R be a weakly periodic ring in which every zero-divisor is an idempotent. Then R is Boolean or a field.*

Proof. Let $a \in R$. Then there exists a potent $p \in R$ such that $a - p$ is nilpotent. In view of Lemma 11.1.3, R is reduced; hence that $a = p$. That is, R is potent. By virtue of Corollary 11.1.2, R is Boolean or a domain. If R is a domain, then R is a division ring. In view of Lemma 10.2.1, R is commutative, and therefore it is a field, as asserted. □

Let $P \in Spec(R)$. Then $R\backslash P$ is a multiplicative closed set of R and the corresponding localization is denoted by R_P. We note that R_P is a local ring with the unique maximal ideal is $\{\frac{r}{s} \in R_P \mid r \in P, s \in R\backslash P\}$.

Lemma 11.1.6. *Let P be a prime ideal of a commutative ring R. Then P is a minimal prime ideal if and only if for any $x \in P$ there exists $y \in R\backslash P$ such that xy is nilpotent.*

Proof. Suppose that P is a minimal prime ideal of R. Let K be a prime ideal of R_P. Set $L = \{x \in R \mid \frac{x}{1} \in K\}$. Then L is a prime ideal of R. For any $x \in L$, we see that $\frac{x}{1} \in K$. This implies that $x \in P$; otherwise, $\frac{x}{1} \in U(R_P)$, a contradiction. Therefore $L \subseteq P$, and so $L = P$. It follows that $K = LR_P = PR_P = P_P$; hence that P_P is the only prime ideal of R_P. We infer that P_P is nil. For any $x \in P$, we have $\frac{x}{1} \in P_P$, and then $\left(\frac{x}{1}\right)^n = \frac{0}{1}$. Thus, $x^n y = 0$ for some $y \in R - P$. Accordingly, $(xy)^n = 0$, as required.

Conversely, we shall prove that P is a minimal prime ideal of R. If there exists a prime ideal Q of R such that $Q \subsetneq P$. Choose $x \in P\backslash Q$. By hypothesis, we have $y \in R\backslash P$ such that $(xy)^n = 0$ for some $n \in \mathbb{N}$. As $x^n y^n \in Q$ and $y \notin Q$, we deduce that $x \in Q$, a contradiction, the result follows. □

The following theorem shows that zero-divisors in a commutative ring are abundant.

Theorem 11.1.7. *Let P be a minimal prime ideal of a commutative ring R. Then every element in P is a zero-divisor.*

Proof. Let $x \in P$. In view of Lemma 11.1.6, there exists $y \in R\backslash P$ such that $y^n x^n = 0$ for some $n \in \mathbb{N}$. Clearly, $y^n \neq 0$; otherwise, $y \in P$, a contradiction. Thus, we can find the small integer k such that $y^n x^k = 0 (k \geq 1)$. If $k = 1$, we are done. Otherwise, $(y^n x^{k-1})x = 0$ and $y^n x^{k-1} \neq 0$, as required. □

Corollary 11.1.8. *Let R be a commutative ring in which every zero-divisor is potent. Then $a \in R$ is a zero-divisor if and only if there exists a minimal prime ideal P of R such that $a \in P$.*

Proof. One direction is obvious, by Theorem 11.1.7. Conversely, assume $a \in R$ is a zero-divisor. Suppose that a belongs to no minimal prime ideal. Let P be a minimal prime ideal of R. If $x \in r(a)$, then

$ax = 0 \in P$; hence that $x \in P$. This shows that $r(a) \subseteq P$. Therefore $r(a) \subseteq \bigcap\{I \mid I \text{ is a minimal prime ideal of } R\}$. That is, $r(a) \subseteq P(R)$, the prime radical of R. In view of Lemma 11.1.3, R is reduced, and so $P(R) = 0$. Therefore $r(a) = 0$, and then a is not a zero-divisor. This gives a contradiction, and we are through. □

Next, we are concerned with rings in which every zero-divisor is an idempotent or a nilpotent.

Lemma 11.1.9. *Let R be a periodic ring in which every zero-divisor is an idempotent or a nilpotent. Then R contains entirely idempotents, nilpotents or units.*

Proof. Let $a \in R$ be neither an idempotent nor a nilpotent. Since R is periodic, $a^m = a^{m+n}$ for some distinct $m, n \in \mathbb{N}$. Hence, $a \cdot a^{m-1}(1 - a^n) = 0$. If $a^{m-1}(a - a^n) \neq 0$, then $a \in R$ is a zero-divisor. Hence, $a^{m-1}(1 - a^n) = 0$. By iteration of this process, we get $1 - a^n = 0$, and then $a \in U(R)$, the result follows. □

Lemma 11.1.10. *Let R be a periodic ring in which every zero-divisor is an idempotent or a nilpotent. If $N(R)$ is commutative, then $N(R)$ is an ideal of R.*

Proof. Let $a, b \in R$. Then $a + b \in N(R)$, as $N(R)$ is commutative.

Let $a \in N(R), r \in R$. We shall prove $ar \in N(R)$. In view of Lemma 11.1.9, $r \in N(R)$, r is an idempotent, or $r \in U(R)$.

If $r \in N(R)$, then $ar \in N(R)$.

If $r \in R$ is an idempotent, then $(1 - r)ar \in N(R)$, and so $a(1 - r)ar = (1-r)ara$. Hence, $a^2r - arar = ara - rara$, and so $ra^2r - rarar = 0$. Thus, $ra^2r = (rar)^2$. Write $a^m = 0 (m \geq 1)$. By induction, we get $(rar)^{2^m} = ra^{2^m}r = 0$, and then $rar \in N(R)$. This implies that $ar = rar + (1-r)ar \in N(R)$.

If $r \in U(R)$, we claim that $ar \in N(R)$. If not, $ar \in R$ is an idempotent or a unit.

Case 1. $ar \in R$ is an idempotent. Then $ara = a$. Consider $1 + r \in R$. If $1 + r \in R$ is an idempotent, then $r(1 + r) = 0$, and so $r = -1$. This implies that $ar = -a \in N(R)$, a contradiction. If $1 + r \in N(R)$, then $a(1 + r) = (1 + r)a$, and so $ar = ra$. Hence, $ar \in N(R)$, a contradiction. Thus, $1 + r \in U(R)$. Consider $a(1 + r) \in R$. If $a(1 + r) \in U(R)$, then $a \in U(R)$, a contradiction. If $a(1 + r) \in R$ is an idempotent, then $a^2 + a^2r = 0$,

and so $a^2 = 0$. As $a, r^{-1}ar \in N(R)$, we see that $ar^{-1}ar = r^{-1}ara$, and so $ar^{-1}ara = a^2r^{-1}ar = 0$. This implies that $ar^{-1}a = 0$. It follows that $r^{-1}a = (r^{-1}(ara) = (r^{-1}ar)a = ar^{-1}ar = 0$. This implies that $a = 0$, hence $ar \in N(R)$.

Case 2. $ar \in U(R)$. Then $a \in U(R)$, a contradiction.

Thus, $ar \in N(R)$. Likewise, $ra \in N(R)$. Therefore $N(R)$ forms an ideal of R. □

Lemma 11.1.11. *A ring R is a Boolean ring or a field if and only if*

(1) *R is commutative;*
(2) *Every element in R is an idempotent or a unit.*

Proof. One direction is obvious.

Conversely, assume that (1) and (2) hold. If R is not a field, then we have a nonzero non-unit $x \in R$. Hence, $x \neq 0, 1$. By hypothesis, $x = x^2$. Let $a \in R$. Then xa is an idempotent or a unit.

Case 1. $xa \in R$ is an idempotent. Hence, $xa(1-a) = 0$. If $1-a \in U(R)$, then $xa = 0$, and so $a \in R$ is an idempotent; otherwise, $a \in U(R)$, and then $x = 0$, a contradiction.

Case 2. $xa \in U(R)$. Then $x \in U(R)$, and so $x = 1$, which is absurd.

Thus $a \in R$ is an idempotent. Accordingly, R is Boolean, as desired.

□

Theorem 11.1.12. *Let R be a periodic ring in which every zero-divisor is an idempotent or a nilpotent. If $N(R)$ is commutative, then $R/J(R)$ is a Boolean ring or a field.*

Proof. In view of Lemma 11.1.10, $N(R)$ forms an ideal of R. Hence $J(R) = N(R)$. Thus, every element in $R/J(R)$ is an idempotent or a unit. Let $\overline{a} \in R/J(R)$. If $\overline{a}$ is a unit, then $\overline{a}^m = \overline{1}$ for some $m \in \mathbb{N}$, as R is periodic. Thus, $R/J(R)$ is potent. In light of Lemma 10.2.1, $R/J(R)$ is commutative. According to Lemma 11.1.11, $R/J(R)$ is a Boolean ring or a field, as asserted. □

Proposition 11.1.13. *Let R be a periodic ring in which every zero-divisor can be uniquely written as the sum of an idempotent and a nilpotent. Then $R/J(R)$ is a Boolean ring or a field.*

Proof. Let $e \in R$ be an idempotent, and let $a \in R$. Set $f = e+ea(1-e)$. If $f = 1$, then $ea = eae$. If $f \neq 1$, then $f \in R$ is a zero-divisor, as $f(1-f) = (1-f)f = 0$. One easily checks that $f = e+ea(1-e) = e+ea(1-e)+0$. The

uniqueness forces that $ea(1-e)=0$, and so $ea=eae$. Likewise, $ae=eae$. Therefore $ea=eae=ae$, i.e., R is abelian. In view of Theorem 5.3.2, $N(R)$ forms an ideal. Hence $J(R)=N(R)$. Clearly, every element in $R/J(R)$ is an idempotent or a unit. As in the proof of Theorem 11.1.12, $R/J(R)$ is commutative. According to Lemma 11.1.11, $R/J(R)$ is a Boolean ring or a field. □

Lemma 11.1.14. *Let R be a ring in which every element is a weak idempotent or a nilpotent. Then R is abelian.*

Proof. Let $e\in R$ be an idempotent, and let $x\in R$. Then $1-ex(1-e)\in U(R)$. If $\big(1-ex(1-e)\big)^2=1-ex(1-e)$, then $ex(1-e)=0$, and so $ex=exe$. If $\big(1-ex(1-e)\big)^2=-\big(1-ex(1-e)\big)$, then $ex(1-e)=2$. and so $ex(1-e)=2e(1-e)=0$. Hence, $ex=exe$. If $1-ex(1-e)\in N(R)$, this will be a contradiction. Thus, $ex=exe$. Likewise, $xe=exe$. Therefore $ex=xe$, hence the result. □

Theorem 11.1.15. *Let R be a ring. Then the following are equivalent:*

(1) $R=N(R)\cup Id(R)\cup -Id(R)$.
(2) $R=J(R)\cup Id(R)\cup -Id(R)$
(3) *R is isomorphic to one of the following:*

   (a) $\mathbb{Z}_3$, $\mathbb{Z}_4$,
   (b) *a Boolean ring, or*
   (c) $\mathbb{Z}_3\times B$ *where B is a Boolean.*

Proof. (1) ⇒ (3) In view of Lemma 11.1.14, R is abelian.

Case I. R is indecomposable. Then $R=N(R)\cup\{0,1,-1\}$. Let $a\in N(R)$. Then $1-a\in U(R)$. If $(1-a)^2=1-a$, then $a=0$. If $(1-a)^2=-(1-a)$, then $a=2$. If $1-a\in N(R)$, then we get a contradiction. Therefore $R=\{0,1,-1,2\}$, and so $R\cong\mathbb{Z}_2,\mathbb{Z}_3$ or $\mathbb{Z}_4$.

Case II. R is decomposable. Write $R=S\times T$. For any $t\in T$, $(1,t)\in R$ is not nilpotent. Then $(1,t)\in R$ is a weak idempotent. This implies that $t\in T$ is a weak idempotent. Hence, every element in T is a weak idempotent. Likewise, every element in S is a weak idempotent. In view of Theorem 9.2.23, S and T are isomorphic to one of the following:

(i) $\mathbb{Z}_3$,
(ii) a Boolean ring, or
(iii) $\mathbb{Z}_3\times B$ where B is a Boolean.

One easily checks that $(1,2) \in \mathbb{Z}_3 \times \mathbb{Z}_3$ and $(1,2,0) \in \mathbb{Z}_3 \times \mathbb{Z}_3 \times B$ where B is Boolean are neither a weak idempotent nor a nilpotent. Therefore R is isomorphic to one of $(a) - (c)$.

(3) $\Rightarrow$ (2) This is obvious.

(2) $\Rightarrow$ (1) Let $a \in J(R)$. Then $1 - a \in U(R)$, and so $(1-a)^2 = 1 - a$ or $(1-a)^2 = -(1-a)$. If $(1-a)^2 = 1 - a$, then $a(1-a) = 0$, and so $a = 0$. If $(1-a)^2 = -(1-a)$, then $(1-a)(2-a) = 0$, and so $a = 2$. Thus, $J(R) \subseteq \{0, 2\}$. If $2 \in J(R)$, then $J(R) = \{0, 2\}$. As $4 \in J(R)$, we see that $4 = 0$ or $4 = 2$. In any case, $2 \in N(R)$. This implies that $J(R) \subseteq N(R)$. If $2 \notin J(R)$, then $J(R) = 0 \subseteq N(R)$. This completes the proof. □

11.2 D-Rings and Their Generalizations

We say that a ring R is a D-ring if every zero-divisor in R is nilpotent. For instance, $\mathbb{Z}_{p^k}(p \text{ is prime}, k \in \mathbb{N})$. This concept coincides with that introduced by Nassif for a commutative ring (see [426]).

Lemma 11.2.1. *Every D-ring is connected.*

Proof. Let $e \in R$ be an idempotent. If $e \neq 1$, then $e \in R$ is a zero-divisor as $e(1-e) = (1-e)e = 0$. By hypothesis, $e \in N(R)$, and so $e = 0$. Therefore every idempotent in R is trivial, as required. □

Lemma 11.2.2. *Let R be a strongly π-regular ring. Then the following are equivalent:*

(1) *R is a D-ring.*
(2) *$R = U(R) \cup N(R)$.*

Proof. (1) $\Rightarrow$ (2) Let $a \in R$. Then $a^n = a^{n+1}b$ and $ab = ba$; hence, $a^n(ab - 1) = 0$. If $a \in R$ is not a zero-divisor, then $ab = ba = 1$, and so $a \in U(R)$. Otherwise, $a \in R$ is a zero-divisor. By hypothesis, $a \in N(R)$. Consequently, $R = U(R) \cup N(R)$.

(2) $\Rightarrow$ (1) This is obvious as every zero-divisor is not invertible. □

Theorem 11.2.3. *Let R be a strongly π-regular ring. Then the following are equivalent:*

(1) *R is a D-ring.*
(2) *R is local.*

Proof. (1) $\Rightarrow$ (2) By Lemma 11.2.1, R is abelian. In view of Theorem 5.3.2, $N(R)$ forms an ideal of R. Since R is strongly π-regular, $J(R)$ is nil. Hence, $J(R) = N(R)$. By virtue of Lemma 11.2.2, $R = U(R) \cup N(R)$, and so $R = U(R) \cup J(R)$. Therefore R is local.

(2) $\Rightarrow$ (1) Since R is local, $R = U(R) \cup J(R)$. As R is strongly π-regular, $J(R) \subseteq N(R)$. Hence, $R = U(R) \cup N(R)$, and therefore the result follows by Lemma 11.2.2. □

Corollary 11.2.4. *Every artinian D-ring is local.*

Proof. Since every artinian ring is strongly π-regular, the result follows by Theorem 11.2.3. □

Corollary 11.2.5. *Every finite D-ring is local.*

Proof. Let R be a finite D-ring. Then it is artinian, whence the result by Corollary 11.2.4. □

We now turn to consider commutative D-rings.

Proposition 11.2.6. *Let R be a commutative D-ring. Then $N(R)$ is prime.*

Proof. Suppose that $ab \in N(R)$. If $a \notin N(R)$, then $a \in R$ is not a zero-divisor. Write $(ab)^n = 0$ and $(ab)^{n-1} \neq 0 (n \geq 1)$. Then $b(ab)^{n-1}a = 0$, and so $b(ab)^{n-1} = 0$. This shows that $b \in R$ is a zero-divisor. By hypothesis, $b \in N(R)$. This completes the proof. □

Let $f(x) = 4x^2 + 8x + 4 \in \mathbb{Z}_{12}[x]$. Then it is a zero-divisor, as $f(x)$ is annihilated by a polynomial of the first degree, namely $3x + 6$. But it is obvious that $f(x)$ is in fact annihilated by an element of $\mathbb{Z}_{12}$, e.g., 3. The following preliminary result is contained in [409, Theorem 2].

Lemma 11.2.7. *Let R be a commutative ring. Then $f(x) \in Z\big(R[x]\big)$ if and only if there exists some $0 \neq r \in R$ such that $f(x)r = 0$.*

Proof. One direction is obvious. We now assume that $f(x) \in Z\big(R[x]\big)$. Then $f(x)g(x) = 0$ for some $g(x) \neq 0$. For the convenience, we illustrate a special case. Let $f(x) = a_0x^3 + a_1x^2 + a_2x + a_3$ and $g(x) = b_0x^2 + b_1x + b_2$.

Then

$$\begin{aligned} a_0b_0 &= 0 \\ a_1b_0 + a_0b_1 &= 0 \\ a_2b_0 + a_1b_1 + a_0b_2 &= 0 \\ a_3b_0 + a_2b_1 + a_1b_2 &= 0 \\ a_3b_1 + a_2b_2 &= 0 \\ a_3b_2 &= 0 \end{aligned}$$

It is easy to check that

$$0 = b_0 \begin{vmatrix} a_1 & a_0 & \\ a_2 & a_1 & a_0 \\ a_3 & a_2 & a_1 \end{vmatrix} = b_0 a_1^3.$$

Then there exists an integer $j_1 (0 \leq j_1 \leq 2)$ such that $b_0 a_1^{j_1} \neq 0$ but $b_0 a_1^{j_1+1} = 0$. Further, we have

$$0 = b_0 a_1^{j_1} \begin{vmatrix} a_2 & a_1 & a_0 \\ a_3 & a_2 & a_1 \\ 0 & a_3 & a_2 \end{vmatrix} = b_0 a_1^{j_1} a_2^3.$$

So we can choose an integer $j_2 (0 \leq j_2 \leq 2)$ such that $b_0 a_1^{j_1} a_2^{j_2} \neq 0$ but $b_0 a_1^{j_1} a_2^{j_2+1} = 0$. Finally, we get

$$0 = b_0 a_1^{j_1} a_2^{j_2} \begin{vmatrix} a_3 & a_2 & a_1 \\ 0 & a_3 & a_2 \\ 0 & 0 & a_3 \end{vmatrix} = b_0 a_1^{j_1} a_2^{j_2} a_3^3.$$

So, we can choose an integer $j_3 (0 \leq j_3 \leq 2)$ such that $b_0 a_1^{j_1} a_2^{j_2} a_3^{j_3} \neq 0$ but $b_0 a_1^{j_1} a_2^{j_2} a_3^{j_3+1} = 0$. Choose $r = b_0 a_1^{j_1} a_2^{j_2} a_3^{j_3} \in R$. Then $f(x)r = 0$, as required. □

Theorem 11.2.8. *Let R be a commutative ring. Then the following are equivalent:*

(1) *R is a D-ring.*
(2) *$R[x]$ is a D-ring.*

Proof. (1) ⇒ (2) Let $f(x) \in Z\big(R[x]\big)$. In view of Lemma 11.2.7, we can find $0 \neq r \in R$ such that $f(x)r = 0$. Write $f(x) = \sum_{i=0}^{n} a_i x^{n-i}$ where $a_0, \ldots, a_n \in R$. Then each $a_i r = 0$. This shows that each $a_i \in R$ is a zero-divisor. Hence, all a_i are nilpotent. Therefore $f(x) \in R[x]$ is nilpotent. This means that $R[x]$ is a D-ring.

(2) $\Rightarrow$ (1) Let $a \in Z(R)$. Then $a \in Z\big(R[x]\big)$. By hypothesis, $a \in R[x]$ is nilpotent. Hence, $a \in R$ is nilpotent, as desired. □

Example 11.2.9. Let p be prime and $k \in \mathbb{N}$. Then $\mathbb{Z}_{p^k}[x_1, \ldots, x_n]$ is a D-ring. By induction, we are done from Theorem 11.2.8.

Let R be a commutative ring, and let M be an R-module. Set $Z(M) = \{r \in R \mid rm = 0 \text{ for some } 0 \neq m \in M\}$, i.e., the set of zero-divisor on M.

Lemma 11.2.10. *Let R be a commutative ring, and let M be an R-module. Then $(r, m) \in Z\big(T(R, M)\big)$ if and only if $r \in Z(R) \cup Z(M)$.*

Proof. Let $(r, m) \in Z\big(T(R, M)\big)$. Then we have $(0, 0) \neq (x, s) \in T(R, M)$ such that $(r, m)(x, s) = (0, 0)$. Hence, $rx = 0$ and $rs + mx = 0$. If $x \neq 0$, then $r \in Z(R)$. If $x = 0$ and $s \neq 0$, then $r \in Z(M)$. Thus, $r \in Z(R) \cup Z(M)$.

Let $r \in Z(R) \cup Z(M)$. If $r \in Z(R)$, then $rx = 0$ for some $0 \neq x \in R$. Hence $(r, m)(x, 0) = (0, 0)$. If $r \in Z(M)$, then $rs = 0$ for some $0 \neq s \in M$. This implies that $(r, m)(0, s) = (0, 0)$. Accordingly, $r \in Z\big(T(R, M)\big)$. □

Let R be a commutative ring, and let M be an R-module. Denote $T_1(R, M) = T(R, M)$ and $T_n(R, M) = T\big(R, T_{n-1}(R, M)\big) (n \geq 2)$.

Theorem 11.2.11. *Let R be a commutative ring. Then $T_n(R, M)$ is a D-ring if and only if*

(1) *R is a D-ring;*
(2) *$Z(M) \subseteq N(R)$.*

Proof. $\Longrightarrow$ Let $a \in Z(R)$. Then $\begin{pmatrix} a & \\ & a \end{pmatrix} \in T_n(R, M)$ is a zero-divisor. Hence, $\begin{pmatrix} a & \\ & a \end{pmatrix}$ is nilpotent, and therefore $a \in R$ is nilpotent. This implies that R is a D-ring. Let $r \in Z(M)$. Then $rm = 0$ for some $0 \neq m \in M$. Hence, $\begin{pmatrix} r & 0 \\ & r \end{pmatrix} \in Z\big(T_n(R, M)\big)$. By hypothesis, $r \in N(R)$. Thus, $Z(M) \subseteq N(R)$.

$\Longleftarrow$ By induction and Lemma 11..2.10, we see that $Z\big(T_{n-1}(R, M)\big) = Z(R) \cup Z(M)$. For any $\begin{pmatrix} a & x \\ & a \end{pmatrix} \in Z\big(T_n(R, M)\big)$, it follows by Lemma 11.2.10 that $a \in Z(R) \cup Z\big(T_{n-1}(R, M)\big) = Z(R) \cup Z(M) = Z(R)$. Hence,

$a \in N(R)$, and so $\begin{pmatrix} a & x \\ & a \end{pmatrix} \in N\big(T_n(R,M)\big)$. Therefore $T_n(R,M)$ is a D-ring. □

Corollary 11.2.12. *Let R be a commutative ring. Then $T_n(R,R)$ is a D-ring if and only if so is R.*

Proof. This is obvious by Theorem 11.2.11. □

Theorem 11.2.13. *A ring R is an abelian ring in which every zero-divisor is a weak idempotent or a nilpotent if and only if R is isomorphic to one of the following:*

(a) *a D-ring,*
(b) *a Boolean ring,*
(c) $\mathbb{Z}_3 \times \mathbb{Z}_3$,
(d) $\mathbb{Z}_3 \times B$ *where B is Boolean.*

Proof. $\Longrightarrow$ Case I. R is indecomposable. Then every weak idempotent is $0, 1$ or -1. Hence, every zero-divisor in R is nilpotent, and so R is a D-ring.

Case II. R is decomposable. Write $R = S \times T$. For any $t \in T$, $(0,t)$ is a weak idempotent or a nilpotent. We infer that every element in T is a weak idempotent or a nilpotent. Similarly, every element in S is a weak idempotent or a nilpotent. By virtue of Theorem 11.1.15, S and T are both isomorphic to one of the following:

(a) $\mathbb{Z}_3$, $\mathbb{Z}_4$,
(b) a Boolean ring, or
(c) $\mathbb{Z}_3 \times B$ where B is Boolean.

But one easily checks that $Z(R) \neq Id(R) \cup -Id(R) \cup N(R)$ for any of those types

(1) $\mathbb{Z}_3 \times \mathbb{Z}_4$,
(2) $\mathbb{Z}_3 \times \mathbb{Z}_3 \times B$ where B is a Boolean ring,
(3) $\mathbb{Z}_4 \times \mathbb{Z}_4$,
(4) $\mathbb{Z}_4 \times B$ where B is a Boolean ring, and
(5) $\mathbb{Z}_3 \times \mathbb{Z}_4 \times B$ where B is a Boolean ring.

Therefore R is isomorphic to one of (a) – (d).

$\Longleftarrow$ Clearly, R is abelian. One easily verifies that any of these four types of rings satisfy the desired condition. □

Corollary 11.2.14. *Let R be a ring. Then the following are equivalent:*

(1) *R is an abelian ring in which every zero-divisor is an idempotent or a nilpotent.*
(2) *R is a D-ring or a Boolean ring.*

Proof. (1) $\Rightarrow$ (2) In view of Theorem 11.2.13, R is isomorphic to one of the following:

(a) a D-ring,
(b) a Boolean ring,
(c) $\mathbb{Z}_3 \times \mathbb{Z}_3$,
(d) $\mathbb{Z}_3 \times B$ where B is Boolean.

But in the case $\mathbb{Z}_3 \times \mathbb{Z}_3$, $(0,2) \notin Id(R) \cup N(R)$. In the case $\mathbb{Z}_3 \times B$, $(2,0) \notin Id(R) \cup N(R)$. Therefore proving (2).

(2) $\Rightarrow$ (1) This is obvious. □

The following example shows that the abelian condition in Corollary 11.2.14 is necessary.

Example 11.2.15. Every zero-divisor in $T_2(\mathbb{Z}_2)$ is an idempotent or a nilpotent. But $T_2(\mathbb{Z}_2)$ is neither a Boolean ring nor a D-ring.

11.3 Decompositions of Zero-divisors

The goal of this section is to investigate when every zero-divisor in a ring can be uniquely written as the sum or difference of a nilpotent and an idempotent. A ring R is called uniquely D-nil-clean provided that for every zero-divisor $a \in R$ there exists a unique idempotent $e \in R$ such that $a - e \in N(R)$. Clearly, every uniquely nil-clean ring and every domain are uniquely D-nil-clean. Also we see that every D-ring is uniquely D-nil-clean. We start this section by characterizing uniquely D-nil-clean rings.

Lemma 11.3.1. *Every uniquely D-nil-clean ring is abelian.*

Proof. Let R be a uniquely D-nil-clean ring, and let $e^2 = e, x \in R$. Then $e + ex(1-e) \in R$ is an idempotent. If $e + ex(1-e) = 1$, then $ex = exe$. If $e + ex(1-e) \neq 1$, then $e + ex(1-e) \in R$ is a zero-divisor. Since $e + ex(1-e) = e + ex(1-e) + 0$, we get $ex(1-e) = 0$, and so $ex = exe$. Likewise, $xe = exe$. Thus, $ex = xe$, hence the result. □

Theorem 11.3.2. *A ring R is a uniquely D-nil-clean ring if and only if R is a D-ring or R is uniquely nil-clean.*

Proof. Suppose R is a uniquely D-nil-clean ring. Then R is abelian by Lemma 11.3.1.

Case I. R is indecomposable. Let $a \in R$ be a zero-divisor. Then $a \in R$ is nilpotent or $a \in U(R)$. This shows that every zero-divisor is nilpotent, i.e., R is a D-ring.

Case II. R is decomposable. Write $R = A \times B$. For any $x \in A$, $(x, 0) \in R$ is a zero-divisor. Hence, we can find a unique idempotent $(e, f) \in R$ such that $(x, 0) - (e, f) \in N(R)$. So $x - e \in N(A)$ for an idempotent $e \in A$. If there exists an idempotent $g \in A$ such that $x - g \in N(A)$. Then $(x, 0) - (g, f) \in N(R)$. By the uniqueness, we get $g = e$, and therefore A is uniquely nil-clean. Similarly, B is uniquely nil-clean, and then R is uniquely nil-clean.

Conversely, if R is a D-ring, then R is a uniquely D-nil clean ring. So we assume that R is uniquely nil-clean, and therefore R is a uniquely D-nil-clean ring. □

Corollary 11.3.3. *A ring R is uniquely D-nil-clean if and only if R is a D-ring, or*

(1) *R is abelian;*
(2) *R is periodic;*
(3) *$R/J(R)$ is Boolean.*

Proof. This is obvious by Theorem 11.3.2 and Corollary 5.3.6. □

Corollary 11.3.4. *Let R be uniquely D-nil-clean. Then the ring $T(R, R) = \left\{ \begin{pmatrix} a & b \\ & a \end{pmatrix} \middle| \, a, b \in R \right\}$ is uniquely D-nil clean.*

Proof. By virtue of Theorem 11.3.2, R is a D-ring or R is uniquely nil-clean.

Case I. R is a D-ring. Let $\begin{pmatrix} a & x \\ & a \end{pmatrix} \in Z\big(T(R, R)\big)$. Then we have $0 \neq \begin{pmatrix} b & y \\ & b \end{pmatrix}, \begin{pmatrix} c & z \\ & c \end{pmatrix} \in T(R, R)$ such that

$$\begin{pmatrix} a & x \\ & a \end{pmatrix} \begin{pmatrix} b & y \\ & b \end{pmatrix} = \begin{pmatrix} c & z \\ & c \end{pmatrix} \begin{pmatrix} a & x \\ & a \end{pmatrix} = 0.$$

If $b, c \neq 0$, then $ab = 0 = ca$. If $b = c = 0$, then $y, z \neq 0$ and $ay = 0 = za$. If $b = 0, c \neq 0$ or $b \neq 0, c = 0$, then $ca = 0 = ay$ or $ab = 0 = za$. Hence $a \in R$ is a zero-divisor, and so it is nilpotent. Therefore $\begin{pmatrix} a & x \\ & a \end{pmatrix}$ is nilpotent, and then $T(R, R)$ is a D-ring.

Case II. R is uniquely nil-clean. In light of Example 5.3.13, $T(R, R)$ is uniquely nil-clean.

Therefore $T(R, R)$ is uniquely D-nil-clean, in terms of Theorem 11.3.2. □

A ring R is called uniquely weakly D-nil-clean provided that R is weakly D-nil-clean and every nil-clean zero-divisor is uniquely nil-clean. As in the proof of Lemma 11.3.1, we derive

Lemma 11.3.5. *Every uniquely weakly D-nil-clean ring is abelian.*

Theorem 11.3.6. *Every uniquely weakly D-nil-clean ring is a D-ring or the product of two uniquely weakly nil-clean rings.*

Proof. Let R be a uniquely weakly D-nil-clean ring. In view of Lemma 11.3.5, R is abelian.

Case I. R is indecomposable. Then every zero-divisor is nilpotent or invertible. The later is impossible, and so R is a D-ring.

Case II. R is decomposable. Write $R = A \times B$. Let $a \in A$. Then $(a, 0) \in R$ is a zero-divisor. By hypothesis, there exists an idempotent $(e, f) \in R$ such that $(a, 0) \pm (e, f) \in N(R)$. Thus, $a \pm e \in N(A)$. If there exist idempotents $g, h \in A$ such that $a - g, a - h \in N(A)$. Then $(a, 0) - (g, 0), (a, 0) - (h, 0) \in N(R)$. By the uniqueness, we get $(g, 0) = (h, 0)$, and so $g = h$. Therefore A is uniquely weakly nil-clean. Similarly, B is uniquely weakly nil-clean, as asserted. □

The following observation is an obvious extension of [13, Theorem 2.12].

Lemma 11.3.7. *A ring R is an abelian ring in which every zero-divisor is a weak idempotent if and only if R is isomorphic to one of the following:*

(1) *a domain,*
(2) $\mathbb{Z}_3 \times \mathbb{Z}_3$,
(3) $\mathbb{Z}_3 \times B$ *where B is Boolean, or*
(4) *a Boolean ring.*

Proof. $\Longrightarrow$ If R is a D-ring, then R has no trivial zero-divisor, and so R is a domain. This implication is obtained by Theorem 11.2.13.

$\Longleftarrow$ If R is a domain, then R has no nontrivial zero-divisor. If $R = \mathbb{Z}_3 \times \mathbb{Z}_3$, then $N_D(R) = \{(1,1),(1,2),(2,1)\}$, $Id(R) = \{(0,0),(0,1),(1,0),(1,1)\}$ and $-Id(R) = \{(0,0),(0,2),(2,0),(2,2)\}$. Here, $N_D(R) = R \setminus Z(R)$. Therefore $R = N_D(R) \cup Id(R) \cup -Id(R)$. If $R = \mathbb{Z}_3 \times B$ where B is Boolean, then $Id(R) = \{(0,b),(1,b) \mid b \in B\}$ and $-Id(R) = \{(0,b),(2,b) \mid b \in B\}$. Therefore $R = Id(R) \cup -Id(R)$. If R is a Boolean ring, then every element in R is an idempotent. In any case, every element in R is a weak idempotent, and we are done. □

We come now to the main result of this section.

Theorem 11.3.8. *A ring R is uniquely weakly D-nil-clean if and only if R is a D-ring, or*

(1) *R is abelian;*
(2) *R is periodic;*
(3) *$R/J(R)$ is isomorphic to one of the following:*

  (a) *a field,*
  (b) *$\mathbb{Z}_3 \times \mathbb{Z}_3$,*
  (c) *$\mathbb{Z}_3 \times B$ where B is Boolean, or*
  (d) *a Boolean ring.*

Proof. $\Longrightarrow$ Suppose that R is not a D-ring. In view of Theorem 11.3.6, R is the product of two uniquely weakly nil-clean rings R_1 and R_2. By Corollary 9.4.2, R_1 and R_2 are abelian periodic rings, and then so is R. Thus, $N(R)$ is an ideal of R; hence, $J(R) = N(R)$. As every idempotent lifts modulo $J(R)$, $R/J(R)$ is abelian. Let $\overline{a} \in R/J(R)$ be a zero-divisor. If $a \in R$ is not a zero-divisor, then $a \in U(R)$, and so $\overline{a} \in U\big(R/J(R)\big)$, a contradiction. Thus, $a \in R$ is a zero-divisor. By hypothesis, it is the sum of a weak idempotent and a nilpotent. Hence, $\overline{a}$ is a weak idempotent. That is, every zero-divisor in $R/J(R)$ is a weak idempotent. In light of Lemma 11.3.7, $R/J(R)$ is isomorphic to one of the following:

(i) a domain,
(ii) $\mathbb{Z}_3 \times \mathbb{Z}_3$,
(iii) $\mathbb{Z}_3 \times B$ where B is Boolean, or
(iv) a Boolean ring.

If $R/J(R)$ is a domain, then for any $a \in R/J(R)$, $a = 0$ or $a^m = 1$ for some $m \in \mathbb{N}$. This shows that $R/J(R)$ is a field, as required.

$\Longleftarrow$ Suppose that R is not a D-ring. In view of Theorem 5.3.2, $N(R)$ forms an ideal of R. As R is periodic, $J(R)$ is nilpotent; hence, $J(R) = N(R)$. Let $a \in R$ be a zero-divisor. Since $R/J(R)$ is periodic, $\overline{a}$ is a zero-divisor; otherwise, $\overline{a} \in R/J(R)$ is invertible, and so $a \in R$ is invertible, a contradiction. According to Lemma 11.3.7, $\overline{a}$ is a weak idempotent in $R/J(R)$. Since every idempotent lifts modulo $J(R)$, we see that $a \pm e \in N(R)$ for some idempotent $e \in R$.

Assume that $a - e, a - f \in N(R)$ for some idempotents $e, f \in R$. Since R is abelian, we see that $e - f = (a - f) - (a - e) \in N(R)$. It follows from $(e - f)^3 = e - f$ that $e = f$. Hence, every nil-clean element is uniquely nil-clean. This completes the proof. □

Corollary 11.3.9. *Let R be uniquely weakly D-nil-clean. Then $T(R, R)$ is uniquely weakly D-nil-clean.*

Proof. By virtue of Theorem 11.3.8, R is a D-ring, or R is an abelian periodic ring and $R/J(R)$ is isomorphic to one of the following:

(a) a field,
(b) $\mathbb{Z}_3 \times \mathbb{Z}_3$,
(c) $\mathbb{Z}_3 \times B$ where B is Boolean, or
(d) a Boolean ring.

Case 1. R is a D-ring. As in the proof of Corollary 11.3.4, $T(R, R)$ is a D-ring.

Case 2. R is not a D-ring. Clearly, $T(R, R)$ is an abelian periodic ring. Since $T(R, R)/J\big(T(R, R)\big) \cong R/J(R)$, we see that $T(R, R)/J\big(T(R, R)\big)$ is isomorphic to one of the preceding items. Consequently, $T(R, R)$ is weakly D-nil-clean, in terms of Theorem 11.3.8. □

11.4 Almost Clean Rings

Throughout this section, all rings are commutative with 1. Let $N_D(R) = R \setminus Z(R)$, i.e., $x \in N_D(R)$ if and only if $xy = 0 \Longrightarrow y = 0$ in R. A ring R is an almost clean ring if every element $a \in R$ can be written in the form $a = e + r$ where $e \in Id(R)$ and $r \in N_D(R)$. In this section we establish some elementary properties of almost clean rings. Clearly, we have

Proposition 11.4.1. *A ring R is almost clean if and only if for every zero-divisor $a \in R$ there exists an idempotent $e \in R$ such that $a - e \in N_D(R)$.*

Example 11.4.2. Every weakly D-nil-clean ring is almost clean.

Example 11.4.3.

(1) Every integral domain is almost clean.
(2) Every clean ring is almost clean. But the converse is not true, e.g., $\mathbb{Z}$.
(3) A direct product $R = \prod_i R_i$ of rings R_i is almost clean if and only if each R_i is almost clean.

Proposition 11.4.4. *Let R be a ring, and let $S \subseteq Id(R)$. Then the following are equivalent:*

(1) *For any $a \in R$, there exists $e \in S$ such that $a - e \in N_D(R)$.*
(2) *R is almost clean and $S = Id(R)$.*

Proof. (1) $\Rightarrow$ (2) Clearly, R is almost clean. For any idempotent $e \in R$, we can find some $f \in S$ such that $(1-e)-f \in N_D(R)$. As $\big((1-e)-f\big)(e-f) = 0$, we see that $e = f \in S$. This shows that $Id(R) \subseteq S$, as required.

(2) $\Rightarrow$ (1) This is obvious. □

Lemma 11.4.5. *Let R be a ring. Then $Id(R) = Id\big(R[[X]]\big)$.*

Proof. Obviously, $Id(R) \subseteq Id\big(R[[X]]\big)$. Let $f(x) = \sum_{i=0}^{\infty} a_i x^i \in Id\big(R[[X]]\big)$. Then $e_0^2 = e_0, e_0e_1 + e_1e_0 = e_1, e_0e_2 + e_1e_1 + e_2e_0 = e_2, \ldots$. This shows that $(2e_0 - 1)e_1 = 0$, and so $e_1 = 0$. Hence, $(2e_0 - 1)e_2 = 0$, and so $e_2 = 0$. Likewise, $e_3 = e_4 = \cdots = 0$. Therefore $f(x) = e_0 \in Id(R)$, hence the result. □

Theorem 11.4.6. *Let R be a ring. Then the following are equivalent:*

(1) *R is almost clean.*
(2) *$R[x]$ is almost clean.*
(3) *$R[[x]]$ is almost clean.*

Proof. (1) $\Rightarrow$ (2) Let $f(x) \in R[x]$. Then we have an idempotent $e \in R$ such that $f(0) - e \in N_D(R)$. Write $f(x) - e = \sum_{i=0}^{n} r_i x^i$. If $\big(f(x) - e\big)\left(\sum_{i=0}^{m} s_i x^i\right) = 0$,

then $r_0 s_0 = 0$, and so $s_0 = 0$. This shows that $f(x)\Big(\sum\limits_{i=1}^{m} s_i x^i\Big) = 0$, and so $r_0 s_1 = 0$; hence that $s_1 = 0$. Similarly, we get $s_2 = \cdots = s_m = 0$. Thus, $\sum\limits_{i=0}^{m} s_i x^i = 0$, i.e., $f(x) - e \in N_D\big(R[x]\big)$, as desired.

(2) $\Rightarrow$ (1) Let $a \in R$. Then we can find an idempotent $e(x) \in R[x]$ such that $a - e(x) \in N_D\big(R[x]\big)$. As $e(x) \in R[x] \subseteq R[[x]]$, we see that $e(x) \in Id(R)$. Moreover, $a - e(x) \in N_D(R)$, as needed.

(1) $\Rightarrow$ (3) Let $f(x) \in R[[x]]$. Then there exists an idempotent $e \in R$ such that $f(0) - e \in N_D(R)$. Write $f(x) - e = \sum\limits_{i=0}^{\infty} r_i x^i$. If $\big(f(x) - e\big)\Big(\sum\limits_{i=0}^{\infty} s_i x^i\Big) = 0$, analogously to the preceding discussion, we get $s_1 = s_2 = \cdots = 0$. Thus, $\sum\limits_{i=0}^{\infty} s_i x^i = 0$, i.e., $f(x) - e \in N_D\big(R[[x]]\big)$, as required.

(3) $\Rightarrow$ (1) Let $a \in R$. Then we have an idempotent $e(x) \in R[[x]]$ such that $a - e(x) \in N_D\big(R[[x]]\big)$. By virtue of Lemma 11.4.5, $e(x) \in R$ is an idempotent. Therefore $a - e(x) \in N_D(R)$, the result follows. □

Theorem 11.4.7. *Let R be a ring, and let M be an R-module. Then $T_n(R, M)$ is almost clean if and only if each $a \in R$ can be written in the form $a = e + r$ where $e \in Id(R)$ and $r \in R\backslash\big(Z(R) \cup Z(M)\big)$.*

Proof. As in the proof of Theorem 11.2.11, we claim that $Z\big(T_{n-1}(R, M)\big) = Z(R) \cup Z(M)$.

Suppose that $T_n(R, M)$ is almost clean. For any $a \in R$, we have $(a, 0) \in T_n(R, M)$. Then there exists an idempotent $(e, s) \in T_n(R, M)$ such that $(a, 0) - (e, s) \in N_D\big(T_n(R, M)\big)$. Hence $(a - e, -s) \notin Z\big(T_n(R, M)\big)\big)$. By virtue of Lemma 11.2.10, $a - e \in R\backslash\big(Z(R) \cup Z(M)\big)$, as desired.

Conversely, assume that $(a, x) \in T_n(R, M)$. By hypothesis, $a = e + r$ where $e \in Id(R)$ and $r \in R\backslash\big(Z(R) \cup Z(M)\big)$. Hence $(a, x) - (e, 0) = (a - e, x) \in N_D\big(T_n(R, M)\big)$. Therefore $T_n(R, M)$ is almost clean. □

We claim that $T_n\big(\mathbb{Z}, \mathbb{Z}_6\big)$ is not almost clean. As $Z(\mathbb{Z}) \cup Z(\mathbb{Z}_6) = 2\mathbb{Z} \cup 3\mathbb{Z}$, we see that $3 - e \notin \mathbb{Z}\backslash Z(\mathbb{Z}) \cup Z(\mathbb{Z}_6)$ for any idempotent $e \in \mathbb{Z}$, and we are done by Theorem 11.4.7.

Corollary 11.4.8. *Let R be a ring. Then the following are equivalent:*

(1) *R is almost clean.*
(2) *$T_n(R, R)$ is almost clean for some $n \in \mathbb{N}$.*
(3) *$T_n(R, R)$ is almost clean for all $n \in \mathbb{N}$.*

Proof. (1) $\Rightarrow$ (3) This is obvious by Theorem 11.4.7.

(2) $\Rightarrow$ (3) This is trivial.

(3) $\Rightarrow$ (1) For any $a \in R$, it follows by Theorem 11.4.7 that $a = e + r$ where $e \in Id(R)$ and $r \in R\backslash Z(R) \subseteq N_D(R)$, so the corollary is true. □

Recall that a ring R is called a π-Rickart ring if for any $a \in R$ there exists some $n \in \mathbb{N}$ such that $a^n R$ is projective, equivalently, $r(a^n) = eR$ for an idempotent $e \in R$.

Lemma 11.4.9. *Let R be a ring. Then each $a \in R$ can be written in the form $a^n = er$ where $n \in \mathbb{N}, e \in Id(R)$ and $r \in N_D(R)$ if and only if R is a π-Rickart ring.*

Proof. Suppose R is a π-Rickart ring. Let $a \in R$. Then there exists some $n \in \mathbb{N}$ such that $\varphi : a^n R \cong eR$ where $e \in R$ is an idempotent. Write $a^n = \varphi(ex)$. Then $a^n = e\varphi(ex)$. Let $r = \varphi(ex) + 1 - e$. Then $a^n = er$. If $rz = 0$, then $\big(e\varphi(ex) + 1 - e\big)z = 0$, and so $(1 - e)z = 0$. Moreover, $e\varphi(ex)z = 0$, and so $a^n z = 0$. Write $e = \varphi^{-1}(a^n y)$. Then $ez = \varphi^{-1}(a^n zy) = 0$. This implies that $z = 0$, and so $r \in N_D(R)$, as desired.

Conversely, assume that $a^n = er$ where $n \in \mathbb{N}, e \in Id(R)$ and $r \in N_D(R)$. Then we have an R-isomorphism $eR \cong a^n R, ex \mapsto erx$ for any $x \in R$. Hence, $a^n R$ is projective, as required. □

Theorem 11.4.10. *Every π-Rickart ring is almost clean.*

Proof. Let R be a π-Rickart ring, and let $a \in R$. Then it follows from Lemma 11.4.9 that $a^n = er$ where $n \in \mathbb{N}, e \in Id(R)$ and $r \in N_D(R)$. Hence $a = (1 - e) + (a - 1 + e)$. If $(a - 1 + e)x = 0$, then $(a - 1 + e)\big(a^{n-1} + a^{n-2}(1 - e) + \cdots + a(1 - e) + (1 - e)\big)x = 0$. That is, $(a^n - 1 + e)x = 0$. Hence, $(1 - e)x = 0$. Furthermore, $rex = 0$, and so $ex = 0$, as $r \in N_D(R)$. Therefore $x = ex + (1 - e)x = 0$. This means that $a - 1 + e \in N_D(R)$, hence the result. □

Clearly, every π-regular ring is a π-Rickart ring. Thus, every finite ring is π-Rickart. Also we note that every ring with finitely many zero-divisor is finite (see [259, Theorem 1]). A ring R is a p.p. ring if every principal ideal is projective, e.g., every regular ring.

Corollary 11.4.11. *Every p.p ring is almost clean.*

Proof. This is obvious, by Theorem 11.4.10. □

A ring R is nonsingular if 0 is the only element of R which is annihilated by an essential ideal of R.

Lemma 11.4.12. *A ring R is a nonsingular ring if and only if R is reduced.*

Proof. $\Longrightarrow$ Let $x^2 = 0$. If there exists an ideal I of R such that $r(x) \cap I = 0$, it follows from $xI \subseteq r(x) \cap I$ that $I \subseteq r(x)$. This implies that $I = 0$, and so $r(x) \subseteq_e R$. Since R is nonsingular, we get $x = 0$. Thus, R is reduced.

$\Longleftarrow$ Let I be an essential ideal of R such that $xI = 0$ for some $x \in I$. For any $y \in I \cap xR$, we have $xy = 0$ and $y = xr$ for some $r \in R$. Then $x^2 r = 0$, and so $xr = 0$. This implies that $I \cap xR = 0$; hence, $x = 0$. Therefore R is nonsingular. □

An ideal I of a ring R is complement if I has no a proper essential extension in R. A ring R is called CS if each complement ideal of R is a direct summand of R.

Theorem 11.4.13. *Every nonsingular CS ring is almost clean.*

Proof. Let R be a nonsingular CS ring. Then R is reduced by Lemma 11.4.12. Let $a \in R$. Assume that $r(a) \subseteq_e I \subsetneq R$. Then $r(a) \cap aI = 0$. As $aI \subseteq I$, we get $aI = 0$, and so $I \subseteq r(a)$. We infer that $r(a) = I$. Hence, $r(a)$ is a complement ideal of R. Since R is CS, we see that $r(a) = eR$ for some idempotent e of R. This implies that aR is projective. Hence, R is a p.p. ring. Therefore we obtain the result by Corollary 11.4.11. □

Let X be a topological space, and let $f \in C(X)$. We define the cozero set of f by $coz(f) = \{x \in X \mid f(x) \neq 0\}$. The closure of a set S is the set of all points of closure of S, that is, the set S together with all of its limit points. The closure of S is denoted $\overline{S}$. $\overline{S}$ is the intersection of all closed sets containing S. A topological space X is basically disconnected if every cozero set has an open closure. The following lemma was contained in [268, Theorem 8.4.4].

Lemma 11.4.14. *Let X be a topological space. If X is basically disconnected, then $C(X)$ is a p.p. ring.*

Proof. Let $f \in C(X)$. Then we have an exact sequence

$$0 \to K \to C(X) \to fC(X) \to 0,$$

where $K = \{g \in C(X) \mid fg = 0\}$. Since $\overline{coz(f)}$ is clopen the characteristic function $\chi_{X-\overline{coz(f)}}$ of $X - \overline{coz(f)}$ is continuous. Clearly, $\chi_{X-\overline{coz(f)}} \in K$. Let $g \in K$. Then $g(x)f(x) = 0$, and so $g(x) = 0$ for all $x \in coz(f)$. This shows that $g(x) = g(x)\chi_{X-\overline{coz(f)}}$. Therefore K is generated by the idempotent $\chi_{X-\overline{coz(f)}} \in C(X)$. Hence, the exact sequence $0 \to K \to C(X) \to fC(X) \to 0$ splits. This implies that $fC(X) \oplus K \cong C(X)$, and so $fC(X)$ is projective. Therefore $C(X)$ is a p.p. ring, as asserted. □

Theorem 11.4.15. *Let X be a topological space. If X is basically disconnected, then $C(X)$ is almost clean.*

Proof. Sine X is basically disconnected, it follows by Lemma 11.4.14 that $C(X)$ is a p.p. ring. Therefore we complete the proof, by Corollary 11.4.11. □

Let $q(R)$ be the classical ring of quotients of a ring R, i.e., $q(R) = \{\frac{r}{s} \mid r \in R, s \in N_D(R)\}$. Here, we give a characterization of π-Rickart rings.

Theorem 11.4.16. *A ring R is a π-Rickart ring if and only if*

(1) *R_M is a D-ring for all $M \in Max(R)$;*
(2) *$q(R)$ is π-regular.*

Proof. $\Longrightarrow$ Let $M \in Max(R)$. Then R_M is local, and that R_M is a π-Rickart ring. Set $S = R_M$. Let $a \in Z(S)$. Write $ab = 0$ with $0 \neq b \in S$. Then we can find some $n \in \mathbb{N}$ such that $\varphi : a^nS \cong eS$ for an idempotent $e \in S$. If $e = 0$, then $a^n = 0$, and so $a \in N(S)$. If $e = 1$, then $\varphi : a^nS \cong S$. Write $1 = \varphi(a^ns)$ for some $s \in S$. Then $b = \varphi(a^ns)b = \varphi(a^nbs) = 0$, a contradiction. Therefore, S is a D-ring.

Let $\frac{a}{b} \in q(R)$. Then a^nR is projective for some $n \in \mathbb{N}$, and so $r(a^n) = eR$ for an idempotent $e \in R$. Hence, $a^n(a^n + e) = a^{2n}$. We claim that $a^n + e \in Z_D(R)$. If $(a^n + e)x = 0$ for some $x \in R$, then $a^nx = ex$, and so $ex = e(a^nx) = 0$. This implies that $x \in r(a^n)$. Write $x = ey$. Then $x = ex = 0$, and so $a^n + e \in N_D(R)$. Therefore, $\left(\frac{a}{b}\right)^n = \left(\frac{a}{b}\right)^{2n}\left(\frac{b^n}{a^n+e}\right)$. This shows that $q(R)$ is π-regular, as desired.

$\Longleftarrow$ Let $\delta_M = \{r \in R \mid \exists\, n \in \mathbb{N}$ such that $sr^n = 0$ for some $s \in R \setminus M\}$ for any maximal ideal M of R. Then δ_M is an ideal of R. If $ab \in \delta_M$,

then we can find some $n \in \mathbb{N}, s \in R \setminus M$ such that $sa^m b^m = 0$. Hence, $\frac{a^m}{1} \cdot \frac{b^m}{1} = \frac{0}{1}$ in R_M. If $\frac{b^m}{1} = \frac{0}{1}$, then $b \in \delta_M$. If $\frac{b^m}{1} \neq \frac{0}{1}$, then $\frac{a^m}{1} \in Z(R_M)$. As R_M is a D-ring, we have some $k \in \mathbb{N}$ such that $\left(\frac{a^m}{1}\right)^k = \frac{0}{1}$; hence that, $a \in \delta_M$. That is, δ_M is a prime ideal of R.

If $\frac{c}{d} \cdot \frac{v}{w} \in \delta_M q(R)$, then $\frac{cv}{dw} = \frac{xp}{q}$ for some $x \in \delta_M$. Thus, $cvq = xpdw$; hence, $s(cvq)^l = 0$ for some $l \in \mathbb{N}, s \in R \setminus M$. That is, $s(qcv)^l = 0$, and so $qcv \in \delta_M$. As δ_M is a prime ideal of R, we get $qc \in \delta_M$ or $v \in \delta_M$. Therefore $\frac{c}{d} \in \delta_M q(R)$ or $\frac{v}{w} \in \delta_M q(R)$. That is, $\delta_M q(R)$ is a prime ideal of $q(R)$. Since $q(R)$ is π-regular, it follows by Theorem 6.1.10 that $\delta_M q(R)$ is a maximal ideal of $q(R)$.

Let I be a maximal ideal of $q(R)$. Set $M = \{r \in R \mid \frac{r}{1} \in I\}$. Then M is an ideal of R. Thus, we can find a maximal ideal K of R such that $M \subseteq K$. We claim that $\delta_K q(R) \subseteq I$. For any $x \in \delta_K$, we have some $m \in \mathbb{N}, s \in R \setminus K$ such that $sx^m = 0$. Clearly, $s \in R \setminus M$, and so $\frac{s}{1} \notin I$. But $\frac{s}{1} \cdot \left(\frac{x}{1}\right)^m = \frac{0}{1} \in I$. As I is prime, we deduce that $\frac{x}{1} \in I$. We infer that $\delta_K q(R) \subseteq I$. As $\delta_K q(R)$ is a maximal ideal of R, we get $I = \delta_K q(R)$. Therefore the set of all $\delta_K q(R)$ coincides with the set of all maximal ideals of $q(R)$.

Let $f \in q(R)$ be an idempotent. If $f \in \delta_M q(R)$, then $sf = 0$ for some $s \in R \setminus M$. If $f \notin \delta_M q(R)$, then $1 - f \in \delta_M q(R)$. Hence, $s'f = s'$ for some $s' \in R \setminus M$. Set $I = \{r \in R \mid rf \in R\}$. Then I is an ideal of R which is not contained in any maximal ideal of R. Thus, $I = R$, and so $f \in R$. Let $a \in R$. Then $a \in q(R)$ is π-regular. We have some $i \in \mathbb{N}$ such that $a^i = fu$ where $f \in q(R)$ is an idempotent and $u \in q(R)$ is invertible. Construct an R-morphism $\varphi : a^i R \to fR, a^i x \to fx$ for any $x \in R$. Then $a^i R \cong fR$, and therefore R is a π-Rickart ring. □

Corollary 11.4.17. *A ring R is a π-Rickart ring if and only if*

(1) *If $ab = 0$, then there exists some $n \in \mathbb{N}$ such that $a^n x = 0, bx = b$ is solvable in R;*
(2) *$q(R)$ is π-regular.*

Proof. Suppose that R is a π-Rickart ring. Then $q(R)$ is π-regular, by Theorem 11.4.16. If $ab = 0$, then we have $n \in \mathbb{N}$ such that $r(a^n) = eR$ for an idempotent $e \in R$. Hence, $a^n e = 0$. Further, $b \in r(a^n)$, and so $be = b$, as desired.

Conversely, assume that (1) and (2) hold. By Theorem 11.4.16, it suffices to prove that R_M is a D-ring for any maximal ideal M of R. Let $\frac{a}{b} \in Z(R_M)$. Then we can find $0 \neq \frac{c}{d} \in R_M$ such that $\frac{a}{b} \cdot \frac{c}{d} = 0$. Hence,

$\frac{ac}{bd} = 0$, and so $(sa)c = 0$ for some $s \in R \setminus M$. By hypothesis, we have some $n \in \mathbb{N}, x \in R$ such that $(sa)^n x = 0, cx = c$. If $x \in M$, then $1 - x \in R \setminus M$. But $(1 - x)c = 0$, and so $\frac{c}{d} = 0$. This gives a contradiction. Therefore $x \in R \setminus M$. Hence, $(s^n x)a^n = 0$ with $s^n x \in R \setminus M$. This implies that $\left(\frac{a}{b}\right)^n = 0$. Accordingly, R_M is a D-ring, as needed. □

In the proofs of Theorem 11.4.16 and Corollary 11.4.17, we choose all indices to be 1, then we have

Proposition 11.4.18. *Let R be a ring. Then the following are equivalent:*

(1) *R is a p.p. ring.*
(2) *$q(R)$ is regular, and R_M is a domain for all $M \in Max(R)$.*
(2) *$q(R)$ is regular, and $ab = 0$ implies that $ax = 0, bx = b$ is solvable in R.*

Lemma 11.4.19. *Let R be a ring, and let G be an abelian group. If RG is a p.p. ring, then so is R.*

Proof. Let $a \in R$. Then we can find an idempotent $e = \sum\limits_{g \in G} e_g g$ such that $r_{RG}(a) = eRG$. Let $\omega : RG \to R, \sum\limits_{g \in G} r_g g \to \sum\limits_{g \in G} r_g$. Clearly, $ae = 0$, and so $a\omega(e) = 0$. This implies that $\omega(e)R \subseteq r_R(a)$. For any $x \in r_R(a)$, we can find some $n \in \mathbb{N}$ such that $x = ey$ for $y \in RG$. Hence, $x = \omega(e)\omega(y)$. It follows that $r_R(a) \subseteq \omega(e)R$. Therefore $r_R(a) = \omega(e)R$. Here $\omega(e) \in R$ is an idempotent, hence the result. □

With this information we now provide a class of p.p. group rings, which is due to Zan (see [570, Example 3.1.11]).

Proposition 11.4.20. *Let G be a finite abelian group, and let $n \geq 2$ be an integer. Then the following are equivalent:*

(1) *$\mathbb{Z}_n G$ is a p.p. ring.*
(2) *$gcd\big(n, |G|\big) = 1$ and n is square-free.*

Proof. (1) $\Rightarrow$ (2) Write $n = p_1^{s_1} \cdots p_m^{s_m}$ where $p_1, \ldots, p_m$ are primes, $s_1, \ldots, s_m \in \mathbb{N}$. Then $\mathbb{Z}_n G \cong \bigoplus\limits_{i=1}^{m} \mathbb{Z}_{p_i^{s_i}} G$. This implies that each $\mathbb{Z}_{p_i^{s_i}} G$ is a p.p. ring, and then so is $\mathbb{Z}_{p_i^{s_i}}$, in terms of Lemma 11.4.19.

Assume that $s_i \geq 2$. Write $r(p_i) = e\mathbb{Z}_{p_i^{s_i}}$ for an idempotent $e \in \mathbb{Z}_{p_i^{s_i}}$. As $\mathbb{Z}_{p_i^{s_i}}$ is local, we see that $e = 0$ or 1. We infer that $r(p_i) = 0$ or $\mathbb{Z}_{p_i^{s_i}}$.

This gives a contradiction. Therefore $s_i = 1$, and so n is square-free. Hence, $n = p_1 \cdots p_m$.

If $p_i \mid |G|$, then $|G| = p_i^{k_i} q$ with $\big(p_i^{k_i}, q\big) = 1$. Hence, $G = G_{p_i} \oplus H$, where G_{p_i} is a p_i-group. Then $\mathbb{Z}_{p_i} G \cong \big(\mathbb{Z}_{p_i} G_{p_i}\big) H$. In view of Lemma 11.4.19, $\mathbb{Z}_{p_i} G_{p_i}$ is a p.p. ring. Let $\omega : \mathbb{Z}_{p_i} G_{p_i} \to \mathbb{Z}_{p_i}, \sum r_g g \mapsto \sum r_g$. As in the proof of Proposition 2.4.5, $Ker(\omega) \subseteq J(RG)$. Let $g \in G_{p_i}$ be an element of order p_i^k. Then $r(1-g) = \big(1 + g + \cdots + g^{p_i^k - 1}\big)\mathbb{Z}_{p_i} G$. Thus, we can find an idempotent $e \in \mathbb{Z}_{p_i} G$ such that $\big(1 + g + \cdots + g^{p_i^k - 1}\big)\mathbb{Z}_{p_i} G = e\mathbb{Z}_{p_i} G$. In light of Lemma 4.4.33, $e \in \mathbb{Z}_{p_i}$; hence, $e = 0$ or 1, a contradiction. Accordingly, $p_i \nmid |G|$, and therefore $gcd\big(n, |G|\big) = 1$.

(2) $\Rightarrow$ (1) Write $n = p_1 \cdots p_m$ where $p_1, \ldots, p_m$ are primes and each $p_i \nmid |G|$. Then $\mathbb{Z}_n G \cong \bigoplus_{i=1}^{m} \mathbb{Z}_{p_i} G$. As $p_i \nmid |G|$, by virtue of Maschke's Theorem, $\mathbb{Z}_{p_i} G$ is a semisimple ring, and so it is regular. Finally, each $\mathbb{Z}_{p_i} G$ is a p.p. ring, as desired. □

Let $\Phi_N = \{P \in Spec(R) \mid P \cap N_D(R) = \emptyset\}$. Then

Theorem 11.4.21. *For a ring R, the following conditions are equivalent:*

(1) *R is an indecomposable almost clean ring.*
(2) *For $a \in R$, either $a \in N_D(R)$ or $1 - a \in N_D(R)$.*
(3) *For ideals I and J of R consisting of zero-divisors, $I + J \neq R$.*
(4) *For all $I, J \in \Phi_N$, $I + J \neq R$.*

Proof. (1) $\Rightarrow$ (2) Let $a \in R$. If $a \in Z(R)$, then we can find an idempotent $e \in R$ such that $a - e \in N_D(R)$. If $e = 0$, then $a \in N_D(R)$, a contradiction. If $e = 1$, then $1 - a \in N_D(R)$, as desired.

(2) $\Rightarrow$ (3) Let I and J be ideals of R consisting of zero-divisors. If $I + J = R$, then $1 = a + b$ with $a \in I, b \in J$. This shows that $a \in Z(R)$ and $1 - a = b \in Z(R)$, a contradiction. Therefore $I + J \neq R$.

(3) $\Rightarrow$ (4) This is trivial.

(4) $\Rightarrow$ (2) Let $a \in R$. Suppose that a and $1 - a$ are both zero-divisors. We now show that there exists a prime ideal of R consisting of zero-divisors contains (a).

Let $\Omega = \{I \lhd R \mid (a) \subseteq I \subseteq Z(R)\}$. Then Ω is a nonempty inductive set. By using Zorn's Lemma, there exists an ideal M which is maximal in Ω. We claim that M is prime. If not, we have $xy \in M$, while $x, y \notin M$. Since $M \subsetneq (x) + M, (y) + M$, we see that $(x) + M$ and $(y) + M$ contain

non-zero-divisors. Thus, $\big((x)+M\big)\big((y)+M\big)$ contains a non-zero-divisor. But $\big((x)+M\big)\big((y)+M\big) \subseteq (xy)+M \subseteq M$. This shows that M contains a non-zero-divisor. This gives a contradiction. Therefore M is prime.

Similarly, we can find a prime ideal K such that $(1-a) \subseteq K \subseteq Z(R)$. By hypothesis, $M+K \neq R$. But $R=(a)+(1-a) \subseteq M+K \subseteq R$, and so $M+K=R$, a contradiction. Therefore $a \in N_D(R)$ or $1-a \in N_D(R)$, as desired.

(2) $\Rightarrow$ (1) Let $e \in R$ be an idempotent. Then $e \in N_D(R)$ or $1-e \in N_D(R)$. As $e(1-e)=0$, we get $e=0$ or $e=1$. This shows that R is indecomposable. Let $a \in R$. Then $a \in N_D(R)$ or $a=1+(a-1)$ with $a-1 \in N_D(R)$. Therefore R is almost clean. □

Corollary 11.4.22. *Let R be a présimplifiable ring, i.e., $Z(R) \subseteq J(R)$. Then R is an indecomposable almost clean ring.*

Proof. Let $a \in R$. If $a \in Z(R)$, then $a \in J(R)$; hence, $1-a \in U(R)$. This implies that $1-a \in N_D(R)$. Therefore $a \in N_D(R)$ or $1-a \in N_D(R)$. According to Theorem 1.4.21, R is indecomposable almost clean ring. □

An R-module is called uniserial if the set of its submodules is totally ordered by inclusion. A ring R is a valuation ring if it is a uniserial R-module. A ring R is arithmetic if R_M is a valuation ring for all maximal ideal M. For instance, every Bézout ring is arithmetic.

Corollary 11.4.23. *Let R be an indecomposable arithmetic ring. Then the following are equivalent:*

(1) *R is almost clean.*
(2) *$q(R)$ is a local ring.*

Proof. (1) $\Rightarrow$ (2) Let I, J be prime ideals of R containing zero-divisors. Then $I+J \neq R$, by Theorem 11.4.21. Hence, we can find a maximal ideal M of R such that $I+J \subseteq M$. By hypothesis, R_M is a valuation ring, either $I_M \subseteq J_M$ or $J_M \subseteq I_M$. If $I_M \subseteq J_M$, then for any $r \in I$ we have $\frac{r}{1} = \frac{y}{s}$ for some $y \in J, s \in R \setminus M$. Hence, $t(sr-y)=0 \in J$ for some $t \in R \setminus M$. This implies that $sr-y \in J$, as $t \notin J$. Thus, $sr \in J$, and so $r \in J$. We infer that $I \subseteq J$. If $J_M \subseteq I_M$, similarly, we get $J \subseteq I$.

Let K and L be prime ideals of $q(R)$. Let $P=\{r \in I \mid \frac{r}{1} \in K\}$ and $Q=\{r \in I \mid \frac{r}{1} \in L\}$. Then P and Q are prime ideals of R, and that $K=Pq(R)$ and $L=Qq(R)$. Obviously, P and Q contain zero-divisors. By the preceding discussion, $P \subseteq Q$ or $Q \subseteq P$. Then $K \subseteq L$ or $L \subseteq K$.

We infer that $q(R)$ has only one maximal ideal, and therefore $q(R)$ is a local ring.

(2) $\Rightarrow$ (1) Let I and J be ideals of R containing zero-divisors. Then $Iq(R)$ and $Jq(R)$ be ideals of $q(R)$. If $Iq(R) = q(R)$, then $\frac{1}{1} = \frac{r}{s}$ for some $r \in I$ and $s \in N_D(R)$. Thus, we can find some $t \in N_D(R)$ such that $t(r-s) = 0$, and so $r = s \in N_D(R)$, a contradiction. Hence, $Iq(R) \neq q(R)$. Likewise, $Jq(R) \neq q(R)$. Since $q(R)$ is a local ring, $Iq(R) + Jq(R) \subseteq J\big(q(R)\big) \subsetneq q(R)$. Therefore $I + J \neq R$. In light of Theorem 11.4.21, R is almost clean. □

Example 11.4.24. Let $R = K[x, y, z]$ where K a field. Then R is almost clean as it is an integral domain. Let $\overline{R} = K[x, y, z]/\big(x(x-1)y\big)$. Note that $\overline{R}$ is indecomposable. Here $\overline{x}$ and $\overline{1} - \overline{x}$ are both zero-divisors, so $\overline{R}$ is not almost clean by Theorem 11.4.21.

A ring R is a torch ring if R is an arithmetic ring with at least two maximal ideals, and R has a unique minimal prime ideal which is a nonzero uniserial module. A module E has a canonical form if $E \cong R/I_1 \oplus \cdots \oplus R/I_n$, where $I_1 \subseteq \cdots \subseteq I_n \neq R$. Call a ring R a CF-ring if every direct sum of finitely many cyclic modules has a canonical form. We end this chapter by noting that every CF-ring is almost clean (see [209, Corollary 4.3]).

Notes. Theorem 11.1.7 is due to Hernriksen and Jerison [309, Corollary 1.2]. Theorem 11.1.12 was first proved in [426, Theorem 4.1]. Theorem 11.1.15 was proved by Danchev and McGovern [230, Proposition 1.19 and Proposition 1.21]. Theorem 11.4.7 is an obvious extension of [13, Theorem 2.11]. Theorem 11.4.13 is essentially due to Beidar et al. [52, Theorem 3.2]. Theorem 11.4.21 is due to [13, Theorem 2.3] and [209, Proposition 4.1].

Chapter 12

Topological Structures

A topological space is a set X together with a topology on X which is a set of subsets of X satisfying a few requirements with respect to their unions and intersections. The problem that motivates this chapter is to discover which relations must exist between strong cleanness and properties of the topological spaces of all prime ideals and all maximal ideals of a ring. In Section 12.1, we find necessary and sufficient conditions under which an abelian ring is nearly clean. In Section 12.2 we discuss separative properties for the special case at the commutativity. Namely, we take R to be commutative and then discuss what happen. In Section 12.3, we establish some elementary properties of pm rings. We enrich the supply of various such rings. In Section 12.4 we describe a fundamental work on noncommutative rings and determine equivalent conditions under which a ring is feckly clean.

12.1 Nearly Clean Rings

An ideal P of a ring R is prime provided that for any ideals I and J, $IJ \subseteq P$ implies that $I \subseteq P$ or $J \subseteq Q$. We use $Spec(R)$ to stand for the set of all prime ideals of R (the prime spectrum of R). As is well known, $P \in Spec(R)$ if and only if for any $a, b \in R$, $aRb \subseteq P$ implies that $a \in P$ or $b \in P$. For instance, every maximal ideal of a ring is prime. Let I be an ideal of R,

and let $E(I) = \{P \in Spec(R) \mid I \not\subseteq P\}$. Then $E(R) = Spec(R), E(0) = \emptyset, E(I) \cap E(J) = E(IJ)$ and $\bigcup_i E(I_i) = E\left(\sum_i I_i\right)$. So $Spec(R)$ is a topological space, where $\{E(I) \mid I \trianglelefteq R\}$ is the collection of its open sets. Let $V(I) = Spec(R) \setminus E(I)$. Then $V(I) = \{P \in Spec(R) \mid I \subseteq P\}$. In addition, $V(0) = Spec(R), V(R) = \emptyset, V(I) \cup V(J) = V(IJ)$ and $\bigcap_i V(I_i) = V\left(\sum_i I_i\right)$. Let $a \in R$, and let $V(a) := V(RaR)$. Then $V(I) = \bigcap_{a \in I} V(a)$.

Lemma 12.1.1. *$Spec(R)$ is a compact space.*

Proof. Suppose that $\bigcup_i E(I_i) = Spec(R)$. Then $E\left(\sum_i I_i\right) = Spec(R)$. If $R \neq \sum_i I_i$, then the set $\Omega = \left\{I \trianglelefteq R \;\middle|\; \sum_i I_i \subseteq I\right\}$ is non-empty. By using Zorn's Lemma, there exists $J \in Max(R)$ such that $\sum_i I_i \subseteq J$. Hence, $J \in Spec(R)$, while $J \notin E\left(\sum_i I_i\right)$. This implies that $\sum_i I_i = R$, and so we have $r_1 \in I_1, \ldots, r_n \in I_n$ such that $\sum_{i=1}^{n} r_i = 1$. Thus, $R = \sum_{i=1}^{n} I_i$. This shows that $\bigcup_{i=1}^{n} E(I_i) = E\left(\sum_{i=1}^{n} I_i\right) = Spec(R)$, and therefore $Spec(R)$ is a compact space. □

An element $u \in R$ is a full element if the principal ideal generated by u is R, i.e., $RuR = R$. An element $a \in R$ is nearly clean provided that there exists an idempotent $e \in R$ and a full element $u \in R$ such that $a = e + u$. We say that R is nearly clean in the case where every element of R is nearly clean. For a commutative ring, the exchange property, the cleanness and the nearly clean property coincide. Now we characterize the nearly clean ring R by means of the separation property of the topological space $Spec(R)$.

Lemma 12.1.2. *Let R be an abelian ring. Then* (1) *$a \in R$ is nearly clean if and only if there exists an idempotent $e \in R$ such that $V(a-1) \subseteq V(e) \subseteq Spec(R) \setminus V(a)$; and* (2) *$-a \in R$ is nearly clean if and only if there exists an idempotent $e \in R$ such that $V(a+1) \subseteq V(e) \subseteq Spec(R) \setminus V(a)$.*

Proof. (1) Let $a \in R$ be nearly clean. Then we have an idempotent $e \in R$ and a full element $u \in R$ such that $a = e + u$. For any $P \in V(a-1)$,

we have $P \notin V(1-e)$; otherwise, $u = (a-1)+(1-e) \in P$. Clearly, $Spec(R) = V(e) \sqcup V(1-e)$, as R is abelian. Hence $P \in V(e)$. This implies that $V(a-1) \subseteq V(e)$. If $P \in Spec(R)$ and $P \notin Spec(R) \setminus V(a)$, then $P \in V(a)$, and so $P \notin V(e)$; otherwise, $u = a - e \in P$. We infer that $V(e) \subseteq Spec(R) - V(a)$, as required.

Conversely, assume that there exists an idempotent $e \in R$ such that $V(a-1) \subseteq V(e) \subseteq Spec(R) \setminus V(a)$. Let $u = a - e$. If $RuR \neq R$, by using Zorn's Lemma, we have a maximal ideal P such that $RuR \subseteq P \subsetneqq R$. Clearly, $P \in Spec(R)$. Since $eR(1-e) \subseteq P$, we have $e \in P$ or $1-e \in P$. Thus, $P \in V(e)$ or $P \in V(1-e)$. If $P \in V(e)$, then $a = e + u \in P$, whence, $P \in V(a)$. This gives a contradiction. If $P \in V(1-e)$, then $a-1 = (e-1)+u \in P$, whence, $P \in V(a-1)$. This implies that $P \in V(e)$, a contradiction. Therefore $a - e \in R$ is a full element, as required.

(2) By (1), $-a \in R$ is nearly clean if and only if there exists an idempotent $e \in R$ such that $V(-a-1) \subseteq V(e) \subseteq Spec(R) \setminus V(-a)$. Obviously, $V(-a-1) = V(a+1)$ and $V(-a) = V(a)$, hence the result. □

A topological space X is said to be strongly zero-dimensional provided that any two disjoint closed sets are separated by clopen sets, that is, if A and B are disjoint closed sets, then there exist disjoint clopen sets C_1, C_2 such that $A \subseteq C_1$ and $B \subseteq C_2$ (see [400]). Let X be strongly zero-dimensional. It follows from Urysohn's Lemma that for any two disjoint closed sets A and B, there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0, x \in A; f(x) = 1, x \in B$. It follows from Tietze's Theorem that for any closed set A and a continuous map $f : A \to \mathbb{R}$, there exists a continuous map $g : X \to \mathbb{R}$ such that $g \mid_A = f$.

Theorem 12.1.3. *Let R be an abelian ring. Then the following are equivalent:*

(1) *R is nearly clean.*
(2) *For any $a \in R$, there exists an idempotent $e \in R$ such that $V(a) \subseteq V(e)$ and $V(1-a) \subseteq V(1-e)$.*
(3) *For any disjoint closed sets A and B of $Spec(R)$, there exists an idempotent $e \in R$ such that $A \subseteq V(e)$ and $B \subseteq V(1-e)$.*

Proof. (1) ⇒ (3) Let A and B be disjoint closed sets of $Spec(R)$. Then $A \cap B = \emptyset$. Clearly, there exist two ideals I and J such that $A = V(I)$ and $B = V(J)$, thus $V(I) \cap V(J) = \emptyset$. If $I + J \neq R$, then we have a maximal ideal P of R such that $I + J \subseteq P \subsetneqq R$. This shows that

$P \in V(I+J) = V(I) \cap V(J)$, a contradiction. Accordingly, $I + J = R$, and then $a + b = 1$ for some $a \in I$ and $b \in J$. Since $a \in R$ is nearly clean, it follows from Lemma 12.1.2 that there exists an idempotent $e \in R$ such that

$$V(a-1) \subseteq V(1-e) \subseteq Spec(R) \setminus V(a).$$

One easily checks that

$$\begin{aligned} B = V(J) &\subseteq V(b) \\ &= V(a-1) \subseteq V(1-e) \subseteq Spec(R) \setminus V(a) \\ &\subseteq Spec(R) \setminus V(I) = Spec(R) \setminus A. \end{aligned}$$

Obviously, $B \subseteq V(1-e)$. As $V(1-e) \subseteq Spec(R) \setminus A$, we conclude that $A \subseteq V(e)$, as desired.

(3) $\Rightarrow$ (2) For any $a \in R$, $V(a) \cap V(1-a) = \emptyset$. That is, $V(a)$ and $V(a-1)$ are disjoint closed sets of $Spec(R)$. Therefore, we can find an idempotent $e \in R$ such that $A \subseteq V(e)$ and $B \subseteq V(1-e)$.

(2) $\Rightarrow$ (1) For any $a \in R$, we have an idempotent $e \in R$ such that $V(a) \subseteq V(1-e)$ and $V(a-1) \subseteq V(e)$. If $P \in Spec(R)$ and $P \notin Spec(R) \setminus V(a)$, then $P \in V(a)$, and so $P \in V(1-e)$. We infer that $P \notin V(e)$; hence, $V(e) \subseteq Spec(R) \setminus V(a)$. Thus, $V(a-1) \subseteq V(e) \subseteq Spec(R) \setminus V(a)$. In light of Lemma 12.1.2, $a \in R$ is nearly clean, as required. □

Proposition 12.1.4. *Let R be a ring. Then the following are equivalent:*

(1) *R is nearly clean.*
(2) *$R/P(R)$ is nearly clean.*

Proof. (1) $\Rightarrow$ (2) This is obvious.

(2) $\Rightarrow$ (1) For any $a \in R$, there exists an idempotent $\overline{e} \in R/P(R)$ and a full element $\overline{u} \in R/P(R)$ such that $\overline{a} = \overline{e} + \overline{u}$. Hence, $a - e - u \in P(R)$. Write $a = e + u + s$ for some $s \in P(R)$. In view of Lemma 4.3.1, $P(R)$ is nil. Since idempotents lift modulo any nil ideal, we may assume that $e = e^2 \in R$. Write $1 = \sum_{i=1}^{n} s_i u t_i + r$ for some $r \in P(R)$. Then $1 = \sum_{i=1}^{n} s_i(u+s)t_i + t$ for some $t \in P(R)$. Hence, $\sum_{i=1}^{n} s_i(u+s)t_i(1-t)^{-1} = 1$. This means that $u + s \in R$ is a full element. This completes the proof. □

Example 12.1.5. Let F be a field of characteristic 0 and $R = F[x, y]$ be the additive group of polynomials in two indeterminates. Define multiplication in R by requiring that multiplication be distributive, that $ax = xa, ya = ay$

for all $a \in F$, that the product of x and y be the polynomial xy as usual, but that the product of y and x be the polynomial $xy + 1$. Then R is a noncommutative simple domain. In addition, R is not a division ring. If $a = 1$, then $a \in R$ is nearly clean. If $a \neq 1$, then $a - 1 \in R$ is a full element. Hence, $a \in R$ is nearly clean. Clearly, R has no non-trivial idempotents. Therefore R is an abelian nearly clean ring.

Evidently, every simple domain is a nearly clean abelian ring.

Lemma 12.1.6. *Let R be an abelian ring. Then A is a clopen subset of $Spec(R)$ if and only if there exists a unique idempotent $e \in R$ such that $A = V(e)$.*

Proof. One direction is obvious. Conversely, assume that A is an clopen subset of $Spec(R)$. Then we have two ideals I and J of R such that $A = V(I)$ and $Spec(R) = V(I) \sqcup V(J)$. Thus, $V(IJ) = Spec(R)$; hence that $IJ \subseteq P(R)$. Since $V(I) \cap V(J) = \emptyset$, $V(I + J) = \emptyset$. If $I + J \neq R$, then we have a maximal ideal P of R such that $I + J \subseteq P \subsetneqq R$. As $P \in Spec(R)$, we get $P \in V(I+J)$, a contradiction. Hence $I+J = R$. Write $1 = a_1 + a_2, a_1 \in I, a_2 \in J$. Then $a_1a_2 \in P(R)$. In view of Lemma 4.3.1, $P(R)$ is nil. Obviously, $a_1a_2 = a_1(1 - a_1) = (1 - a_1)a_1 = a_2a_1$, and so $a_1^n(1 - a_1)^n = (a_1a_2)^n = 0 (n \in \mathbb{N})$. Let $f(x) = \sum\limits_{i=0}^{n} \binom{2n}{i} x^{2n-i}(1 - x)^i$. Then $f(x) \equiv 0 \,(mod\, x^n)$. It follows from

$$f(x) + \sum_{i=n+1}^{2n} \binom{2n}{i} x^{2n-i}(1 - x)^i = \big(x + (1 - x)\big)^n = 1$$

that $f(x) \equiv 1 \,\big(mod\,(1 - x)^n\big)$. Thus, $f(x)\big(1 - f(x)\big) \equiv 0 \,\big(mod\, x^n(1 - x)^n\big)$. Let $e_1 = f(a_1)$. Then $e_1(1 - e_1) = 0$; hence, $e_1 = e_1^2 \in R$. Let $e_2 = 1 - e_1$. Then $e_2 = e_2^2$ and $e_1e_2 = 0$. Obviously, $a_1^m = (a_1 + a_2)a_1^m \equiv a_1^{m+1} \,(mod\, IJ)$ for any $m \in \mathbb{N}$. Thus, $e_1 \equiv a_1^{2n} \equiv \cdots \equiv a_1 \,(mod\, IJ)$. Likewise, $e_2 \equiv a_2 \,(mod\, IJ)$. Hence $e_1 \in I$ and $e_2 \in J$, and so

$$Spec(R) = V(e_1) \sqcup V(e_2), V(I) \subseteq V(e_1), V(J) \subseteq V(e_2).$$

Since $Spec(R) = V(I) \sqcup V(J)$, we get $V(I) = V(e_1)$ and $V(J) = V(e_2)$. If $A = V(e)$, then $V(e_1) = V(e)$; hence, $V(1 - e_1) = V(1 - e)$. If $P \in V(e_1)$, then $P \in V(e)$, and so $e_1 - e \in P$. If $P \notin V(e_1)$, then $P \in V(1 - e_1)$, and so $P \in V(1 - e)$. Therefore $e_1 - e = (1 - e) - (1 - e_1) \in P$. This shows that $e_1 - e \in P(R)$, whence $(e_1 - e)^{2k+1} = e_1 - e = 0$ for some $k \in \mathbb{N}$. Accordingly, $e_1 = e$, as desired. □

Theorem 12.1.7. *Let R be an abelian ring. Then the following are equivalent:*

(1) *R is nearly clean.*
(2) *$Spec(R)$ is strongly zero-dimensional.*

Proof. (1) $\Rightarrow$ (2) This is obvious from Theorem 12.1.3.

(2) $\Rightarrow$ (1) Let $a \in R$. Then $V(a) \cap V(a-1) = \emptyset$, i.e., $V(a)$ and $V(a-1)$ are disjoint closed sets of $Spec(R)$. Thus we have distinct clopen sets U and V such that $V(a) \subseteq U$ and $V(a-1) \subseteq V$. In view of Lemma 12.1.6, we have idempotents $e, f \in R$ such that $U = V(e)$ and $V = V(f)$. Since $V(e) \cap V(f) = V(eR + fR) = \emptyset$, we get $eR + fR = R$. Hence $(1-e)R \subseteq (1-e)fR \subseteq fR$. If $P \in Spec(R)$ and $P \notin Spec(R) \setminus V(a)$, then $P \in V(a)$; hence that $P \in V(e)$. We infer that $P \notin V(1-e)$. Clearly, $V(f) \subseteq V(1-e)$, and so $P \notin V(f)$. Therefore $V(f) \subseteq Spec(R) \setminus V(a)$. Obviously, $V(a-1) \subseteq V(f)$. In light of Lemma 12.1.2, $a \in R$ is nearly clean. □

Lemma 12.1.8. *Let R be a commutative ring. Then the following are equivalent:*

(1) *$a^2 \in R$ is clean.*
(2) *$a, -a \in R$ are clean.*

Proof. (1) $\Rightarrow$ (2) Straightforward.

(2) $\Rightarrow$ (1) Let $a, -a \in R$ be clean. Then we have clopen sets U, W of $Spec(R)$ such that

$$V(a-1) \subseteq U \subseteq Spec(R) \setminus V(a),$$
$$V(a+1) \subseteq W \subseteq Spec(R) - V(a).$$

Thus,

$$V(a-1) \cup V(a+1) \subseteq U \cup W \subseteq Spec(R) \setminus V(a).$$

Clearly, $V(a^2-1) = V(a-1) \cup V(a+1)$ and $V(a^2) = V(a)$. Thus,

$$V(a^2-1) \subseteq U \cup W \subseteq Spec(R) \setminus V(a^2).$$

In view of Lemma 12.1.6, we find an idempotent $e \in R$ such that $U \cup V = V(e)$. Therefore $a^2 \in R$ is clean, by Lemma 12.1.2. □

Proposition 12.1.9. *Let R be a commutative ring. Then the following are equivalent:*

(1) *$a^2 \in R$ is clean.*
(2) *For any idempotent $e \in R$, $a + e \in R$ is clean.*

Proof. (1) $\Rightarrow$ (2) Let $a^2 \in R$ be clean. It follows by Lemma 12.1.8 that $a, -a \in R$ are clean. Then we have idempotents $e_1, e_2 \in R$ and $u_1, u_2 \in U(R)$ such that $a = e_1 + u_1$ and $-a = e_2 + u_2$. Set $U = \big(V(e) \cap V(e_1)\big) \cup \big(V(1-e) \cap V(1-e_2)\big)$. Let $P \in V(a+e-1)$. Then $P \in V(e)$ or $P \in V(e-1)$. If $P \in V(e)$, then $a - 1 \in P$, and so $e_1 - 1 \notin P$. Thus, $e_1 \in P$; whence, $P \in V(e) \cap V(e_1)$. If $P \in V(e-1)$, then $a \in P$, and so $e_2 \notin P$. Thus, $e_2 - 1 \in P$; whence, $P \in V(e-1) \cap V(e_2 - 1)$. Therefore $V(a+e-1) \subseteq U$. If $P \notin Spec(R) \setminus V(a+e)$, then $P \in V(a+e)$. If $P \in V(e) \cap V(e_1)$, then $u_1 = (a+e) - e - e_1 \in P$, a contradiction. If $P \in V(1-e) \cap V(1-e_2)$, then $u_2 = (1-e_2) - (1-e) - (a+e) \in P$, a contradiction. Thus, $U \subseteq Spec(R) \setminus V(a+e)$. We conclude that $V(a+e-1) \subseteq U \subseteq Spec(R) \setminus V(a+e)$. In view of Lemma 12.1.6, $U = V(f)$ for an idempotent $f \in R$. Therefore $a + e \in R$ is clean, by Lemma 12.1.2.

(2) $\Rightarrow$ (1) Obviously, $a, a+1 \in R$ are both clean. Then we have $e^2 = e \in R$ and $u \in U(R)$ such that $a + 1 = e + u$; hence, $-a = (1-e) + (-u)$ is clean. Therefore we complete the proof, in terms of Lemma 12.1.8. □

Let I be an ideal of R, and let $D(I) = \{P \in Max(R) \mid I \not\subseteq P\}$. Then $Max(R)$ is a topological space, where $\{D(I) \mid I \trianglelefteq R\}$ is the collection of its open sets. Let $U(I) = Max(R) \setminus D(I)$. Then $U(I) = \{M \in Max(R) \mid I \subseteq M\}$. Then $\{U(I) \mid I \trianglelefteq R\}$ is the collection of its closed set. Let $a \in R$, and let $D(a) := D(RaR)$ and $U(a) := U(RaR)$. As in the proof of Lemma 12.1.1, $Max(R)$ is compact. As every maximal ideal of R is prime, we can derive the following lemmas analogously to Lemma 12.1.2 and Theorem 12.1.3.

Lemma 12.1.10. *Let R be an abelian ring. Then $a \in R$ is nearly clean if and only if there exists an idempotent $e \in R$ such that $U(a-1) \subseteq U(e) \subseteq Max(R) \setminus U(a)$.*

Lemma 12.1.11. *Let R be an abelian ring. Then the following are equivalent:*

(1) *R is nearly clean.*
(2) *For any $a \in R$, there exists an idempotent $e \in R$ such that $U(a) \subseteq U(e)$ and $U(1-a) \subseteq U(1-e)$.*
(3) *For any disjoint closed sets A and B of $Max(R)$, there exists an idempotent $e \in R$ such that $A \subseteq U(e)$ and $B \subseteq U(1-e)$.*

Theorem 12.1.12. *Let R be an abelian ring. Then the following are equivalent:*

(1) *R is nearly clean.*
(2) *For any disjoint compact sets A and B of $Max(R)$, there exists an idempotent $e \in R$ such that $A \subseteq U(e)$ and $B \subseteq U(1-e)$.*

Proof. (1) $\Rightarrow$ (2) Let $P, Q \in Max(R), P \neq Q$. Then there exist $a \in P, b \in Q$ such that $a + b = 1$. Since R is nearly clean, we have an idempotent $e \in R$ and a full element $u \in R$ such that $a = e + u$. Then $e \notin P$. Since $b = 1 - a = (1 - e) - u$, we see that $1 - e \notin Q$. Hence, $1 - e \in P$ and $e \in Q$, as every maximal ideal is prime. Thus $P \in U(1 - e)$ and $Q \in U(e)$. Obviously, $U(e)$ and $U(1 - e)$ are clopen sets. That is, $Max(R)$ is an Hausdorff space. Clearly, $Max(R)$ is compact. So every compact subset of $Max(R)$ is closed. Hence, any disjoint compact subsets A and B of $Max(R)$ are closed. In light of Lemma 12.1.11, there exists an idempotent $e \in R$ such that $A \subseteq U(e)$ and $B \subseteq U(1 - e)$.

(2) $\Rightarrow$ (1) As $Max(R)$ is compact, every closed set is compact. Thus the result follows from Lemma 12.1.11. □

Recall that a ring R is a pm (Gelfand) ring provided that each prime ideal is contained in exactly one maximal ideal. We note that any ring of the form $C(X)$, the ring of continuous real valued functions on a (completely regular) topological space X, is a pm ring.

Theorem 12.1.13. *Let R be an abelian ring. Then the following are equivalent:*

(1) *R is nearly clean.*
(2) *R is a pm ring and $Max(R)$ is strongly zero-dimensional.*

Proof. (1) $\Rightarrow$ (2) By virtue of Lemma 12.1.11, $Max(R)$ is strongly zero-dimensional. Let $I \in Spec(R)$. Then I is contained in a maximal ideal of R. If there exist distinct $P, Q \in Max(R)$ such that $I \subseteq P, Q$, then $I \subseteq P \cap Q$. As in the proof of Theorem 12.1.12, $P \in U(1 - e)$ and $Q \in U(e)$ for an idempotent $e \in R$. Here, $e \notin P$ and $1 - e \notin Q$. As $e(1 - e) \in I$, we see that either $e \in I$ or $1 - e \in I$. This implies that $e \in P$ or $1 - e \in Q$, a contradiction. Accordingly, R is a pm ring.

(2) $\Rightarrow$ (1) Since R is a pm ring, there exists a map $\varphi : Spec(R) \to Max(R)$, $\varphi(P) = M$, where M is the unique maximal ideal such that $P \subseteq M$. For any $I \trianglelefteq R$, we let $V(I) := \{P \in Spec(R) \mid I \subseteq P\}$ and $U(I) := \{P \in Max(R) \mid I \subseteq P\}$. It is easy to check that $\varphi\big(V(I)\big) = U(I)$.

For any disjoint closed sets $A, B \subseteq Spec(R)$, there exist two ideals I and J of R such that $A = V(I)$ and $B = V(J)$. Then $\varphi(A)$ and $\varphi(B)$ are both closed. Sine $V(I) \cap V(J) = \emptyset$, we have $V(I+J) = \emptyset$, and so $I+J = R$. This implies that $U(I) \cap U(J) = U(I+J) = U(R) = \emptyset$, i.e., $U(I)$ and $U(J)$ are disjoint closed sets of $Max(R)$. Since $Max(R)$ is strongly zero-dimensional, we have disjoint clopen sets $U, V \subseteq Max(R)$ such that $U(I) \subseteq U, U(J) \subseteq V$. Clearly, $A \subseteq \varphi^{\leftarrow}\varphi(A) \subseteq \varphi^{\leftarrow}(U)$ and $B \subseteq \varphi^{\leftarrow}\varphi(B) \subseteq \varphi^{\leftarrow}(V)$. As is well known, φ is continuous; hence, $\varphi^{\leftarrow}(U)$ and $\varphi^{\leftarrow}(V)$ are clopen. For any $P \in \varphi^{\leftarrow}(U) \cap \varphi^{\leftarrow}(V)$, there exists a unique $M \in Max(R)$ such that $P \subseteq M$. Hence, $M \in U \cap V$, a contradiction. So $\varphi^{\leftarrow}(U) \cap \varphi^{\leftarrow}(V) = \emptyset$. Therefore $Spec(R)$ is strongly zero-dimensional, hence the result by Theorem 12.1.7. □

12.2 Commutative Rings

Throughout this section, all rings are commutative with 1. We shall describe more explicit topological characterizations of commutative clean rings.

Lemma 12.2.1. *Let R be a ring. Then the following are equivalent:*

(1) *R is clean.*
(2) *$Spec(R)$ is strongly zero-dimensional.*
(3) *R is a pm ring and $Max(R)$ is strongly zero-dimensional.*

Proof. As R is commutative, every full element in R is invertible. Therefore the result follows by Theorem 12.1.7 and Theorem 12.1.13. □

Lemma 12.2.2. *Let R be a ring. Then the following are equivalent:*

(1) *R is a clean ring.*
(2) *For each $a \in R$ there exists an idempotent $e \in R$ such that $U(a) \subseteq U(e)$ and $U(1-a) \subseteq U(1-e)$ in $Max(R)$.*
(3) *For each $a \in R$ there exists an idempotent $e \in R$ such that $U(a-1) \subseteq U(e) \subseteq Max(R) \setminus U(a)$.*

Proof. This is obvious by Lemma 12.1.10 and Lemma 12.1.11. □

A base B for a topological space X is a collection of open sets such that every open set can be written as a union of elements of B. A topological

space X is said to be zero-dimensional if it has a base of clopen sets, i.e., that are at the same time open and closed. Every discrete space is zero-dimensional. Thus, every bases of topological space $Max(R)$ is of the form $\{D(a) \mid a \in R\}$.

Theorem 12.2.3. *Let R be a commutative ring. Then the following are equivalent:*

(1) *R is clean.*
(2) *R is a pm ring and $Max(R)$ is zero-dimensional.*

Proof. (1) $\Rightarrow$ (2) In view of Lemma 12.2.3, R is a pm ring. Let W be an open set of $Max(R)$. Then $Max(R) \setminus W$ is a closed set, and so $Max(R) \setminus W = U(I)$ for an ideal I. For any $M \in W$, we see that $M \notin U(I)$; hence, $I \nsubseteq M$. As M is maximal, $I + M = R$. Write $x + y = 1$ with $x \in I, y \in M$. In view of Lemma 12.2.1, there exists an idempotent $e \in R$ such that $U(x-1) \subseteq U(e) \subseteq Max(R) \setminus U(x)$. As $x - 1 = -y \in M$, we see that $M \in U(x-1) \subseteq U(e)$. We infer that $W \subseteq U(e)$.

As $x \in I$, we see that $U(I) \subseteq U(x)$. For any $M \in U(e)$, we have $M \notin U(x)$; hence that $M \notin U(I)$. Thus, $M \in W$, and so $U(e) \subseteq W$. Therefore $W = U(e)$ is clopen in $Max(R)$, as required.

(2) $\Rightarrow$ (1) Since $Max(R)$ is zero-dimensional, it has a base of clopen sets. Let A and B be disjoint closed sets. For any $M \in Max(R)$, we see that $M \notin A$ or $M \notin B$. If $M \notin A$, then $M \subseteq Max(R) \setminus A$. Write $Max(R) \setminus A = \bigcup_i D(I_i)$ where $D(I_i)$ is a clopen set for each i. Hence, $M \in D(I_i)$ for some i. Here, $D(I_i) \cap A = \emptyset$. If $M \notin B$, then M is in a clopen set $D(J_i)$, where $D(J_i) \cap B = \emptyset$. Hence, we have a clopen set U_M such that $M \in U_M$ and that $U_M \cap A = \emptyset$ or $U_M \cap B = \emptyset$. Clearly, $Max(R) = \bigcup_{M \in Max(R)} U_M$. Since $Max(R)$ is compact, we may assume that $Max(R) = \bigcup_{i=1}^{n} U_i$, where each U_i is clopen, and that $U_i \cap A = \emptyset$ or $U_i \cap B = \emptyset$.

Let $V_k = U_k \setminus \bigcup_{i<k} U_i$. Then $\{V_1, \ldots, V_n\}$ is a family of disjoint clopen sets whose union is $Max(R)$. For each k, $V_k \cap A = \emptyset$ or $V_k \cap B = \emptyset$.

Let $W = \bigcup\{V_k \mid V_k \cap A = \emptyset\}$. Then W is clopen. Clearly, $W \cap A = \emptyset$, and $\big(Max(R) \setminus W\big) \cap B = \emptyset$ as $V_k \cap A \neq \emptyset$ implies that $V_k \cap B = \emptyset$. Therefore $A \subseteq Max(R) \setminus W, B \subseteq W$ and $\big(Max(R) \setminus W\big) \cap W = \emptyset$, as required. $\square$

Lemma 12.2.4. *Let R be a ring. Then every clopen subset of $Max(R)$ is of the form $U(e)$ for some idempotent $e \in R$ if and only if idempotents lift modulo $J(R)$.*

Proof. Suppose that every clopen subset of $Max(R)$ is of the form $U(e)$ for some idempotent $e \in R$. If $f^2 - f \in J(R)$, then $U(f) \cup U(1-f) = Max(R)$ and $U(f) \cap U(1-f) = \emptyset$. Hence $U(f)$ is clopen in $Max(R)$, and so $U(f) = U(g)$ for an idempotent $g \in R$. For any $M \in Max(R)$, $M \in U(f)$ implies that $f, g \in M$, and so $f - g \in M$. $M \notin U(f)$ implies that $M \in U(1-f)$, as $f(1-f) \in M$. But $U(1-f) = U(1-g)$, and then $M \in U(1-g)$. We infer that $1-f, 1-g \in M$; hence, $f - g = (1-g) - (1-f) \in M$. Therefore $f - g \in J(R)$, as desired.

Suppose that idempotents lift modulo $J(R)$. If $U(I)$ is clopen in $Max(R)$, then there exists an ideal J of R such that $U(I) \cup U(J) = Max(R)$ and $U(I) \cap U(J) = \emptyset$. Hence, $U(IJ) = Max(R)$ and $U(I+J) = \emptyset$. It follows that $I + J = R$. Write $e + f = 1$ with $e \in I, f \in J$. Then $e - e^2 = ef \in IJ \subseteq J(R)$. By hypothesis, we have an idempotent $g \in R$ such that $e - g \in J(R)$. Accordingly, we easily check that $U(e) = U(g)$. Clearly, $U(I) \subseteq U(e)$ and $U(J) \subseteq U(f)$. Further, $U(e) \cup U(f) = Max(R)$ and $U(e) \cap U(f) = \emptyset$. Hence, $U(I) = U(e) = U(g)$, as required. □

Theorem 12.2.5. *Let R be a ring. Then the following are equivalent:*

(1) *R is clean.*
(2) *$Max(R)$ is zero-dimensional and every clopen subset is of the form $U(e)$ for some idempotent $e \in R$.*

Proof. (1) ⇒ (2) In view of Theorem 12.2.3, $Max(R)$ is zero-dimensional. Let $U(I)$ be a clopen subset of $Max(R)$. Since R is clean, every idempotent lifts modulo $J(R)$. It follows by Lemma 12.2.4 that $U(I) = U(e)$ for some idempotent $e \in R$, as desired.

(2) ⇒ (1) Since $Max(R)$ is zero-dimensional, it has a bases of clopen sets $\{U_k\}$. Let $V_k = U_k \setminus \bigcup_{i<k} U_i$. Then $\{V_k\}$ is a family of disjoint clopen sets whose union is $Max(R)$. As $Max(R)$ is compact, we may assume that $Max(R) = \bigcup_{k=1}^{n} V_k$. Let $a \in R$. Then $D(a)$ is a union of some clopen subsets of $\{U_k\}$. Write $D(a) = \bigcup_i U_{k_i}$. Hence,

$$D(a) = D(a) \cap Max(R) = \left(\bigcup_{k=1}^{n} V_k\right) \cap \left(\bigcup_i U_{k_i}\right) = \bigcup_{k=1}^{n} \bigcup_i (V_k \cap U_{k_i}).$$

But $V_k \cap U_{k_i} = V_k$ or $\emptyset$. Thus, we may assume that $D(a) = \bigcup_{i=1}^{n} U(e_i) = \bigcup_{i=1}^{n} D(1-e_i)$. For any idempotents $e, f \in R$, one easily checks that $D(e) \cup D(f) = D\big(e+(1-e)f\big)$. Hence $D(a) = U(e)$ for an idempotent $e \in R$. This implies that $U(a) = U(1-e)$. We shall prove that $a - e \in U(R)$. If not, there exists a maximal ideal M of R such that $a-e \in M$. If $M \in U(1-e)$, then $a \in M$, and so $e = a-(a-e) \in M$. But $1-e \in M$, which is impossible. If $M \notin U(1-e)$, then $e \in M$, and so $a = e-(a-e) \in M$. This implies that $M \in U(a) = U(1-e)$, a contradiction. Therefore $a - e \in U(R)$, as needed. □

Corollary 12.2.6. *Let R be a ring. Then the following are equivalent:*

(1) *R is clean.*
(2) *$Max(R)$ has a bases $\{U(e) \mid e \in R$ is an idempotent$\}$.*

Proof. (1) ⇒ (2) This is clear by Theorem 12.2.5.

(2) ⇒ (1) As $U(e)$ is clopen in $Max(R)$, $Max(R)$ is zero-dimensional. Let $U(I)$ be a clopen subset of $Max(R)$. As in the proof of Theorem 12.2.5, we can find some idempotents $e_1, \ldots, e_n \in R$ such that $U(I) = \bigcup_{i=1}^{n} U(e_i) = \bigcup_{i=1}^{n} D(1-e_i)$. Therefore we can find an idempotent $g \in R$ such that $U(I) = D(g) = U(1-g)$, hence the result by Theorem 12.2.5. □

Corollary 12.2.7. *Let R be a ring. Then the following are equivalent:*

(1) *R is clean.*
(2) *Every idempotent in R lifts modulo $J(R)$ and $Max(R)$ is zero-dimensional.*

Proof. (1) ⇒ (2) This is obvious.

(2) ⇒ (1) In view of Lemma 12.2.4, every clopen subset of $Max(R)$ is of the form $U(e)$ for some idempotent $e \in R$. This completes the proof by Theorem 12.2.5. □

Lemma 12.2.8. *Let R be a ring. If $E(a) \subseteq E(b)$ with $a, b \in R$, then $a^n R \subseteq bR$ for some $n \in \mathbb{N}$.*

Proof. If $a^n R \nsubseteq bR$ for all $n \in \mathbb{N}$, then $\overline{a} \in R/bR$ is not nilpotent. Thus $\overline{a} \notin P\big(R/bR\big)$. Hence, we can find a prime ideal P/bR of R/bR such that $\overline{a} \notin P/bR$. Clearly, $P \in Spec(R)$. Furthermore, $a \notin P$, but $b \in P$. That is,

$bR \subseteq P$, while $aR \nsubseteq P$. This shows that $P \in V(b)$ and $P \in E(a)$. Hence $P \in E(b)$, which is absurd. Therefore $a^n R \subseteq bR$ for some $n \in \mathbb{N}$. □

We include an analog for prime spectrum which is due to [400, Theorem 2.3].

Proposition 12.2.9. *Let R be a ring. Then the following are equivalent:*

(1) *R is strongly π-regular.*
(2) *$Spec(R)$ is zero-dimensional.*

Proof. (1) $\Rightarrow$ (2) Let $E(I)$ be an open set of $Spec(R)$. For any $a \in I$, there exists $b \in R$ such that $a^n = a^{2n}b$ for some $n \in \mathbb{N}$. Set $e_a = a^n b$. Then $e_a = e_a^2 \in R$. Furthermore, $E(a) = E(e_a)$ is clopen, and therefore $U(I) = \bigcup_{a \in I} U(e_a)$. Therefore $Spec(R)$ has a bases of clopen sets $\{U(e) \mid e = e^2 \in R\}$, as required.

(2) $\Rightarrow$ (1) Let $a \in R$. Then $E(a)$ is the union of some clopen sets $\{E(I_i) \mid i \in K\}$. As in the proof of Theorem 12.2.5, we may assume that $E(a) = \bigcup_{i=1}^{n} E(I_i)$. This shows that $E(a)$ is clopen. In view of Lemma 12.1.6, we can find an idempotent $e \in R$ such that $E(a) = V(e) = E(1-e)$. Hence, there exists some $m \in \mathbb{N}$ such that $a^m \in (1-e)R$ and $1 - e \in aR$, in terms of Lemma 12.2.8. Therefore $a^m R = (1-e)R$, as required. □

Lemma 12.2.10. *Let R be an almost clean ring, and let I be an ideal generated by idempotents. Then R/I is almost clean. In particular, the Pierce stalks of an almost clean ring are almost clean.*

Proof. Let $f, g \in R$ be idempotents. Then $fR+gR = (f+(1-f)g)R$. This observation deduces that $I = \{r \in R \mid r = re$ for some idempotent $e \in I\}$. For any $a \in R$, there exists an idempotent $e \in R$ such that $s := a - e \in N_D(R)$. If $\overline{s} \in Z(R/I)$, then we can find $t \notin I$ such that $st \in I$. Thus we have an idempotent $e \in I$ such that $st = ste$; hence, $st(1-e) = 0$. It follows that $t(1-e) = 0$, and so $t = te \in I$. This gives a contradiction. Therefore $\overline{s} \in N_D(R/I)$, as required. □

Theorem 12.2.11. *A ring R is clean if and only if*

(1) *R is a pm ring;*
(2) *R is almost clean.*

Proof. $\Longrightarrow$ This is obvious.

$\Longleftarrow$ In view of Lemma 12.2.10, the stalks of R are almost clean, and that they are pm. Thus, we will suffice to prove that an indecomposable and almost pm ring is local. So we assume that R is indecomposable.

A prime ideal is said to be a minimal prime ideal if it is minimal among all prime ideals. Suppose that $P_1 \neq P_2$ are minimal primes of R. Then there exists a unique maximal ideal M_i containing P_i for each i. We note that the set of zero-divisors of a given commutative ring contains the union of the minimal prime ideals. Thus, $P_1, P_2 \subseteq Z(R)$, we see that $P_1 + P_2 \neq R$, by Theorem 11.4.21. Hence there exists a maximal ideal M of R such that $P_1 + P_2 \subseteq M$. This shows that $M_1 = M = M_2$. Therefore we conclude that all the minimal prime ideals are contained in the same maximal ideal. As every prime ideal contains a minimal prime ideal, we show that there is only one maximal ideal, as desired. □

Example 12.2.12. $\mathbb{Z}$ is not a pm-ring. Since $\mathbb{Z}$ is almost clean and not clean, we are through by Theorem 12.2.11.

Theorem 12.2.13. *Let R be a ring. Then the following are equivalent:*

(1) *$R/J(R)$ is clean.*
(2) *$aR + bR = R$ implies that there exist $x, y \in R$ such that $a|x, b|y, xy \in J(R)$ and $xR + yR = R$.*
(3) *$Max(R)$ is zero-dimensional.*

Proof. (2) $\Rightarrow$ (3) Let $M \in D(a), a \in R$. Then $a \notin M$; hence, $aR + M = R$. Write $ar + b = 1$ with $r \in R, b \in M$. Then $aR + bR = R$. By hypothesis, there exist $x, y \in R$ such that $a|x, b|y, xy \in J(R)$ and $xR + yR = R$. Thus, $D(x) \cup D(y) = D(xR + yR) = D(R) = Max(R)$ and $D(x) \cap D(y) = D(xy) = \emptyset$ as $xy \in J(R)$. This shows that $D(x) = Max(R) \backslash D(y)$, and so $D(x)$ is clopen.

As $b|y$, it follows from $b \in M$ that $y \in M$. Hence, $x \notin M$, i.e., $M \in D(x)$. Thus $D(a) \subseteq D(x)$. As $a|x$, we have $D(x) \subseteq D(a)$. Therefore $D(a) = D(x)$ is clopen, hence the result.

(3) $\Rightarrow$ (2) Given $aR + bR = R$ with $a, b \in R$, then $D(aR) \cup D(bR) = D(aR + bR) = D(R) = Max(R)$. Since $Max(R)$ is zero-dimensional, it has a bases $\{D(I) \mid I \lhd R, D(I) \text{ is closed}\}$. Thus $D(a) = \bigcup_i D(I_i)$. As $Max(R)$ is compact, we say $D(a) = \bigcup_{i=1}^{n} D(I_i) = D(I)$. Likewise, we say $D(b) = D(J)$. Thus, $D(I + J) = D(I) \cup D(J) = Max(R)$. This shows that $I + J = R$. Write $x + y = 1$ with $x \in I, y \in J$. Then $xR + yR = R$.

Further, $D(xy) = D(x) \cap D(y) \subseteq D(I) \cap D(J) = \emptyset$, hence that $D(xy) = \emptyset$. We infer that $xy \in J(R)$. Notice that $D(x) \subseteq D(I) = D(a)$, we get $D(x) = D(x) \cap D(a) = D(ax)$. Likewise, $D(y) = D(by)$. As $xR + yR = R$, we have $D(x) \cup D(y) = D(xR + yR) = D(R) = Max(R)$; hence, $D(ax) \cup D(by) = D(axR + byR) = Max(R)$. This shows that $axR + byR = R$. It is easy to verify that $(ax)(by) \in J(R), a|(ax)$ and $b|(by)$, as desired.

(1) $\Rightarrow$ (2) Let $S = R/J(R)$. In view of Theorem 12.2.5, $Max(S)$ is zero-dimensional. Given $aR + bR = R$, then $\overline{a}S + \overline{b}S = S$. By (3) $\Leftrightarrow$ (2), we can find $\overline{x}, \overline{y}$ such that

$$\overline{a}|\overline{x}, \overline{b}|\overline{y}, \overline{xy} \in J(S) \text{ and } \overline{x}S + \overline{y}S = S.$$

Then there exist $r, s \in R$ such that $c := x - ar, d := y - bs \in J(R)$. Thus, $a|(x-c), b|(y-d), (x-c)(y-d) \in J(R)$. Furthermore, $\overline{(x-c)}S + \overline{(y-d)}S = S$. This implies that $(x-c)R + (y-d)R = R$, as desired.

(2) $\Rightarrow$ (1) Let $S = R/J(R)$. Let $a \in S$ and $M \in D(a)$. Then $a \notin M$; hence, $aS + M = S$. Thus, $aS + bS = S$ for $b \in M$. This shows that $cR + dR = R$ with $\overline{c} = a$ and $\overline{d} = b$. So we can find some $s, t \in R$ such that $c|s, d|t, st \in J(R)$ and $sR + tR = R$. Set $x = \overline{s}$ and $y = \overline{t}$. Then $a|x, b|y, xy = 0$ and $xS + yS = S$. It follows that $xS + bS = S$. This implies that $x \notin M$, i.e., $M \in D(x)$. Then $D(a) \subseteq D(x)$. As $a|x$, we have $D(x) \subseteq D(a)$. Hence, $D(a) = D(x)$. As $x^2S = xS$ and $y^2S = yS$, we may assume that $x, y \in S$ are idempotents. Therefore the collection of clopen idempotents subsets of $Max(S)$ forms a base. According to Corollary 12.2.6, S is clean. □

Corollary 12.2.14. *Let R be a ring. If $J(R)$ is nil, then the following are equivalent:*

(1) *R is clean.*
(3) *$Max(R)$ is zero-dimensional.*

Proof. (1) $\Rightarrow$ (2) This is obvious by Theorem 12.2.3.

(2) $\Rightarrow$ (1) In view of Theorem 12.2.13, $R/J(R)$ is clean, and so it is an exchange ring. As $J(R)$ is nil, every idempotent lifts modulo $J(R)$. Thus R is an exchange ring by Lemma 4.4.5. The result follows by Corollary 1.1.14. □

We now record an elementary result which is due to Burgess and R. Raphael (see [88, Proposition 3.2]).

Corollary 12.2.15. *A ring R is clean if and only if*

(1) *R is a pm ring;*
(2) *R is a weakly clean ring.*

Proof. One direction is clear by Theorem 12.2.3. Conversely, assume that (1) and (2) hold. Let S be an indecomposable homomorphic image of R. Then S is an indecomposable weakly clean ring. Hence, S is $\{0,1\}$-weakly clean. By virtue of Theorem 9.1.2, S is semilocal, and then $S/J(S)$ is clean. It follows from Theorem 12.2.13 that $Max(S)$ is zero-dimensional. On the other hand, one easily checks that S is a pm ring. According to Theorem 12.2.3, S is clean. Therefore R is clean, as asserted. □

Lemma 12.2.16. *Let R be a ring, and let G be a finitely generated abelian group. If RG is clean, then R is clean.*

Proof. Let $\omega : RG \to R, \sum\limits_{g\in G} r_g g \mapsto \sum\limits_{g\in G} r_g$. Then $R \cong RG/Ker(\omega)$, and therefore R is clean. □

Let R be a ring, and let $G = \{1, g\}$ be a group. Then $RG \cong R[x]/\big(x^2 - 1\big)$. In the sequel, we consider cleanness of such group rings.

Lemma 12.2.17. *Let R be a ring, and let $G = \{1, g\}$ be a group. Then*

(1) *$a + bg \in Id(RG)$ if and only if $a^2 + b^2 = a$ and $2ab = b$;*
(2) *$a + bg \in U(RG)$ if and only if $a + b, a - b \in U(R)$.*

Proof. (1) $a+bg \in Id(RG)$ if and only if $a+bg = (a+bg)^2 = a^2+b^2+2abg$, i.e., $a = a^2 + b^2$ and $b = 2ab$.

(2) If $a + bg \in U(RG)$, then there exist $x, y \in R$ such that $(a + bg)(x + yg) = 1$; hence, $ax + by = 1$ and $ay + bx = 0$. Thus, $(a + b)(x + y) = 1$ and $(a - b)(x - y) = 1$, whence $a + b, a - b \in U(R)$.

If $a + b, a - b \in U(R)$, we choose $x = \frac{a}{a^2-b^2}$ and $y = -\frac{b}{a^2-b^2}$. Then $ax + by = 1$ and $ay + bx = 0$. Therefore $(a + bg)(x + yg) = (ax + by) + (ay + bx)g = 1$, as desired. □

Lemma 12.2.18. *Let R be a ring, and let $G = \{1, g\}$ be a group. Suppose $a + bg \in Id(RG)$. then both $U(a), U(b) \setminus U(a)$ are idempotent clopen sets of $Max(R)$*

Proof. In view of Lemma 12.2.17, $a^2 + b^2 = a, 2ab = b$, and so $a + b, a - b \in Id(R)$. This shows that $U(2), U(a) \subseteq U(b)$. Hence $U(a) \subseteq U(a + b) \cap U(a - b)$. If $M \in U(a + b) \cap U(a - b)$, then $2a, a^2 - b^2 \in M$. This implies that $a \in M$ or $2 \in M$. If $2 \in M$, then $a = (a^2 - b^2) + 2b^2 \in M$.

Thus $a \in M$, and so $U(a+b) \cap U(a-b) = U(a)$. Accordingly, $U(a) = U\big((a+b)+(1-a-b)(a-b)\big)$ is idempotent clopen.

Next, we prove that $U(a+b) \cup U(a-b) = U(a) \cup D(b)$. Obviously, $U(a+b) \cup U(a-b) \subseteq U(a) \cup D(b)$. If $M \in U(a)$, then $a \in M$; hence, $b = 2ab \in M$. This implies that $a+b, a-b \in M$, and so $M \in U(a+b) \cup U(a-b)$. If $M \in D(b)$, then $1-2a \in M$ as $(1-2a)b = 0$. It follows from $a^2+b^2 = a$ that $1-4b^2 = 1-4(a-a^2) = (1-2a)^2 \in M$. Thus, $(a+b)(a-b) = a^2-b^2 = a^2-4a^2b^2 = a^2(1-4b^2) \in M$, and so $M \in U(a+b) \cup U(a-b)$. Consequently, $U(a+b) \cup U(a-b) = U(a) \cup D(b)$. As $U(b) \setminus U(a) = Max(R) \setminus \big(U(a) \cup D(b)\big) = Max(R) \setminus U\big((a+b)(a-b)\big) = D\big((a+b)(a-b)\big)$, we see that $U(b) \setminus U(a)$ is idempotent clopen, as asserted. □

Lemma 12.2.19. *Let R be a ring. Then RG is clean if and only if for each $a, b \in R$ there exist idempotents $e_1, e_2, e_3, e_4 \in R$ such that*

$$\sum_{i=1}^{4} e_i = 1, D(e_i) \cap D(e_j) = \emptyset, i \neq j \quad (1)$$

$$U(2) \subseteq D(e_3) \cup D(e_4) \quad (2)$$

$$U(a+b) \subseteq D(e_3) \cup D(e_1) \quad (3)$$

$$U(a+b-1) \subseteq D(e_4) \cup D(e_2) \quad (4)$$

$$U(a-b) \subseteq D(e_3) \cup D(e_2) \quad (5)$$

$$U(a-b-1) \subseteq D(e_4) \cup D(e_1) \quad (6)$$

Proof. Suppose that RG is clean. Let $a, b \in R$. Then $a+bg \in RG$ is clean. Write $a+bg = (c+dg)+(r+sg)$ where $c+dg \in Id(RG)$ and $r+sg \in U(RG)$. By virtue of Lemma 12.2.17, $c^2+d^2 = c, 2cd = d$ and $r+s, r-s \in U(R)$. Clearly, $c+d, c-d \in R$ are idempotents. In light of Lemma 12.2.18, we can find idempotents e_3, e_4 such that $D(e_4) = U(c), D(e_3) = U(d) \setminus U(c)$. Further, $U(c-d) \setminus U(c) = U(c-d) \cap U(e_4) = D(e_1)$, $e_1 = 1 - \big((c-d)+(1-c+d)e_4\big)$. Likewise, we have an idempotent $e_2 \in R$ such that $U(c+d) \setminus U(c) = D(e_2)$. One easily checks that $D(e_i) \cap D(e_j) = \emptyset$ for each $i \neq j$ and that $\bigcup_{i=1}^{4} D(e_i) = Max(R)$. Thus, we have $e_i e_j = 0 (i \neq j)$ and $\sum_{i=1}^{4} e_i = 1$, and so (1) holds. Furthermore, $U(2) \subseteq U(d) = D(e_3) \cup D(e_4)$, proving (2).

As $(a+b)-(c+d) = r+s \in U(R)$, we see that $U(a+b) \subseteq D(c+d) \subseteq D(e_1) \cup D(e_3)$, as $D(c+d) \cap \big(D(e_2) \cup D(e_4)\big) = \emptyset$. Likewise, it follows from $(a-b)-(c-d) = r-s$ that $U(a-b) \subseteq D(c-d) \subseteq D(e_2) \cup D(e_3)$. Thus, proving (3) and (5).

Let $M \in U(a+b-1)$. If $M \in D(e_1) = U(c-d) \setminus U(c)$, then $c+d \notin M$; otherwise, $d = (2c)d \in M$, and so $c \in M$, a contradiction. As $c+d \in M$ is an idempotent, we deduce that

$$M = (a+b-1) + M = (a+b) - (c+d) + M = (r+s) + M.$$

This gives a contradiction. If $M \in D(e_3) = U(d) \setminus U(c)$, then $d \in M$ and $c \notin M$. As $c(1-c) = d^2 \in M$, we get $1-c \in M$. Hence, we also have $M = (a+b-1)+M = (a+b)-(c+d)+M = (r+s)+M$, a contradiction. Therefore $M \in D(e_4) \cup D(e_2)$, proving (4). Similarly, proving (6).

Conversely, letting $a+bg \in RG$, we have idempotents $e_1, e_2, e_3, e_4 \in R$ satisfying (1) – (6). We shall produce $c, d, r, s \in R$ such that $c^2 + d^2 = c, 2cd = d, r = a-c, s = b-d$ and $r-s, r+s \in U(R)$. Then $a+bg = (c+dg) + (r+sg) \in RG$ is clean, by Lemma 12.2.17.

We claim that $e_1 + e_2 \in 2R$. If not, by using Zorn's Lemma, there exists a maximal ideal M of R such that $e_1 + e_2 \notin M$ and $2R \subseteq M$. Then $1-(e_3+e_4) \notin M$; hence that $e_3 + e_4 \in M$, as $(e_3+e_4)^2 = e_3 + e_4$. It follows that $e_3, e_4 \in M$ as $e_3e_4 = 0$. Since $M \in U(2) \subseteq D(e_3) \cup D(e_4)$, we see that $e_3 \notin M$ or $e_4 \notin M$, a contradiction. Therefore $e_1 + e_2 \in 2R$.

Write $e_1 + e_2 = 2x, x(e_1+e_2) = x$. Let $y_1 = xe_1$ and $y_2 = -xe_2$. Set $c = x + e_3$ and $d = y_1 + y_2$. Then $U(c) = D(e_4)$ and $U(d) = D(e_4) \cup D(e_3)$. It is easy to verify that $c^2 + d^2 = c$ and $2cd = d$. Hence, $c + dg \in Id(RG)$. Since $c + d = 2xe_1 + e_3 = e_1 + e_3$, we get $U(c+d) = D(e_2) \cup D(e_4)$. Let $r = a-c$ and $s = b-d$. Then $r+s = (a+b)-(c+d)$ and $r-s = (a-b)-(c-d)$.

If $r+s \notin U(R)$, then there exists a maximal ideal M such that $r+s \in M$. If $M \in U(a+b)$, then $M \in U(c+d)$. This contradicts (3). We infer that $M \in D(a+b)$, and so $M \in D(c+d)$. As $(c+d)^2 = c+d$, we get $a+b+M = c+d+M = 1+M$; hence that $M \in U(a+b-1)$. By (4), $M \in D(e_2)$; otherwise, $M \in D(e_4) = U(c)$, a contradiction. As $e_2 \notin M$, we see that $1-e_2 = e_1 + e_3 + e_4 \in M$; whence, $e_1, e_3 \in M$. Then

$$1+M = c+d+M = x+e_3+y_1+y_2+M = 2xe_1+e_3+M = e_1+e_3+M = M,$$

a contradiction. Therefore, we have $r+s \in U(R)$. Similarly, we have $r-s \in U(R)$. This implies that $r+sg \in U(RG)$ thus completing the proof. □

We have at our disposal all the information necessary to prove the following result which was proved by McGovern [413, Theorem 3.13].

Theorem 12.2.20. *Let R be a ring, and let $G = \{1, g\}$ be a group. Then RG is clean if and only if R is clean.*

Proof. One direction is obvious by Lemma 2.2.16. Conversely, assume that R is clean. Let $a, b \in R$. In view of Lemma 12.2.2, we have idempotents $f_1, f_2 \in R$ such that $U(a+b) \subseteq D(f_1), U(a+b-1) \subseteq U(f_1)$ and $U(a-b) \subseteq D(f_2), U(a-b-1) \subseteq U(f_2)$. Let e_1, e_2, e_3, e_4 be idempotents such that

$$\begin{aligned} D(e_1) &= D(f_1) \cap U(f_2), \\ D(e_2) &= D(f_2) \cap U(f_1), \\ D(e_3) &= D(f_1) \cap D(f_2), \\ D(e_4) &= Max(R) \setminus \big(D(e_1) \cup D(e_2) \cup D(e_3)\big). \end{aligned}$$

Then all conditions in Lemma 12.2.19 are satisfied except for possibly (2).

We claim that $U(2) \cap D(e_1)$ and $U(a+b) \cup U(a-b-1)$ are disjoint closed sets. Let $M \in U(2) \cap D(e_1)$. If $M \in U(a+b)$, then $M \in U(a-b)$ as $2 \in M$. This implies that $M \in D(f_2)$, a contradiction. If $M \in U(a-b-1)$, then $M \in U(a+b-1)$ as $2 \in M$. This implies that $M \in U(f_1)$, a contradiction. The claim is true. Analogously, $U(2) \cap D(e_2)$ and $U(a-b) \cap U(a+b-1)$ are disjoint closed sets.

Since R is clean, we can find an idempotent clopen subsets $K_1 \subseteq D(e_1)$ and $K_2 \subseteq D(e_2)$ such that

$$\begin{aligned} U(2) \cap D(e_1) \subseteq K_1, \big(U(a+b) \cup U(a-b-1)\big) \cap K_1 = \emptyset; \\ U(2) \cap D(e_2) \subseteq K_2, \big(U(a-b) \cup U(a+b-1)\big) \cap K_2 = \emptyset. \end{aligned}$$

By combing $D(e_4)$ together with K_1, K_2 and shrinking down $D(e_1)$ and $D(e_2)$ in the appropriate manner, we obtain idempotent clopen subsets which satisfy (1)−(6) of Lemma 11.2.19. Therefore RG is clean, as asserted. □

Corollary 12.2.21. *Let R be a ring, and let G be an elementary 2-group. Then RG is clean if and only if R is clean.*

Proof. One direction is obvious by Lemma 12.2.16.

Conversely, assume that R is clean. Since G is an elementary 2-group, it follows that for any $f \in RG$ there exists a finite subgroup H of G such that $f \in RH$ and H is a finite direct product of copies of $\mathbb{Z}_2$. In view of Theorem 12.2.20, $R\mathbb{Z}_2$ is clean, and then so is RH. Therefore f can be written as the sum of a unit and an idempotent in RG, as asserted. □

12.3 PM Rings

Throughout this section, all rings are commutative with 1. Clearly, a ring R is a pm ring if and only if R/P is a local ring for all prime ideal P of R. In this section, we study various characterizations of pm rings.

Example 12.3.1. The class of pm rings contains many familiar examples.

(1) Any ring in which every prime ideal is maximal is a pm ring.
(2) Every local ring is a pm ring.
(3) Every regular ring is a pm ring.

A ring R is VNL if for each $a \in R$, either a or $1-a$ is regular. Further, every VNL ring is a pm ring. Thus, $\mathbb{Z}_n$ is a pm ring if whenever p and q are distinct primes, $(pq)^2 \nmid n$, e.g., $\mathbb{Z}_{12}$ (see [444, Proposition 2.9]).

The following provides new classes of pm rings which is due to Mahdou et al. (see [405, Proposition 2.1]).

Proposition 12.3.2. *Let R be a ring, and let M be an R-module. Then $T(R,M)$ is a pm ring if and only if so is R.*

Proof. Let $I \in Spec\big(T(R,M)\big)$, and let $P = \left\{ r \in R \;\middle|\; \begin{pmatrix} r & \\ & r \end{pmatrix} \in I \right\}$. Then $P \in Spec(R)$. For any $\begin{pmatrix} r & m \\ & r \end{pmatrix} \in I$, we see that $\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} \in I$, as $\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}^2 = 0$. Hence,

$$\begin{pmatrix} r & 0 \\ & r \end{pmatrix} = \begin{pmatrix} r & m \\ & r \end{pmatrix} - \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} \in I,$$

and so $r \in P$. This implies that $I \subseteq T(P,M)$. For any $r \in P, m \in M$, we see that

$$\begin{pmatrix} r & m \\ & r \end{pmatrix} = \begin{pmatrix} r & 0 \\ & r \end{pmatrix} + \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} \in I.$$

Thus $T(P,M) \subseteq I$, and so $Spec(R) = \{T(P,M) \mid P \in Spec(R)\}$. Clearly, $T(R,M)/T(P,M) \cong R/P$ for each prime ideal P of R, whence $T(R,M)/T(P,M)$ is local if and only if R/P is local. Therefore $T(R,M)$ is a pm ring if and only if so is R. □

Theorem 12.3.3. *Let R be a ring. Then the following are equivalent:*

(1) *R is a pm ring.*
(2) *For any distinct $M, N \in Max(R)$, there exists $a \notin M$ and $b \notin N$ such that $ab = 0$.*
(3) *$a + b = 1$ in R implies that $(1 + ar)(1 + bs) = 0$ for some $r, s \in R$.*

Proof. (1) $\Rightarrow$ (3) Let $a, b \in R$ be such that $a+b=1$. Set $A = \{1+ax \mid x \in R\}, B = \{1+bx \mid x \in R\}$ and $T = AB$. We claim that T contains 0. If not, we have a nonempty set $\Omega = \{I \lhd R \mid I \cap T = \emptyset\}$. Given $I_1 \subseteq I_2 \subseteq \cdots$ in Ω, we have $I_i \cap T = \emptyset$. Thus, $\Big(\bigcup_i I_i\Big) \cap T = \bigcup_i \big(I_i \cap T\big) = \emptyset$; hence, $\bigcup_i I_i \in \Omega$. This means that Ω is inductive. By using Zorn's Lemma, there exists an ideal P of R, which is maximal in Ω. If $P \notin Spec(R)$, there exist $c, d \in R$ such that $c, d \notin P$, while $cd \in P$. Hence, $(RcR+P) \cap T, (RdR+P) \cap T \neq \emptyset$. This shows that $\big((RcR+P)(RdR+P)\big) \cap T \neq \emptyset$, and then $(RcdR+P) \cap T \neq \emptyset$. This gives a contradiction. Hence $P \in Spec(R)$.

If $RaR+P = R$, then $az+p=1$ for some $z \in R, p \in P$. So, $p = 1-az \in A \subseteq T$. This contradicts the choice of P. Then, $RaR+P \subseteq M$ for some $M \in Max(R)$. Similarly, $RbR+P \subseteq N$ for some $N \in Max(R)$. It follows that $P \subseteq M \cap N$. But $M \neq N$; otherwise, $1 = a+b \in M = N$. This deduces that R is not a pm ring. Accordingly, $0 \in AB$, and therefore $(1+ar)(1+bs) = 0$ for some $r, s \in R$.

(3) $\Rightarrow$ (2) For any distinct $M, N \in Max(R)$, we have $M+N = R$. Write $x+y=1$ with $x \in M$ and $y \in N$. By hypothesis, there exist $r, s \in R$ such that $(1+xr)(1+ys) = 0$. Set $a = 1+xr$ and $b = 1+ys$. Then $ab = 0$ with $a \notin M$ and $b \notin N$.

(2) $\Rightarrow$ (1) Let $P \in Spec(R)$ be contained in two distinct maximal ideals M and N. Then there exists $a \notin M$ and $b \notin N$ such that $ab = 0$. As P is prime, we get $a \in P$ or $b \in P$. If $a \in P$, then $a \in M$, a contradiction. If $b \in P$, then $b \in N$, a contradiction. Therefore P is contained only one maximal, as required. □

Corollary 12.3.4. *A direct product of pm rings is a pm ring.*

Proof. Let $R = \prod_i R_i$, where each R_i is a pm ring. Let $a = (a_i), b = (b_i) \in R$ be such that $a+b = 1_R$. Then $a_i + b_i = 1_{R_i}$. By virtue of Theorem 12.3.3, we can find $r_i, s_i \in R_i$ such that $(1_{R_i}+a_ir_i)(1_{R_i}+b_is_i) = 0$. Choose $r = (r_i)$ and $s = (s_i)$. Then $(1_R+ar)(1_R+bs) = 0$. By using Theorem 12.3.3 again, R is a pm ring. □

Let X be a topological space. If p is a point in X, a neighbourhood of p is a subset W of X, which includes an open set U containing p, $p \in U \subseteq W$. Two points x and y in X can be separated by neighbourhoods if there exists a neighbourhood U of x and a neighbourhood V of y such that U and V are disjoint ($U \cap V = \emptyset$). X is a Hausdorff space if any two distinct points of X can be separated by neighborhoods.

Corollary 12.3.5. *Let R be a pm ring. Then $Max(R)$ is Hausdorff.*

Proof. Take $M, N \in Max(R), M \neq N$. By virtue of Theorem 12.3.3, we can find $a \notin M, b \notin N$ such that $ab = 0$. Thus, $M \in D(a), N \in D(b)$ and $D(a) \cap D(b) = D(ab) = D(0) = \emptyset$, as required. □

Note that $Max(R)$ is Hausdorff does not imply R is a pm ring as $D(a) \cap D(b) = \emptyset$ in $Max(R)$ implies only that $ab \in J(R)$. But if $J(R)$ is nil, then the converse of Corollary 12.3.5 holds.

A topological space X is a normal space if, given any disjoint closed sets E and F, there are open neighbourhoods U of E and V of F that are also disjoint.

Proposition 12.3.6. *Let R be a ring. Then the following are equivalent:*

(1) *R is a pm ring.*
(2) *$Spec(R)$ is normal.*

Proof. (1) ⇒ (2) Take $V(I), V(J) \subseteq Spec(R), V(I) \cap V(J) = V(I + J) = \emptyset$. Then $I + J = R$. Write $a + b = 1$ with $a \in I, b \in J$. In view of Theorem 12.3.3, there exist $s, t \in R$ such that $(1 + as)(1 + bt) = 0$. Thus, $E(1 + as) \cap E(1 + bt) = E\big((1 + as)(1 + bt)\big) = E(0) = \emptyset$. As $1 + as \notin I$, we get $I \subseteq E(1 + as)$. Likewise, $J \subseteq E(1 + bt)$. Therefore $Spec(R)$ is normal.

(2) ⇒ (1) Let $M, N \in Max(R)$ be distinct. Then $M + N = R$. Write $a + b = 1$ with $a \in M, b \in N$, then $V(a) \cap V(b) = V(aR + bR) = V(R) = \emptyset$. Further, $M \in V(a)$ and $N \in V(b)$. As $Spec(R)$ is normal, we have some open neighbourhoods U of $V(a)$ and V of $V(b)$ that are also disjoint. Write $U = E(I)$ and $V = E(J)$. Then $E(I) \cap E(J) = E(IJ) = \emptyset$. Hence, $IJ \subseteq P(R)$. This shows that IJ is nil. Say $(IJ)^n = 0 (n \in \mathbb{N})$. But $M \in V(a) \subseteq E(I)$, we see that $I \nsubseteq M$. Similarly, $J \nsubseteq N$. Take $c \in I, c \notin M$ and $d \in J, d \notin N$. Then $c^n \notin M, d^n \notin N$, while $c^n d^n = (cd)^n = 0$. In light of Theorem 12.3.3, R is a pm ring. □

For two idempotents e and f in a ring R the meet is given by $e_1 \vee e_2 = e_1 + e_2 - e_1 e_2$. $e_1 \vee \cdots \vee e_n$ is defined by induction, i.e., $e_1 \vee \cdots \vee e_n = \big(e_1 \vee \cdots \vee e_{n-1}\big) \vee e_n (n \geq 2)$.

Lemma 12.3.7. *Let R be a ring, let $r_1, \ldots, r_n \in R$, and let $e_1, \ldots, e_n \in R$ be idempotents. Then*

$$\sum_{i=1}^{n} r_i e_i = tz, \text{ where } z = e_1 \vee \cdots \vee e_n.$$

Proof. It is easy to verify that

$$r_1 e_1 + r_2 e_2 = \big(r_1 e_1 + r_2 e_2\big)\big(e_1 + (1 - e_1)e_2\big).$$

Thus, the result holds for $n = 2$. By induction, we complete the proof. □

As proved before, every clean ring is a pm ring. Analogously to Theorem 12.3.3, we could characterize clean rings as follows.

Theorem 12.3.8. *Let R be a ring. Then the following are equivalent:*

(1) *R is a clean ring.*
(2) *For any distinct $M, N \in Max(R)$, there exists an idempotent $e \in R$ such that $e \in M$ and $1 - e \in N$.*
(3) *$a + b = 1$ in R implies that there exist $r, s \in R$ such that $1 + ar \in eR, 1 + bs \in (1 - e)R$ for an idempotent $e \in R$.*

Proof. (1) ⇒ (2) For any distinct $M, N \in Max(R)$, $M + N = R$. Write $a + b = 1$ with $a \in M, b \in N$. In view of Lemma 7.1.9, there exists an idempotent $e \in aR$ such that $1 - e \in (1 - a)R$. Hence, $e \in M$ and $1 - e \in bR \subseteq N$.

(2) ⇒ (3) For any distinct $M, N \in Max(R)$, there exists an idempotent $e \in R$ such that $e \in M$ and $1 - e \in N$. Hence, $e \in M \backslash N, 1 - e \in N \backslash M$ and $e(1 - e) = 0$. Let $a + b = 1$ in R. Let $C = \{M \in Max(R) \mid a \in M\}$ and $D = \{N \in Max(R) \mid b \in N\}$. For every pair $(M, N) \in C \times D$, there exist two idempotents $e_M^N \in M \backslash N$ and $e_N^M \in N \backslash M$ such that $e_M^N e_N^M = 0$. Fix M and let N run over D. We claim that the ideal I generated by b and all the idempotents e_M^N is R; otherwise, $I \subseteq N$ for some $N \in D$. Hence, $e_M^N \in N$ and this is impossible. Therefore, by Lemma 12.3.7,

$$1 + \lambda_M b = \sum_{i=1}^{m} \mu_i e_M^{N_i} = h_M z_M, \text{ where } z_M = e_M^{N_1} \vee \cdots \vee e_M^{N_m}.$$

Set $y_M = \prod_{i=1}^{m} e_{N_i}^M$. Then

$$D \subseteq D(z_M), M \in D(y_M), y_M z_M = 0.$$

Let M run over C. We claim that the ideal J generated by a and the $y'_M s$ is R. Thus,

$$1 + ar = \sum_{i=1}^{n} \nu_i y_{M_i} = ke \in eR, \text{ where } e = y_{M_1} \vee \cdots \vee y_{M_n}.$$

Let $f = \prod_{i=1}^{n} z_{M_i}$. Then $f \in R$ is an idempotent and $ef = 0$. Furthermore, $D \subseteq \bigcap_{i=1}^{m} D(z_{M_i}) = D(f)$. Therefore, we get

$$\prod_{i=1}^{n} (1 + \lambda_{M_i} b) = \prod_{i=1}^{n} h_{M_i} z_{M_i};$$

hence, $1 + bs = lf$ for some $b, l \in R$. As $ef = 0$, we get $1 + bs = lf(1 - e) \in (1 - e)R$, proving (3).

(3) $\Rightarrow$ (1) For any $a \in R$, there exist $r, s \in R$ such that $1 + ar \in eR, 1 + (1 - a)s \in (1 - e)R$ for an idempotent $e \in R$. Write $1 + ar = ec$ and $1 + (1 - a)s = (1 - e)d$. Then $1 - e = -ar(1 - e) \in aR$ and $e = -(1 - a)se \in (1 - a)R$. This means that R is an exchange ring, proving (1) by Corollary 1.1.9. □

Corollary 12.3.9. *Let R be a ring. Then the following are equivalent:*

(1) *R is a clean ring.*
(2) *For any distinct $M, N \in Max(R)$, there exists an idempotent $e \in R$ such that $e \in M$ and $e \notin N$.*

Proof. (1) $\Rightarrow$ (2) For any distinct $M, N \in Max(R)$, there exists an idempotent $e \in R$ such that $e \in M$ and $1 - e \in N$. This shows that $e \notin N$, as required.

(2) $\Rightarrow$ (1) For any distinct $M, N \in Max(R)$, there exists an idempotent $e \in R$ such that $e \in M$ and $e \notin N$. As $N \in Max(R)$, N is prime. Since $e(1 - e) = 0 \in N$, we get $1 - e \in N$, as asserted. □

Let I be an ideal in a ring R. The "pure part" of I is the set

$$\varepsilon(I) = \{x \in I \mid \exists\ \eta \in I \text{ such that } x = x\eta\}.$$

Proposition 12.3.10. *Let I be an ideal of a ring R. Then $\varepsilon(I)$ is an ideal of R.*

Proof. It suffices to show that $\varepsilon(I)$ is closed under the addition. Let $a, b \in \varepsilon(I)$. Then we have some $\zeta, \eta \in I$ such that $a = a\zeta$ and $b = b\eta$.

Hence, $(a+b)(1-\zeta)(1-\eta) = a(1-\zeta)(1-\eta)+b(1-\eta)(1-\zeta) = 0$, and then $(a+b)(1-\zeta-\eta+\zeta\eta) = 0$. This implies that $(a+b)(\zeta+\eta-\zeta\eta) = a+b$, hence that $a+b \in \varepsilon(I)$. Therefore $\varepsilon(I)$ is an ideal of R. □

Proposition 12.3.11. *Let I be an ideal of a ring R. Then*

$$\varepsilon(I) = \bigcap\{\varepsilon(M) \mid M \in Max(R), I \subseteq M\}.$$

Proof. Obviously, $\varepsilon(I) \subseteq \bigcap\{\varepsilon(M) \mid M \in Max(R), I \subseteq M\}$, as $\varepsilon(I) \subseteq I$. Suppose $x \notin \varepsilon(I)$. Then $r(x) \neq R$; otherwise, $x = 0 \in \varepsilon(I)$, a contradiction. Thus, $r(x) \subseteq M$ for some $M \in Max(R)$. If $x \in \varepsilon(M)$, then $x = x\zeta$ for $\zeta \in M$. This implies that $1-\zeta \in r(x)$. Hence, $R = M+r(x) \subseteq M$, which is impossible. Therefore $x \notin \varepsilon(M)$, and so $x \notin \bigcap\{\varepsilon(M) \mid M \in Max(R), I \subseteq M\}$. This completes the proof. □

Proposition 12.3.12. *Let I be an ideal of a pm ring R. If $M \in Max(R)$, then*

$$\varepsilon(M) \subseteq I \Rightarrow I \subseteq M.$$

Proof. Suppose $\varepsilon(M) \subseteq I$. If $I \nsubseteq M$, then we can find some $N \in Max(R)$ such that $I \subseteq N$ with $M \neq N$. By virtue of Theorem 12.3.3, we have some $a \notin M, b \notin N$, while $ab = 0$. Hence $M + aR = R$. Write $s + at = 1$ with $s \in M, t \in R$. Then $sb = b$, and so $b \in \varepsilon(M) \subseteq I \subseteq N$. This is impossible. Accordingly, $I \subseteq M$, as asserted. □

Let R be a pm ring. As a consequence, we deduce that $M = N$ if and only if $\varepsilon(M) = \varepsilon(N)$ for any $M, N \in Max(R)$.

Theorem 12.3.13. *Let R be a ring. Then the following are equivalent:*

(1) *R is a pm ring.*
(2) *For any ideals I, J, $I + J = R \Rightarrow \varepsilon(I) + J = R$.*
(3) *For any ideals I, J, $\varepsilon(I + J) = \varepsilon(I) + \varepsilon(J)$.*

Proof. (1) ⇒ (2) Let I and J be ideals of R. If $I+J = R$, then $a+b = 1$ for some $a \in I, b \in J$. In light of Theorem 12.3.3, we can find $s, t \in R$ such that $(1-as)(1-bt) = 0$. Thus, $as(1-bt)+bt = 1$. Clearly, $as(1-bt)(1-as) = 0$ and $as \in I$, and so $as(1-bt) \in \varepsilon(I)$. Thus, $1 \in \varepsilon(I) + J$, as needed.

(2) ⇒ (1) Given $a + b = 1$ in R, then $aR + bR = R$. By hypothesis, $\varepsilon(aR) + bR = R$. Write $x + y = 1$, where $x \in \varepsilon(aR)$ and $y \in bR$. Hence, $1 - y = x \in \varepsilon(aR)$. This means that $(1 - y)(1 - z) = 0$ for $z \in aR$. Write

$z = -as, y = -bt$ for some $s, t \in R$. Then $(1 + as)(1 + bt) = 0$, proving (1) by Theorem 12.3.3.

(3) $\Rightarrow$ (2) Let I and J be ideals of R. If $I + J = R$, then $\varepsilon(I + J) = \varepsilon(R) = R$. Hence $R = \varepsilon(I) + \varepsilon(J) \subseteq \varepsilon(I) + J$, as required.

(2) $\Rightarrow$ (3) Let I and J be ideals of R. If $a \in \varepsilon(I)$ and $b \in \varepsilon(J)$, then there exist some $\zeta \in I, \eta \in J$ such that $a(1 - \zeta) = 0 = b(1 - \eta)$. Hence, $(a + b)(1 - \zeta)(1 - \eta) = a(1 - \zeta)(1 - \eta) + b(1 - \eta)(1 - \zeta) = 0$. That is, $a + b = (a + b)(\zeta + \eta - \zeta\eta)$ with $\zeta + \eta - \zeta\eta \in I + J$. Thus $a + b \in \varepsilon(I + J)$. Accordingly, $\varepsilon(I+J) \supseteq \varepsilon(I)+\varepsilon(J)$. To prove the converse inclusion, consider $x \in \varepsilon(I + J)$. Then $x = x(c + d)$ with $c \in I, d \in J$, and so $1 - c - d \in r(x)$. This implies that $I+J+r(x) = R$, and an iterated application of (2) yields $\varepsilon(I)+\varepsilon(J)+r(x) = R$. Write $1 = y+z+s$ with $y \in \varepsilon(I), z \in \varepsilon(J), s \in r(x)$. Then $x = xy + xz \in \varepsilon(I) + \varepsilon(J)$, as desired. □

For each $M \in Max(R)$, let $O_M = \bigcap\{P \in Spec(R) \mid P \subseteq M\}$.

Corollary 12.3.14. *Let R be a ring. Then the following are equivalent:*

(1) *R is a pm ring.*
(2) *For each $M \in Max(R)$, M is the unique maximal ideal containing O_M.*

Proof. (1) $\Rightarrow$ (2) Let $M \in MaxR$. Then $O_M \subseteq M$. Suppose that there is some maximal ideal N such that $O_M \subseteq N$. If $M \neq N$ then there exists $a \notin M$ and $b \notin N$ such that $ab = 0$, and then the maximality of M gives $M + aR = R$. In virtue of Theorem 12.3.13, $\varepsilon(M) + aR = R$. Write $x + ar = 1$ with $x \in \varepsilon(M), r \in R$. Then $x = xs$ for some $s \in M$. Thus, $b = (x + ar)b = xb$. For any prime ideal P contained in M, we see that $1 - s \notin P$; otherwise, $1 = (1 - s) + s \in P + M \subseteq M$, which is absurd. As $x(1 - s) = 0 \in P$, we have $x \in P$. This implies that $b = xb \in P$. Thus, $b \in O_M$, a contradiction. Therefore M is the unique maximal ideal containing O_M.

(2) $\Rightarrow$ (1) Let P be a prime ideal of R. Then there is a maximal ideal M such that $P \subseteq M$. Hence, $O_M \subseteq P \subseteq M$. If there is some maximal N such that $P \subseteq N$. Then $O_M \subseteq N$. By hypothesis, $M = N$, as required. □

Lemma 12.3.15. *Let M be a maximal ideal of a ring. Then*

$$O_M = \{x \in R \mid \exists y \in R\backslash M \text{ such that } xy \text{ is nilpotent}\}.$$

Proof. Obviously, $O_M \supseteq \{x \in R \mid \exists y \in R\backslash M$ such that xy is nilpotent$\}$.

To prove the converse, let $x \in O_M$. Suppose $xy \in R$ is not nilpotent for any $y \notin M$. Let $S = \bigcup_{y \notin M} \{y, xy, x^2y, \ldots, x^m y, \ldots\}$, and let

$$\Omega = \{I \lhd R \mid S \cap I = \emptyset\}.$$

By using Zorn's Lemma, there exists an ideal P which is maximal in Ω. If P is not prime, there are $c, d \in R$ such that $c, d \notin P$, while $cd \in P$. This implies that $x^s y \in cR + P$ and $x^t z \in dR + P$ for some $s, t \in \mathbb{N}$. It follows that $x^{s+t}(yz) \in P$, which is impossible. Thus P is prime. If $P \not\subseteq M$, then we have $b \in P$ and $b \notin M$. This yields that $b \in S \cap P$, a contradiction. Hence $P \subseteq M$, and so $x \in P$. This shows that $xy \in P$, and then $S \cap P \neq \emptyset$, an absurd. Consequently, the proof is true. □

Theorem 12.3.16. *Let R be a ring. Then the following are equivalent:*

(1) *R is a pm ring.*
(2) *For each $M \in Max(R)$, $\varphi_M : R \to R_M, r \mapsto \frac{r}{1}$ is surjective.*

Proof. (1) $\Rightarrow$ (2) Let $M \in Max(R)$. For any $s \in R \backslash M$, we see that $s \notin M$; hence, $sR + M = R$. Write $sx + y = 1$ with $x \in R, y \in M$. In view of Theorem 12.3.3, there exist $c, d \in R$ such that $(1 + yc)(1 - sxd) = 0$. Clearly, $1 + yc \in R \backslash M$, and so $\frac{1}{s} = \frac{xd}{1}$. Therefore $\frac{r}{s} = \varphi_M(rxd)$ for any $\frac{r}{s} \in R_M$, as desired.

(2) $\Rightarrow$ (1) Let $M \in Max(R)$, and let $\varphi_M : R \to R_M, r \mapsto \frac{r}{1}$. By hypothesis, $R_M \cong R/Ker(\varphi_M)$. In view of Lemma 12.3.15, we get $Ker(\varphi_M) = \{x \in R \mid \exists\ y \notin M \text{ such that } xy = 0\} \subseteq O_M$. Therefore, we have

$$R/O_M \cong R/Ker(\varphi_M)/O_M/Ker(\varphi_M);$$

hence, R/O_M is local. This means that M/O_M is the unique maximal ideal of R/O_M. That is, M is the only maximal containing O_M. According to Corollary 12.3.14, R is a pm ring. □

Corollary 12.3.17. *Let R be a ring. Then the following are equivalent:*

(1) *R is a pm ring.*
(2) *For each $M \in Max(R)$, R_M is a finitely generated R-module.*

Proof. (1) $\Rightarrow$ (2) In view of Theorem 12.3.16, $\varphi_M : R \to R_M$ is surjective. Hence, $R_M \cong R/Ker(\varphi_M) \cong R\big(1 + Ker(\varphi_M)\big)$. Thus R_M is a finitely generated R-module.

(2) $\Rightarrow$ (1) Let $M \in Max(R)$. Then we have $\varphi_M : R \to R_M$ given by $\varphi_M(r) = \frac{r}{1}$. We will prove that φ_M is an R-epimorphism. We assume that

R_M is generated by $\{\frac{1}{s_1}, \ldots, \frac{1}{s_m}\}$. Set $s = s_1 \cdots s_m$. Then R_M is generated by $\frac{1}{s}$. As $s \notin M$, we get $sR + M = R$. Write $sr + x = 1$ with $r \in R, x \in M$. Then $r \notin M$. By hypothesis, there exists $c \in R$ such that $\frac{1}{s^2 r} = \frac{c}{s}$; hence, $t(s - s^2 rc) = 0$ for some $t \notin M$. That is, $ts(1 - src) = 0$ with $ts \notin M$. This implies that $\frac{1}{s} = \frac{rc}{1}$, and so $\varphi_M : R \to R_M$ is an R-epimorphism. Therefore we complete the proof by Theorem 12.3.16. □

Corollary 12.3.18. *A pm ring is a subdirect product of local rings.*

Proof. Let R be a pm ring. For any $M \in Max(R)$, there exists an epimorphism $\varphi_M : R \to R_M$, by Theorem 12.3.16. Hence $R/Ker(\varphi_M) \cong R_M$ is local. Let $0 \neq x \in \bigcap_{M \in Max(R)} Ker(\varphi_M)$. Then $r(x) \lhd R$, and so $r(x) \subseteq N$ for some $N \in Max(R)$. But $x \in Ker(\varphi_N)$; hence, $xt = 0$ for some $t \in R \backslash N$. This implies that $t \in r(x) \subseteq N$, this is impossible. Therefore $\bigcap_{M \in Max(R)} Ker(\varphi_M) = 0$, as required. □

Let I be an ideal of a ring R. There is a smallest radical ideal containing I, namely, the intersection of all prime ideals $\supseteq I$. This radical ideal is denoted by $\sqrt{I}$. As is well known, $\sqrt{I} = \{x \in R \mid x^n \in I \text{ for some } n \in \mathbb{N}\} = \bigcap\{P \lhd R \mid P \text{ is a minimal prime ideal containing } I\}$.

Corollary 12.3.19. *Let R be a ring with a finite number of minimal prime ideals (e.g., R is noetherian). Then the following are equivalent:*

(1) *R is a pm ring.*
(2) *R is a clean ring.*
(3) *R is a finite direct product of local rings.*

Proof. (1) ⇒ (3) Clearly, every maximal ideal contains a minimal prime ideal. As R is a pm ring, R has only finitely many maximal ideals. Say $Max(R) = \{M_1, \ldots, M_n\}$. Let $\varphi_{M_i} : R \to R_{M_i}$ given by $r \mapsto \frac{r}{1}$. Then φ_{M_i} is an epimorphism, by Theorem 12.3.16. Hence, $R/I_i \cong R_{M_i}$, where $I_i = Ker(\varphi_{M_i})$. Therefore each R/I_i is local.

As in the proof of Corollary 12.3.18, $\bigcap_{i=1}^{n} I_i = 0$. Let $P_1, \ldots, P_s$ be all minimal prime ideals contained in M_i. For any k, $P_k \nsubseteq M_j$, and so $P_k + M_j = R$. Write $p_k + c_k = 1$ with $p_k \in P_k, c_k \in M_j$. Then $p_1 p_2 \cdots p_s + c = 1$ for some $c \in M_j$. We claim that

$$\begin{aligned}\sqrt{I_i} &= \bigcap\{P \lhd R \mid P \text{ is minimal prime ideal containg } I_i\} \\ &= \bigcap\{P \lhd R \mid P \text{ is minimal prime ideal contained in } M_i\}.\end{aligned}$$

If P is a minimal prime ideal contained in M_i and $z \in I_i$, then $yz = 0$ for some $y \in R\backslash M_i$. It follows from $yz \in P$ and $y \notin P$ that $z \in P$. Hence, $I_i \subseteq P$. If P is a minimal prime ideal containing I_i, it will suffice to prove that $P \subseteq M_i$. If not, $P+M_i = R$, and so $p+m = 1$ for some $p \in P, m \in M_i$. By Theorem 12.3.3, we can find some $s, t \in R$ such that $(1+ps)(1+mt) = 0$. As $1 + mt \in R\backslash M_i$, we see that $1 + ps \in I_i \subseteq P$. This implies that $1 \in P$, a contradiction. Therefore $P \subseteq M_i$. Write $\left(p_1p_2\cdots p_s\right)^l \in I_i$, we get $1 = \left(p_1p_2\cdots p_s + c\right)^l \in I_i + M_j$. This implies that $I_i + M_j = R$ for $j \neq i$, i.e., $I_i+M_1 = R, \ldots, I_i+M_{i-1} = R, I_i+M_{i+1} = R, \ldots, I_i+M_n = R$. Thus, $I_i + \prod_{j\neq i} M_j = R$. Similarly, $I_j + M_i = R$. It follows that $I_i + \left(\prod_{j\neq i} M_j\right)\left(I_j + M_i\right) = R$; whence, $I_i+I_j+J(R) = R$. Write $a_i+b_j+r = 1$ for $a_i \in I_i, b_j \in I_j$ and $r \in J(R)$. Then $a_i(1-r)^{-1}+b_j(1-r)^{-1} = 1$, and so $I_i+I_j = R(i \neq j)$. By the Chinese Remainder Theorem, $R \cong R/I_1 \oplus \cdots \oplus R/I_n$ is a finite direct product of local rings.

(3) $\Rightarrow$ (2) This is obvious as every local ring is clean.

(2) $\Rightarrow$ (1) Since every clean ring is a pm ring, we are through. □

An ideal I of a ring R is called pure if $I = \varepsilon(I)$, i.e., for any $x \in I$ there exists $y \in I$ such that $x = xy$.

Lemma 12.3.20. *Let M be a maximal ideal of a commutative reduced ring. If R is a pm ring, then O_M is pure.*

Proof. Let $x \in O_M$. In view of Lemma 12.3.15, there exists $y \notin M$ such that $xy = 0$. Hence, $yR + M = R$. Write $yz + c = 1$ with $z \in R, c \in M$. Then $(1 + yzs)(1 + ct) = 0$ for some $s, t \in R$, by Theorem 12.3.3. Clearly, $1+ct \notin M$. By virtue of Lemma 12.3.15, $1+yzs \in O_M$. But $x = x(1+yzs)$, hence that $x \in \varepsilon(O_M)$, which completes the proof. □

Proposition 12.3.21. *Let I be an ideal of a reduced ring R. If R is a pm ring, then the following are equivalent:*

(1) *I is pure.*
(2) *$I = \bigcap\{O_M \mid M \in Max(R), I \subseteq M\}$.*

Proof. (1) $\Rightarrow$ (2) Let $M \in Max(R)$ with $I \subseteq M$, and let $P \in Spec(R)$ with $P \subseteq M$. For any $x \in I$, there exists $y \in I$ such that $x = xy$; hence, $x(1 - y) = 0$. This implies that $1 - y \notin P$; otherwise, $1 = (1 - y) + y \in M$. As P is prime, we have $x \in P$. Therefore $x \in O_M$, and so $I \subseteq \bigcap\{O_M \mid M \in Max(R), I \subseteq M\}$.

Let $x \notin I$, and let $S = \{c \in R \mid x = xc\}$. Clearly, $S \cap I = \emptyset$. Set

$$\Omega = \{J \lhd R \mid I \subseteq J, S \cap J = \emptyset\}.$$

Then Ω is an inductive nonempty set. By using Zorn's Lemma, there exists an ideal P of R which is maximal in Ω. If P is not prime, there are $a, b \notin P$, while $ab \in P$. Then $S \cap (aR + P), S \cap (bR + P) \neq \emptyset$. Thus, we have $c, d \in S$ such that $c \in aR + P$ and $d \in bR + P$. Hence $cd \in (aR + P)(bR + P) \subseteq P$. But $x = xc = (xd)c = x(cd)$, hence that $cd \in S$. This is impossible as $S \cap P = \emptyset$. Consequently, P is prime, and so we have a maximal ideal M of R such that $I \subseteq P \subseteq M$. We shall show that $x \notin O_M$. If not, by virtue of Lemma 12.3.15, we have $y \notin M$ such that $xy = 0$. As $yR + M = R$, we have some $s \in R, t \in M$ such that $ys + t = 1$. Since $ys + (1 - ys) = 1$, we can find some $k, l \in R$ such that $(1 + ysk)\big(1 + (1 - ys)l\big) = 0$, in terms of Theorem 12.3.3. Clearly, $1 + (1 - ys)l = 1 + tl \notin M$, and then $1 + ysk \in P$. But $x(1+ysk) = x$, and so $1+ysk \in S \cap P$, which is absurd. Thus, $x \notin O_M$. Therefore $I \supseteq \bigcap\{O_M \mid M \in Max(R), I \subseteq M\}$, proving (2).

(2) $\Rightarrow$ (1) Let $x \in I$. For each maximal M containing I, $x \in O_M$; hence, there exists $y_M \in O_M$ such that $x = xy_M$ by Lemma 12.3.20. In view of Lemma 12.3.15, there exists $z_M \notin M$ such that $y_M z_M = 0$. As $M + z_M R = R$, we get $\varepsilon(M) + z_M R = R$, by Theorem 12.3.13. Write $s + z_M t = 1$ with $s \in \varepsilon(M), t \in R$. Then $x = xy_M = xy_M(s + z_M t) = xy_M s = xs$. Write $s = sc$ for some $c \in M$. For any prime ideal P contained in M, we have $s(1 - c) = 0 \in P$. But $1 - c \notin P$; otherwise, $1 = (1 - c) + c \in P + M \subseteq M$. As P is prime, we get $s \in P$. Therefore, $s \in O_M$, and so $s \in I$. This implies that I is pure, as asserted. □

We include a result of Khashan and Burhan (see [349, Lemma 2.2]).

Proposition 12.3.22. *Let R be a ring. Then the following are equivalent:*

(1) *R is a pm ring.*
(2) *$R(x)$ is a pm ring.*

Proof. (1) $\Rightarrow$ (2) Let $\widehat{P}$ be a prime ideal of $R(x)$. If there exist two distinct maximal ideals $\widehat{M}, \widehat{N}$ of $R(x)$ such that $\widehat{P} \subseteq \widehat{M}, \widehat{N}$. Then there are two maximal ideals M and N of R such that $\widehat{M} = MR(x)$ and $\widehat{N} = NR(x)$. Set $P = \widehat{P} \cap R$. Then P is a prime ideal of R. As $P \subseteq MR(x) \cap R = M$ and $P \subseteq NR(x) \cap R = N$, we get a contradiction. Therefore proving (2).

(2) $\Rightarrow$ (1) Let P be a prime ideal of R, and $P \subseteq M, N$, where M, N are distinct maximal ideals of R. Then $PR(x)$ is a prime ideal of $R(x)$, $MR(x)$

and $NR(x)$ are maximal ideals of $R(x)$. It follows from $PR(x) \subseteq MR(x)$ that $MR(x) = NR(x)$; hence, $M = MR(x) \cap R = NR(x) \cap R = N$, a contradiction. This completes the proof. □

Corollary 12.3.23. *Let R be a ring with a finite number of minimal prime ideals. Then the following are equivalent:*

(1) *R is a clean ring.*
(2) *$R(x)$ is a clean ring.*

Proof. We note that there is a one to one correspondence between minimal prime ideals of R and those of $R(x)$. It follows that $R(x)$ has finite number of minimal prime ideals. According to Proposition 12.3.22 and Corollary 12.3.19, we obtain the result. □

If R be a pm ring, then so is the trivial extension $T(R,R) = \left\{ \begin{pmatrix} a & b \\ & a \end{pmatrix} \middle| \, a,b \in R \right\}$. If R is a commutative ring, the ring of complexes of R is the ring $\mathbb{C}(R)$ of matrices of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

with $a,b \in R$. This ring is obviously commutative with the identity I_2. We now derive

Example 12.3.24. Let R be a pm ring such that each element of the form $1+a^2$ is invertible. Then the ring $\mathbb{C}(R)$ of complexes of R is a pm ring.

Proof. Let $I, J \in Max\big(\mathbb{C}(R)\big)$ with $I \neq J$. Then we define

$$M = \left\{ a \in R \middle| \, \exists\, b \in R, \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in I \right\}, N = \left\{ a \in R \middle| \, \exists\, b \in R, \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in J \right\}.$$

If $1 \in M$, then $\begin{pmatrix} 1 & c \\ -c & 1 \end{pmatrix} \in I$ for some $c \in R$. Hence,

$$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & c \\ -c & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{1+c^2} & -\frac{c}{1+c^2} \\ \frac{c}{1+c^2} & \frac{1}{1+c^2} \end{pmatrix} \in I,$$

a contradiction. Thus M is an ideal of R. We claim that $M \in Max(R)$. If not, there exists an ideal P of R such that $M \subseteq P \subsetneq R$. Let

$$K = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathbb{C}(R) \middle| \, a,b \in P \right\}.$$

Then K is an ideal of $\mathbb{C}(R)$. Let $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in I$. Then $a \in M$. Further, $\begin{pmatrix} b & -a \\ a & b \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in I$, hence that $b \in M$. This shows that $a, b \in P$, and so $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in K$. Thus, $I \subseteq K \subsetneq \mathbb{C}(R)$, which is impossible. Likewise, $N \in Max(R)$. Moreover, we see that

$$I \subseteq \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathbb{C}(R) \,\middle|\, a, b \in M \right\} \subsetneq \mathbb{C}(R);$$

hence,

$$I = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathbb{C}(R) \,\middle|\, a, b \in M \right\}.$$

Likewise, we have

$$J = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathbb{C}(R) \,\middle|\, a, b \in N \right\}.$$

We infer that $M \neq N$. In view of Theorem 12.3.3, there exist $a \notin M$ and $b \notin N$ such that $ab = 0$. Choose $A = \begin{pmatrix} a & \\ & a \end{pmatrix}, B = \begin{pmatrix} b & \\ & b \end{pmatrix} \in \mathbb{C}(R)$. Then $A \notin I, B \notin J$ and $AB = 0$. Therefore $\mathbb{C}(R)$ is a pm ring, by Theorem 12.3.3. □

An element $a \in R$ is called to be PM if R/aR is a pm ring.

Lemma 12.3.25. *Let R be a Bézout domain, and let $a \in R$. Then the following are equivalent:*

(1) *$a \in R$ is PM.*
(2) *If $bR + cR = R$, then there exist $r, s \in R$ such that $a = rs, rR + bR = sR + cR = R$.*
(3) *If $aR + bR + cR = R$, then there exist $r, s \in R$ such that $a = rs, rR + bR = sR + cR = R$.*

Proof. (1) ⇒ (3) Suppose $aR + bR + cR = R$. Then $\overline{b}(R/aR) + \overline{c}(R/aR) = R/aR$. By virtue of Theorem 12.3.3, we can find some $p, q \in R$ such that $\overline{(1 + bp)(1 + cq)} = \overline{0}$ in R/aR. Set $k = 1 + bp$ and $l = 1 + cq$. Then $kR + bR = lR + cR = R$ and that $kl \in aR$.

Write $kl = ap$ for some $p \in R$. Since R is a Bézout domain, we have an $r \in R$ such that $kR + aR = rR$. Thus there are some $s, t \in R$ such that

$a = rs, k = rt$ and $sR + tR = R$. Clearly, $rR + bR = R$. It follows that $rtl = rsp$, and so $tl = sp$. Write $lu + cv = 1$. Then $tlu + ctv = spu + ctv = t$. But $sR + tR = R$, we get $sR + cR = R$, as required.

(3) $\Rightarrow$ (2) This is obvious.

(2) $\Rightarrow$ (1) Suppose $\overline{b + c} = \overline{1}$ in R/aR. Then $b + c + ax = 1$ for some $x \in R$. By hypothesis, there exist $r, s \in R$ such that $a = rs, rR + bR = sR + (c + ax)R = R$. Write $rk + bl = sp + (c + ax)q = 1$. Then $\overline{(1 - bl)(1 - cq)} = \overline{rskp} = \overline{0}$ in R/aR. Therefore R/aR is a pm ring, by Theorem 12.3.3. □

Theorem 12.3.26. *Let R be a Bézout domain. Suppose $aR + bR = R \Longrightarrow \exists\ y \in R$ such that $a + by \in R$ is PM. Then R is an elementary divisor ring.*

Proof. In view of Theorem 8.3.8, it will suffice to prove that the matrix $A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ with $aR + bR + cR = R$ admits a diagonal reduction. Write $ax + by + cz = 1$ with $x, y, z \in R$. Then we can find some $k \in R$ such that $w := b + (ax + cz)k \in R$ is PM. Hence,

$$\begin{pmatrix} 1 & 0 \\ xk & 1 \end{pmatrix} A \begin{pmatrix} 1 & 0 \\ zk & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ w & c \end{pmatrix}.$$

Clearly, $wR + aR + cR = R$. In view of Lemma 12.3.25, there exist $r, s \in R$ such that $w = rs, rR + aR = sR + cR = R$. Write $sp' + cl = 1$. Then $rsp' + rcl = r$. Set $q' = rl$. Then $wp' + cq' = r$. Write $p'R + q'R = dR$. Then $p' = dp, q' = dq$ and $pR + qR = R$. Hence $(wp + cq)R + aR = R$. Clearly, $cR + pR = qR + pR = R$, we get $cqR + pR = R$. This implies that $(wp + cq)R + pR = R$. Thus $apR + (wp + cq)R = R$. Write $aps + (wp + cq)t = 1$. Then $p(as + wt) + q(ct) = 1$, and so

$$\begin{pmatrix} s & t \\ wp + cq & -ap \end{pmatrix} \begin{pmatrix} a & 0 \\ w & c \end{pmatrix} \begin{pmatrix} p & -ct \\ q & as + wt \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix},$$

where $\begin{vmatrix} s & t \\ wp + cq & -ap \end{vmatrix} = \begin{vmatrix} p & -ct \\ q & as + wt \end{vmatrix} = 1$. Therefore A admits a diagonal reduction, as desired. □

Lemma 12.3.27. *Let I be an ideal of a pm ring R. Then R/I is PM.*

Proof. Given $\overline{a + b} = \overline{1}$ in R/I, then $a + (b + r) = 1$ in R. Hence, there exist some $s, t \in R$ such that $(1 - as)\big(1 - (b + r)t\big) = 1$. Therefore $\overline{(1 - as)(1 - bt)} = \overline{1}$, as required. □

Theorem 12.3.28. *Let R be Bézout domain. Suppose for any $a \in R$, either a or $1-a$ is PM. Then R is an elementary divisor ring.*

Proof. Given $aR + bR = R$ with $a, b \in R$, then $ax + by = 1$ for some $x, y \in R$. Hence $a(x-y) + (a+b)y = 1$. By hypothesis, either $a(1-x)$ or $1 - a(1-x)$ is PM. If $a(1-x) \in R$ is PM, then $R/a(1-x)R$ is pm. As $R/aR \cong \frac{R/a(1-x)R}{aR/a(1-x)R}$, it follows by Lemma 12.3.27 that R/aR is pm. If $(a+b)y \in R$ is PM, analogously, $R/(a+b)R$ is pm. Therefore $R/(a+by)R$ is pm where $y = 0$ or 1. In light of Theorem 12.3.26, R is an elementary divisor ring. □

Corollary 12.3.29. *Every Bézout pm domain is an elementary divisor ring.*

12.4 Feckly Clean Rings

We say that an element $a \in R$ is feckly clean provided that there exists an element $e \in R$ and a full element $u \in R$ such that $a = e + u, eR(1-e) \subseteq J(R)$. A ring R is feckly clean provided that every element in R is feckly clean. Clearly, every nearly clean element is feckly clean. But the converse is not true.

Example 12.4.1. Let $R = \{\frac{m}{n} \mid (m,n) = 1, m, n \in \mathbb{Z} \text{ and } 2, 3 \nmid n\}$. Then $Max(R) = \{2R, 3R\}$, and so $R/J(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_3$. Hence $R/J(R)$ is clean, and so R is a commutative feckly clean ring. As some idempotents do not lift modulo $J(R)$, we see that R is not clean. In fact, $4 \in R$ can not be written as the sum of an idempotent and a unit.

It is easily shown that a ring R is feckly clean if and only if $R/J(R)$ is feckly clean. The main purpose of this section is to consider such rings and characterize them in terms of topological properties.

Lemma 12.4.2. *Let R be a ring, and let M be a maximal ideal of R. Then $J(R) \subseteq P$.*

Proof. Let $x \in J(R)$. If $x \notin M$, then $RxR + M = R$. Write $s_1xt_1 + \cdots + s_nxt_n + y = 1$ where $y \in M$. Then $y = 1 - (s_1xt_1 + \cdots + s_nxt_n) \in U(R)$, a contradiction. Hence $x \in M$, and therefore $J(R) \subseteq P$. □

Lemma 12.4.3. *Let R be a ring. Then the following are equivalent:*

(1) *$a \in R$ is feckly clean.*
(2) *There exists an element $e \in R$ such that $U(a-1) \subseteq U(e) \subseteq Max(R) \setminus U(a), eR(1-e) \subseteq J(R)$.*

Proof. (1) $\Rightarrow$ (2) Let $a \in R$ be feckly clean. Then there exists an $e \in R$ and a full element $u \in R$ such that $a = e + u$ and $eR(1-e) \subseteq J(R)$. For any $P \in U(a-1)$, we see that $P \notin U(1-e)$; otherwise, $u = (a-1)+(1-e) \in P$. Clearly, $U(e) \cap U(1-e) = \emptyset$. For any $M \in Max(R)$, as $eR(1-e) \subseteq J(R)$, we see that $eR(1-e) \subseteq M$ by Lemma 12.4.2. As M is prime, either $e \in M$ or $1-e \in M$. Thus $M \in U(e)$ or $M \in U(1-e)$. Therefore $Max(R) = U(e) \sqcup U(1-e)$; hence, $P \in U(e)$. We infer that $U(a-1) \subseteq U(e)$. If $M \in Max(R)$ and $M \notin Max(R) \setminus U(a)$, then $M \in U(a)$; hence that $M \notin U(e)$; otherwise, $u = a - e \in M$. This implies that $U(e) \subseteq Max(R) \setminus U(a)$. Accordingly, $U(a-1) \subseteq U(e) \subseteq Max(R) \setminus U(a)$.

(2) $\Rightarrow$ (1) Assume that there exists an element $e \in R$ such that $U(a-1) \subseteq U(e) \subseteq Max(R) \setminus U(a), eR(1-e) \subseteq J(R)$. Let $u = a - e$. If $RuR \neq R$, by using Zorn's Lemma, there exists a maximal ideal M of R such that $RuR \subseteq M \subsetneqq R$. In light of Lemma 12.4.2, $eR(1-e) \subseteq J(R) \subseteq M$, we see that $e \in M$ or $1-e \in M$. That is, $M \in U(e)$ or $M \in U(1-e)$. If $M \in U(e)$, then $a = e + u \in M$; whence, $M \in U(a)$. But $M \in U(e) \subseteq Max(R) \setminus U(a)$, a contradiction. If $M \in U(1-e)$, then $a - 1 = (e-1) + u \in M$; whence, $M \in U(a-1) \subseteq U(e)$. This implies that $M \in U(e)$, a contradiction. Hence, $a - e \in R$ is a full element, and therefore $a \in R$ is feckly clean. □

Theorem 12.4.4. *Let R be a ring. Then the following are equivalent:*

(1) *R is feckly clean.*
(2) *For any disjoint closed sets A and B of $Max(R)$, there exists an element $e \in R$ such that $A \subseteq U(e), B \subseteq U(1-e)$ and $eR(1-e) \subseteq J(R)$.*
(3) *For any $a \in R$, there exists an element $e \in R$ such that $U(a) \subseteq U(e), U(1-a) \subseteq U(1-e)$ and $eR(1-e) \subseteq J(R)$.*

Proof. (1) $\Rightarrow$ (2) Let A and B be disjoint closed sets of $Max(R)$. Then there exist two ideals I and J such that $A = U(I)$ and $B = U(J)$. As $A \cap B = \emptyset$, $U(I) \cap U(J) = \emptyset$. If $I + J \neq R$, then there exists a maximal ideal M of R such that $I+J \subseteq M \subsetneqq R$. Hence $M \in U(I+J) = U(I) \cap U(J)$, which is absurd. Thus $I + J = R$. Write $a + b = 1$ for some $a \in I$ and $b \in J$. Since $a \in R$ is feckly clean, it follows from Lemma 12.4.3 that there

exists an element $e \in R$ such that

$$U(a-1) \subseteq U(1-e) \subseteq Max(R) \setminus U(a), (1-e)Re \in J(R).$$

Then

$$\begin{aligned} B = U(J) &\subseteq U(b) \\ &= U(a-1) \subseteq U(1-e) \subseteq Max(R) \setminus U(a) \\ &\subseteq Max(R) \setminus U(I) = Max(R) \setminus A. \end{aligned}$$

Hence $B \subseteq U(1-e)$. Since $U(1-e) \subseteq Max(R) \setminus A$, we see that $A \subseteq U(e)$, as desired.

(2) $\Rightarrow$ (3) For any $a \in R$, $U(a) \cap U(1-a) = \emptyset$. That is, $U(a)$ and $U(1-a)$ are disjoint closed sets of $Max(R)$. By hypothesis, there exists an element $e \in R$ such that $U(a) \subseteq U(e), U(1-a) \subseteq U(1-e)$ and $eR(1-e) \subseteq J(R)$.

(3) $\Rightarrow$ (1) For any $a \in R$, we have $e \in R$ such that $U(a) \subseteq U(1-e)$ and $U(1-a) \subseteq U(e)$ with $(1-e)Re \subseteq J(R)$. If $M \in Max(R)$ and $M \notin Max(R) \setminus U(a)$, then $M \in U(a)$; hence that $M \in U(1-e)$. We infer that $M \notin U(e)$. Thus $U(e) \subseteq Max(R) \setminus U(a)$. Therefore $U(a-1) \subseteq U(e) \subseteq Max(R) \setminus U(a)$. In light of Lemma 12.4.3, $a \in R$ is feckly clean. □

Corollary 12.4.5. *Let R be a ring. Then the following are equivalent:*

(1) *R is feckly clean.*
(2) *For any disjoint compact sets A and B of $Max(R)$, there exists an element $e \in R$ such that $A \subseteq U(e), B \subseteq U(1-e)$ and $eR(1-e) \subseteq J(R)$.*

Proof. (1) $\Rightarrow$ (2) Let $P, Q \in Max(R), P \neq Q$. Then there exist $a \in P, b \in Q$ such that $a + b = 1$. Since R is feckly clean, we have an element $e \in R$ and a full element $u \in R$ such that $a = e + u$ and $eR(1-e) \subseteq J(R)$. This shows that $e \notin P$. It follows from $b = 1 - a = (1-e) - u$ that $1 - e \notin Q$. As P and Q are prime, we get $1 - e \in P$ and $e \in Q$. This implies that $P \in U(1-e)$ and $Q \in U(e)$. As $Max(R) = U(e) \sqcup U(1-e)$, $U(e)$ and $U(1-e)$ are clopen sets. Hence, $Max(R)$ is an Hausdorff space. Similarly to Lemma 12.1.1, $Max(R)$ is compact. Thus every compact subset of $Max(R)$ is closed. For any disjoint compact subsets A and B of $Max(R)$, it follows from Theorem 12.4.4 that there exists an element $e \in R$ such that $A \subseteq U(e), B \subseteq U(1-e)$ and $eR(1-e) \subseteq J(R)$.

(2) $\Rightarrow$ (1) Since $Max(R)$ is compact, every closed set is compact. Therefore we complete the proof by Theorem 12.4.4. □

Lemma 12.4.6. *Let $e, f \in R$. If $eR(1-e), fR(1-f) \subseteq J(R)$, then there exists an element $g \in R$ such that $D(e) \cup D(f) = D(g)$ and $gR(1-g) \subseteq J(R)$.*

Proof. Set $g = e + f - ef$. If $M \notin D(e) \cup D(f)$, then $e, f \in M$, and so $e + f - ef \in M$. Hence $M \notin D(e + f - ef)$, and so $D(g) \subseteq D(e) \cup D(f)$. If $M \in D(e) \cup D(f)$, then $e \notin M$ or $f \notin M$. Since $eR(1-e), fR(1-f) \subseteq J(R) \subseteq M$, we see that $1-e \in M$ or $1-f \in M$. Hence, $(1-e)(1-f) \in M$, and so $1-(1-e)(1-f) \notin M$. This implies that $e+f-ef \notin M$; whence, $M \in D(g)$. We infer that $D(e) \cup D(f) \subseteq D(g)$. Accordingly, $D(e) \cup D(f) = D(g)$. Furthermore, $gR(1-g) \subseteq eR(1-e)(1-f)+(1-e)fR(1-e)(1-f) \subseteq J(R)$, as desired. □

Theorem 12.4.7. *Let R be a ring. Then the following are equivalent:*

(1) *R is feckly clean.*
(2) *For any distinct maximal ideals M and N, there exists an element $e \in R$ such that $e \in M, 1-e \in N$ and $eR(1-e) \subseteq J(R)$.*

Proof. (1) $\Rightarrow$ (2) Let M and N be distinct maximal ideals. Then $M + N = R$. Write $1 = a + b$ with $a \in M, b \in N$. Since R is feckly clean, there exists an element $e \in R$ and a full element $u \in R$ such that $b = e + u$ and $eR(1-e) \subseteq J(R)$. If $M \in U(1-e)$, then $1 - e \in M$. Hence $u = (1-e) - (1-b) \in M$, as $1 - b = a \in M$. This gives a contradiction. So $M \in U(e)$, that is, $e \in M$. Likewise, $1 - e \in N$, as desired.

(2) $\Rightarrow$ (1) Let A and B be disjoint closed sets A and B of $Max(R)$. Take $M \in B$. For any $N \in A$, then $M \neq N$. By hypothesis, there exists an element $g \in R$ such that $g \in M, 1-g \in N$ and $gR(1-g) \subseteq J(R)$. Hence $g \notin N$, and so $N \in D(g)$. Since $Max(R)$ is compact, we can find some $g_1, \ldots, g_n \in R$ such that $A \subseteq \bigcup_{i=1}^{n} D(g_i)$ and $g_i R(1-g_i) \subseteq J(R)$ for each i. By Lemma 12.4.6 and induction, we have an element $f \in R$ such that $\bigcup_{i=1}^{n} D(g_i) = D(f)$ and $fR(1-f) \subseteq J(R)$. Set $e = 1 - f$. Then $(1-e)Re \subseteq J(R)$, and so $eR(1-e) \subseteq J(R)$. Additionally, $A \subseteq D(f) = U(e)$.

Clearly, $M \in U(g_1), U(g_2)$; hence, $M \in U(g_1 + g_2 - g_1 g_2)$. By iteration of this process, $M \in U(f) = U(1-e)$. As B is compact, we can find some $e_1, \ldots, e_m \in R$ such that $A \subseteq U(e_i), B \subseteq \bigcup_{i=1}^{m} D(e_i)$ and $e_i R(1-e_i) \subseteq J(R)$ for each i. In virtue of Lemma 12.4.6, we have $\alpha \in R$ such that $B \subseteq D(\alpha) = U(1-\alpha)$. Clearly, $U(e_1) \cap U(e_2) \subseteq U(e_1 + e_2 - e_1 e_2)$. Furthermore, $A \subseteq \bigcap_{i=1}^{m} U(e_i) = U(\alpha)$. In addition, $\alpha R(1-\alpha) \subseteq J(R)$. Therefore R is feckly clean, in terms of Theorem 12.4.4. □

Corollary 12.4.8. *Suppose $a+b=1$ in $R \Longrightarrow 1+ar \in eR, 1+bs \in (1-e)R$ where $eR(1-e) \subseteq J(R)$. Then R is feckly clean.*

Proof. Let M and N be distinct maximal ideals of R. Then $M+N=R$. Write $a+b=1$ with $a \in M, b \in N$. By hypothesis, there exist $r, s \in R$ such that $1+ar \in eR, 1+bs \in (1-e)R$, where $eR(1-e) \subseteq J(R)$. As $a \in M$, we have that $e \notin M$. It follows from $eR(1-e) \subseteq J(R) \subseteq M$ that $1-e \in M$. Similarly, $1-e \notin N$, and then $e \in N$. Therefore R is feckly clean, by Theorem 12.4.7. □

Let $J\text{-}spec(R) = \{P \in Spec(R) \mid J(R) \subseteq P\}$. As is well known, the Jacobson radical $J(R)$ is semiprime, and so $J(R)$ is the intersection of some prime ideals. Hence, $J(R) = \bigcap_{P \in J\text{-}spec(R)} P$. Let I be an ideal of R, and let $F(I) = \{P \in J\text{-}spec(R) \mid I \not\subseteq P\}$. Then $F(R) = J\text{-}spec(R), F(0) = \emptyset, F(I) \cap F(J) = F(IJ)$ and $\bigcup_i F(I_i) = F\left(\sum_i I_i\right)$. So $J\text{-}spec(R)$ is a topological space, where $\{F(I) \mid I \trianglelefteq R\}$ is the collection of its open sets. Let $W(I) = J\text{-}spec(R) \setminus F(I)$. Then $W(I) = \{P \in J\text{-}spec(R) \mid I \subseteq P\}$ is the collection of its closed sets. We use $W(a)$ to stand for $W(RaR)$ for any $a \in R$.

Lemma 12.4.9. *Let R be a ring. Then $a \in R$ is feckly clean if and only if there exists an element $e \in R$ such that $W(a-1) \subseteq W(e) \subseteq J\text{-}spec(R) \setminus W(a), eR(1-e) \subseteq J(R)$.*

Proof. Let $a \in R$ be feckly clean. Then there exists an element $e \in R$ and a full $u \in R$ such that $a = e+u$ and $eR(1-e) \subseteq J(R)$. For any $P \in J\text{-}spec(R)$, as $eR(1-e) \subseteq J(R) \subseteq P$, we see that $e \in P$ or $1-e \in P$. Thus $P \in W(e)$ or $P \in W(1-e)$. Therefore $J\text{-}spec(R) = W(e) \sqcup W(1-e)$. As in the proof of Lemma 12.4.3, $W(a-1) \subseteq W(e) \subseteq J\text{-}spec(R) \setminus W(a)$, as required.

Conversely, assume that there exists an element $e \in R$ such that $W(a-1) \subseteq W(e) \subseteq J\text{-}spec(R) \setminus W(a), eR(1-e) \subseteq J(R)$ Let $u = a-e$. If $RuR \neq R$, by using Zorn's Lemma, there exists a maximal ideal P such that $RuR \subseteq P \subsetneqq R$. By virtue of Lemma 12.4.2, $eR(1-e) \subseteq J(R) \subseteq P$; hence, $e \in P$ or $1-e \in P$. Thus, either $P \in W(e)$ or $P \in W(1-e)$. As in the proof of Lemma 12.4.3, $a-e \in R$ is a full element, and so $a \in R$ is feckly clean. □

Lemma 12.4.10. *Let R be a ring. If A is a clopen subset of J-spec(R), then there exists an element $e \in R$ such that $A = W(e), eR(1-e) \subseteq J(R)$.*

Proof. Let A be a clopen subset of J-spec(R). Then we can find ideals I and J of R such that $A = W(I)$ and J-spec$(R) = W(I) \sqcup W(J)$. Thus, $W(IJ) = J$-spec(R). This shows that $IJ \subseteq \bigcap_{P \in Spec(R)} \{P \mid J(R) \subseteq P\} = J(R)$. It follows from $W(I) \cap W(J) = \emptyset$ that $W(I+J) = \emptyset$. If $I + J \neq R$, then there exists a maximal ideal P of R such that $I + J \subseteq P \subsetneqq R$. By Lemma 12.4.2, $J(R) \subseteq P$; hence, $P \in J$-spec(R). This implies that $P \in W(I + J)$, which is absurd. Thus $I + J = R$. Write $1 = e + f, e \in I, f \in J$. Then

$$J\text{-spec}(R) = W(e) \sqcup W(f), W(I) \subseteq W(e), W(J) \subseteq W(f).$$

As J-spec$(R) = W(I) \sqcup W(J)$, we deduce that $W(I) = W(e)$, hence the result. □

Theorem 12.4.11. *A ring R is feckly clean if and only if J-spec(R) is strongly zero-dimensional.*

Proof. $\Longrightarrow$ Let A and B be disjoint closed sets of J-spec(R). Then $A \cap B = \emptyset$. Clearly, there exist two ideals I and J such that $A = W(I)$ and $B = W(J)$. Then $I + J = R$. Write $a + b = 1$ with $a \in I$ and $b \in J$. Since $a \in R$ is feckly clean, by virtue of Lemma 12.4.9, there exists an element $e \in R$ such that

$$W(a-1) \subseteq W(1-e) \subseteq J\text{-spec}(R) - W(a), (1-e)Re \subseteq J(R).$$

Clearly,

$$\begin{aligned} B = W(J) &\subseteq W(b) \\ &= W(a-1) \subseteq W(1-e) \subseteq J\text{-spec}(R) \setminus W(a) \\ &\subseteq J\text{-spec}(R) \setminus W(I) = J\text{-spec}(R) \setminus A. \end{aligned}$$

Obviously, $B \subseteq W(1-e)$. As $W(1-e) \subseteq J$-spec$(R) \setminus A$, we see that $A \subseteq W(e)$. Obviously, $eR(1-e) \subseteq J(R)$, and so J-spec$(R) = W(e) \sqcup W(1-e)$. Hence, $W(e)$ and $W(1-e)$ are disjoint clopen sets. Therefore J-spec(R) is strongly zero-dimensional.

$\Longleftarrow$ For any $a \in R$, $W(a)$ and $W(a-1)$ are disjoint closed sets of J-spec(R). Thus, we have two distinct clopen sets U and V such that $W(a) \subseteq U$ and $W(a-1) \subseteq V$. By virtue of Lemma 12.4.10, we can find $e, f \in R$ such that $U = W(e)$ and $V = W(f)$, where $eR(1-e), fR(1-f) \subseteq J(R)$. If $P \in J$-spec(R) and $P \notin J$-spec$(R) \setminus W(a)$, then $P \in W(a)$; hence, $P \in W(e)$. Since $W(e) \cap W(f) = W(ReR + RfR) = \emptyset$, we get

$ReR+RfR = R$. Write $\sum_{i=1}^{m} s_i e t_i + \sum_{j=1}^{n} x_j f y_j = 1$. If $P \in W(f)$, then $f \in P$; hence, $\sum_{i=1}^{m} s_i e t_i \notin P$. It follows that $e \notin P$. Since $eR(1-e) \subseteq J(R) \subseteq P$, $1 - e \in P$. Thus $P \in W(1-e)$, a contradiction. Hence $P \notin W(f)$. Accordingly, $W(a-1) \subseteq W(f) \subseteq J\text{-}spec(R) \setminus W(a)$. Therefore we complete the proof by Lemma 12.4.9. □

Recall that a ring R is a Hilbert ring if every prime ideal of the ring is an intersection of maximal ideals. For instance, any artinian ring, any ring of integers, any polynomial ring in finitely many variables over a field. A general form states that if R is a Hilbert ring, then so is any finitely generated R-algebra S.

Corollary 12.4.12. *Let R be a Hilbert ring. Then R is feckly clean if and only if $Spec(R)$ is strongly zero-dimensional.*

Proof. In view of Lemma 12.4.2, every maximal ideal of R contains the Jacobson radical $J(R)$ of R. By hypothesis, every prime ideal of R contains $J(R)$. Therefore the result follows from Theorem 12.4.11. □

Theorem 12.4.13. *A ring R is feckly clean if and only if*

(1) *$Max(R)$ is strongly zero-dimensional;*
(2) *Every prime ideal of R containing $J(R)$ is contained in a unique maximal ideal.*

Proof. Suppose R is feckly clean. In view of Theorem 12.4.4, $Max(R)$ is strongly zero-dimensional. If there exists an ideal $P \in J\text{-}spec(R)$ such that $P \subseteq M \cap N$ where M and N are distinct maximal ideals. By virtue of Theorem 12.4.7, there exists an element $e \in R$ such that $e \in M, 1-e \in N$ and $eR(1-e) \subseteq J(R)$. Since $eR(1-e) \subseteq J(R) \subseteq P$, we have $e \in P$ or $1 - e \in P$. If $e \in P$, then $e \in N$, a contradiction. If $1 - e \in P$, then $1 - e \in M$, a contradiction. Therefore every prime ideal of R containing $J(R)$ is contained in exactly one maximal ideal.

Conversely, assume (1) and (2) hold. Then there exists a map $\varphi : J\text{-}spec(R) \to Max(R)$, $\varphi(P) = M$, where M is the unique maximal ideal such that $P \subseteq M$. Let I be an ideal of R, we let $V_S(I) = \{P \in J\text{-}spec(R) \mid I \subseteq P\}$ and $U(I) := \{P \in Max(R) \mid I \subseteq P\}$. Then $\varphi\big(V_S(I)\big) = U(I)$. As in the proof of Theorem 12.1.13, we prove that $J\text{-}spec(R)$ is strongly zero-dimensional, and therefore we complete the proof, by Theorem 12.4.11. □

Corollary 12.4.14. *A ring R is feckly clean if and only if*

(1) *$Max(R)$ is strongly zero-dimensional;*
(2) *For any distinct maximal ideals M and N, there exist $a, b \in R$ such that $a \notin M, b \notin N$ and $aRb \subseteq J(R)$.*

Proof. $\Longrightarrow$ Let M and N be distinct maximal ideals of R. By virtue of Theorem 12.4.13, $Max(R)$ is strongly zero-dimensional. It follows from Theorem 12.4.7 that there exists an element $e \in R$ such that $e \in M, 1-e \in N$ and $eR(1-e) \subseteq J(R)$. Set $a = 1-e$ and $b = e$. Then $a \notin M, b \notin N$ and $aRb = (1-e)Re \subseteq J(R)$.

$\Longleftarrow$ Suppose that P containing $J(R)$ is a prime ideal such that $P \subseteq M \cap N$ where M and N are distinct maximal ideals of R. Then $aRb \subseteq J(R) \subseteq P$; hence, either $a \in P$ or $b \in P$. This implies that $a \in M$ or $b \in N$, a contradiction. Therefore every prime ideal of R containing $J(R)$ is contained in a unique maximal ideal. Therefore the proof is completed by Theorem 12.4.13. □

Corollary 12.4.15. *Suppose $a+b=1$ in $R \Longrightarrow (1+ar)R(1+bs) \subseteq J(R)$ for some $r, s \in R$. Then the following are equivalent:*

(1) *R is feckly clean.*
(2) *$Max(R)$ is strongly zero-dimensional.*

Proof. (1) $\Rightarrow$ (2) This is obvious by Theorem 12.4.13.

(2) $\Rightarrow$ (1) Let $P \in J\text{-}spec(R)$ be such that P is contained in two distinct maximal ideals M and N. Write $a+b=1$ with $a \in M, b \in N$. By hypothesis, there are some $r, s \in R$ such that $(1+ar)R(1+bs) \subseteq J(R)$. As $J(R) \subseteq P$, we see that either $1+ar \in P$ or $1+bs \in P$. If $1+ar \in P$, then $1 = (1+ar)-ar \in M$. If $1+bs \in P$, then $1 = (1+bs)-bs \in N$. In any case, we get a contradiction. Therefore every prime ideal of R containing $J(R)$ is contained in exactly one maximal ideal. According to Theorem 12.4.13, R is feckly clean. □

We come now to the demonstrations of examples of such rings.

Proposition 12.4.16. *If $R/J(R)$ is an abelian π-regular ring, then R is feckly clean.*

Proof. Let $a \in R$. Since $R/J(R)$ is π-regular. there exists an idempotent $\overline{e} \in R/J(R)$ and a unit $\overline{u} \in R/J(R)$ such that $\overline{a} = \overline{e} + \overline{u}$. As units lift

modulo $J(R)$, we may assume that $u \in U(R)$. Hence, there exists some $r \in J(R)$ such that $a = e+(u+r)$. Obviously, $u+r \in U(R)$, and so $u+r \in R$ is a full element. Since $R/J(R)$ is abelian, we have $\overline{e}\big(R/J(R)\big)(\overline{1}-\overline{e}) = \overline{0}$, i.e., $eR(1-e) \subseteq J(R)$. Accordingly, R is feckly clean. □

Corollary 12.4.17. *If for any $a \in R$, there exists some $n \geq 2$ such that $a^n - a \in J(R)$, then R is feckly clean.*

Proof. Let $a \in R$. Then we have some $n \geq 2$ such that $a^n - a \in J(R)$; hence, $\overline{a} = \overline{a^n} (n \geq 2)$. That is, $R/J(R)$ is potent. In view of Lemma 10.2.1, $R/J(R)$ is commutative. This completes the proof, by Proposition 12.4.16. □

Corollary 12.4.18. *If $R/J(R)$ is a finite commutative ring, then R is feckly clean.*

Proof. Clearly, $R/J(R)$ is π-regular, and therefore R is feckly clean by Proposition 12.4.16. □

Proposition 12.4.19. *Every right (left) quasi-duo exchange ring is feckly clean.*

Proof. Let R be a right (left) quasi-duo exchange ring. Then $R/J(R)$ is abelian. As R is an exchange ring, then so is $R/J(R)$. Hence $R/J(R)$ is an abelian exchange ring. It follows from Corollary 1.1.9 that $R/J(R)$ is clean. Let $a \in R$. Then we have an idempotent $\overline{e} \in R/J(R)$ and a unit $\overline{u} \in R/J(R)$ such that $\overline{a} = \overline{e} + \overline{u}$. We may assume that $u \in U(R)$. Then we can find $r \in J(R)$ such that $a = e + (u + r)$, where $u + r \in U(R)$. Since $R/J(R)$ is abelian, we have $eR(1 - e) \subseteq J(R)$, the result follows. □

Example 12.4.20. Let $R = \begin{pmatrix} \mathbb{Z}_4 & \mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{pmatrix}$. Then R is a right (left) quasi-duo exchange ring. By virtue of Proposition 12.4.19, R is feckly clean. In this case, R is not abelian.

Example 12.4.21. Let $p, q \neq 2$ be prime. If $(p, q) = 1$, then the ring $\mathbb{Z}_{(p)} \cap \mathbb{Z}_{(q)}$ is feckly clean, but it is not clean.

Proof. Let $R = \mathbb{Z}_{(p)} \cap \mathbb{Z}_{(q)}$. As in the proof of Theorem 9.1.7, R is a commutative ring with exactly two maximal ideals, while it is not clean. Therefore R is semilocal ring. That is, $R/J(R)$ is the finite direct sums of

division rings. This shows that $R/J(R)$ is clean. Accordingly, R is feckly clean. □

A commutative ring R is feckly clean if and only if $R/J(R)$ is clean. In other words, in the commutative case, the difference between "feckly clean" and "clean" is that both are clean modulo the radical but in the former idempotents do not necessarily lift modulo the radical. We will consider more explicit conditions under which a commutative ring is feckly clean.

Theorem 12.4.22. *Let R be a commutative ring. Then the following are equivalent:*

(1) *R is feckly clean*
(2) *For any $a \in R$, there exists $e \in R$ such that $a-e \in U(R), e^2-e \in J(R)$.*
(3) *For any $a \in R$, there exists $e \in R$ such that $a-e \in (a-a^2)R, e-e^2 \in J(R)$.*
(4) *For any $a \in R$, there exists $e \in R$ such that $e \in aR, 1-e \in (1-a)R, e-e^2 \in J(R)$.*

Proof. (1) $\Rightarrow$ (2) This is obvious.

(2) $\Rightarrow$ (3) For any $a \in R$, there exists $f \in R$ and $u \in U(R)$ such that $a = f + u$ and $f - f^2 \in J(R)$. Let $e = (1-f) + (f-f^2)u^{-1}$. Then $e - e^2 \equiv f - f^2 \equiv 0\big(\ mod\ J(R)\big)$. Hence, $e - e^2 \in J(R)$. Further, $a - e = f + u - 1 + f - (f-f^2)u^{-1} = (u - 2fu - u^2 + f - f^2)(-u^{-1}) = (a-a^2)(-u^{-1}) \in (a-a^2)R$.

(3) $\Rightarrow$ (4) For any $a \in R$, there exists $e \in R$ such that $a - e \in (a-a^2)R, e-e^2 \in J(R)$. Write $a-e = (a-a^2)s$. Then $e = a\big(1-(1-a)as\big) \in aR$, and that $1 - e = (1-a)(1+as) \in (1-a)R$, as desired.

(4) $\Rightarrow$ (1) For any $a \in R$, there exists $e \in R$ such that $e \in aR, 1-e \in (1-a)R, e-e^2 \in J(R)$. Write $e = as$ and $1-e = (1-a)x$. Set $f = 1-e$. Then $f-f^2 = e-e^2 \in J(R)$. We may assume that $se = s$ and $x(1-e) = x$. It is easy to verify that $(a-f)(s-x) = as - ax - fs + fx = e - ax - (1-e)es + fx = e + (f-a)x = e + (f-a)fx = e + f(1-a)x = e + f = 1$. Therefore $a - f \in U(R)$, as asserted. □

Corollary 12.4.23. *Let R be a commutative ring. Then the following are equivalent:*

(1) *R is feckly clean.*

(2) *For any $a \in R$, there exist $n \in \mathbb{N}$ and $e \in R$ such that $a^n - e \in U(R)$ and $e - e^2 \in J(R)$.*

Proof. (1) $\Rightarrow$ (2) This is trivial.

(2) $\Rightarrow$ (1) For any $a \in R$, there exist $n \in \mathbb{N}, e \in R$ such that $a^n - e \in U(R)$ and $e-e^2 \in J(R)$. Then $e-e^n = (e-e^2)+(e^2-e^3)+\cdots+(e^{n-1}-e^n) \in J(R)$. Hence $a^n - e^n \in U(R)$. This implies that $a - e \in U(R)$, and the result follows. □

Corollary 12.4.24. *Let R be a commutative ring. Then the following are equivalent:*

(1) *R is feckly clean.*
(2) *For any $a \in R$, there exists $e \in R$ and $u \in U(R)$ such that $e = aue, 1 - e = (1 - a)(-u)(1 - e)$ and $e - e^2 \in J(R)$.*

Proof. (1) $\Rightarrow$ (2) For any $a \in R$, we have $a = f+v, f-f^2 \in J(R), v \in U(R)$. Hence $a(1-f) = v(1-f)$, and so $1-f = v^{-1}a(1-f)$. As $1-a = 1-f-v$, we have $f = (-v^{-1})(1-a)f$. Set $e = 1-f$ and $u = v^{-1}$. Then $e-e^2 \in J(R)$. Further, $e = aue$ and $1 - e = (1 - a)(-u)(1 - e)$, as required.

(2) $\Rightarrow$ (1) For any $a \in R$, there exists $e \in R$ and $u \in U(R)$ such that $e = aue, 1-e = (1-a)(-u)(1-e)$ and $e-e^2 \in J(R)$. Then $e \in aR, 1-e \in (1 - a)R$. In light of Theorem 12.4.22, R is feckly clean. □

Theorem 12.4.25. *Let R be a commutative ring. Then the following are equivalent:*

(1) *R is feckly clean.*
(2) *$Max(R)$ is a strongly zero-dimensional Hausdorff space.*
(3) *$Max(R)$ is strongly zero-dimensional and $a + b = 1$ in R implies that $(1 + ar)(1 + bs) \in J(R)$ for some $r, s \in R$.*
(4) *$a + b = 1$ in R implies that $1 + ar \in eR, 1 + bs \in (1 - e)R$ where $e - e^2 \in J(R)$.*

Proof. (1) $\Leftrightarrow$ (2) By virtue of [406, Theorem 1.3], $Max(R)$ is an Hausdorff space if and only if every prime ideal of R containing $J(R)$ is contained in a unique maximal ideal, and the result follows by Theorem 12.4.13.

(1) $\Rightarrow$ (3) In view of Theorem 12.4.13, $Max(R)$ is strongly zero-dimensional. Given $a + b = 1$ in R, there exists $u \in U(R)$ such that

$a = e + u$ where $e - e^2 \in J(R)$. It is easy to verify that

$$\begin{aligned} &\big(1 + \tfrac{a}{e-a}\big)\big(1 - \tfrac{b}{e-a}\big) \\ &= 1 - \tfrac{b}{e-a} + \tfrac{a}{e-a} - \tfrac{ab}{(e-a)^2} \\ &= \tfrac{e^2-e}{(e-a)^2} \\ &\in J(R). \end{aligned}$$

Choose $r = \frac{1}{e-a}$ and $t = -\frac{1}{e-a}$. Then $(1 + ar)(1 + bs) \in J(R)$.

(3) $\Rightarrow$ (1) Let P be a prime ideal of R such that $J(R) \subseteq P$ and P is contained in two distinct maximal ideals M and N. Write $a + b = 1$ with $a \in M, b \in N$. Then we have some $r, s \in R$ such that $(1+ar)(1+bs) \in J(R)$. It follows from $J(R) \subseteq P$ that $1 + ar \in P$ or $1 + bs \in P$. If $1 + ar \in P$, then $1 = (1 + ar) - ar \in M$. If $1 + bs \in P$, then $1 = (1 + bs) - bs \in N$. In any case, we get a contradiction. Hence every prime ideal of R containing $J(R)$ is contained in exactly one maximal ideal. Therefore R is feckly clean, by Theorem 12.4.13.

(1) $\Rightarrow$ (4) Suppose that R is feckly clean and $a + b = 1$ in R. By virtue of Theorem 12.4.22, we can find an element $f \in R$ such that $f \in aR, 1 - f \in (1 - a)R$ and $f^2 - f \in J(R)$. Write $1 - f = -(1 - a)s$ and $f = -ar$. Then $1 + bs = f$ and $1 + ar = 1 - f$. Set $e = 1 - f$. Then $e^2 - e = f^2 - f \in J(R)$. Further, $1 + ar \in eR$ and $1 + bs \in (1 - e)R$, as required.

(4) $\Rightarrow$ (1) This is obvious by Corollary 12.4.8. □

Corollary 12.4.26. *A commutative ring R is feckly clean if and only if $Max(R)$ is strongly zero-dimensional and J-spec(R) is normal.*

Proof. In view of [406, Theorem 1.3], *J-spec*(R) is normal if and only if $Max(R)$ is a Hausdorff space, and so the result follows by Theorem 12.4.25. □

Theorem 12.4.27. *Let R be a commutative ring. Then the following are equivalent:*

(1) *R is clean.*
(2) *R is feckly clean and R is a pm ring.*

Proof. (1) $\Rightarrow$ (2) This is obvious.

(2) $\Rightarrow$ (1) For any disjoint closed sets A and B of $Max(R)$, it follows from Theorem 12.4.4 that there exists an element $e \in R$ such that $A \subseteq U(e), B \subseteq U(1 - e)$ and $eR(1 - e) \subseteq J(R)$. Since $eR(1 - e) \subseteq J(R)$, we have $Max(R) = U(e) \sqcup U(1 - e)$, and so $U(1 - e)$ and $U(e)$ are clopen sets

of $Max(R)$. Therefore $Max(R)$ is strongly zero-dimensional. According to Lemma 12.2.1, R is clean. □

Notes. Theorem 12.2.3 is due to Johnstone, Samei, Lu and Yu [343, 462, 400]. Theorem 12.2.5 was proved by McGovern [412, 413]. Theorem 12.2.11 was proved by Burgess and Raphael [88, Theorem 3.4]. Theorem 12.2.13 is due to McGovern [414, Proposition 3.12 and Theorem 3.13]. Theorem 12.2.20 was first proved by McGovern in [413, Theorem 3.13]. Theorem 12.3.3 was established in [406] and [206]. Theorem 12.3.8 is due to Contessa [205, Theorem 4.1]. Theorem 12.3.13 was developed in [74]. Theorem 12.3.16 is due to Couchot [207, Proposition 3.1]. Theorem 12.3.26 was discovered by Zabavsky [568, Theorem 3].

Chapter 13

Rings of Continuous Functions

A function between two topological spaces X and Y is continuous if for every open set $V \subseteq Y$, the inverse image $f^{-1}(U)$ is an open subset of X. That is, f is a function between the sets X and Y, but the continuity of f depends on the topologies used on X and Y. A real-valued function is a function whose values are real numbers. Let X be a topological space. Let $C(X)$ denote the set of all functions from X to real numbers $\mathbb{R}$. A topological space X is a completely regular space if given any closed set F and any point x that does not belong to F, then there is a continuous function f from X to the real line $\mathbb{R}$ such that $f(x)$ is 0 and, for every y in F, $f(y)$ is 1. In other terms, this condition says that x and F can be separated by a continuous function. For instance, metric spaces, Tychonoff spaces (e.g.,topological manifolds, CW complexes, Niemytzki planes), topological groups, etc. Points x and y in a topological space X can be separated by neighbourhoods if there exists a neighbourhood U of x and a neighbourhood V of y such that U and V are disjoint, i.e., $U \cap V = \emptyset$. X is a Hausdorff space if any two distinct points of X can be separated by neighborhoods. X is a Tychonoff space if it is both completely regular and Hausdorff. Our objective in this chapter is a systematic study of the cleanness of the ring $C(X)$ of all real-valued continuous functions on a Tychonoff spaces. Throughout this chapter, all rings are commutative with 1 and all spaces are Tychonoff spaces.

13.1 Clean Functions

Let $C(X) = \{f \mid f : X \to \mathbb{R} \text{ is a continuous function}\}$. Here, the addition and multiplication are defined by $(f + g)(x) = f(x) + g(x), (fg)(x) = f(x)g(x)$, and that $0 : X \to \mathbb{R}, x \mapsto 0$ and $1 : X \to \mathbb{R}, x \mapsto 1$ are zero and identity, respectively. Then $C(X)$ is a commutative ring. Clearly, $f \in C(X)$ is an idempotent if and only if $f = \chi_K$ for some clopen $K \subseteq X$, and $f \in C(X)$ is a unit if and only if $f(x) \neq 0$ for all $x \in X$. It is attractive to express a continuous function as the sum of an idempotent and a unit. This paragraph shall be devoted to the investigations of cleanness of continuous functions over topological spaces.

A connected space is a topological space that cannot be represented as the union of two or more disjoint nonempty open subsets. We start by certain elementary properties of such spaces.

Theorem 13.1.1. *Let X be a topological space. Then the following are equivalent:*

(1) *X is connected.*
(2) *$C(X)$ is connected.*

Proof. (1) $\Rightarrow$ (2) Suppose $C(X)$ is not connected. Then we can find an idempotent $f = f^2 \in C(X)$ and $f \neq 0, 1$. As $f, 1 - f$ are continuous, $f^{-1}(0), (1-f)^{-1}(0)$ are open sets of X. Obviously, $f^{-1}(0) \cup (1-f)^{-1}(0) \subseteq X$. If $x \notin f^{-1}(0)$, then $f(x) \neq 0$. As $\big((1-f)f\big)(x) = 0$, we get $(1-f)(x) \cdot f(x) = 0$, and so $(1-f)(x) = 0$. This implies that $x \in (1-f)^{-1}(0)$; hence, $X = f^{-1}(0) \cup (1-f)^{-1}(0)$. If $x \in f^{-1}(0) \cap (1-f)^{-1}(0)$, then $f(x) = 0 = (1-f)(x)$, and thus $x = 0$. This shows that $X = f^{-1}(0) \sqcup (1-f)^{-1}(0)$, a contradiction. Therefore $C(X)$ is connected.

(2) $\Rightarrow$ (1) Suppose X is not connected. Then we have nonempty open sets Y, Z such that $X = Y \sqcup Z$. Construct map $f : X \to \mathbb{R}$, where $\mathbb{R}$ with discrete topology), such that $f|_Y = 1$ and $f|_Z = 0$. It is easy to check that $f = f^2 \in C(X)$ and $f \neq 0, 1$. We infer that $C(X)$ is not connected, a contradiction. This completes the proof. □

Let X be a topological space. We use $C^*(X)$ to denote the ring of real bounded continuous functions on X. That is, $C^*(X) = \{f \mid f : X \to \mathbb{R}$ is bounded continuous function$\}$.

Corollary 13.1.2. *Let X be a topological space. Then the following are equivalent:*

(1) *X is connected.*
(2) *$C^*(X)$ is connected.*

Proof. (1) $\Rightarrow$ (2) By virtue of Theorem 13.1.1, $C(X)$ is connected, and then so is $C^*(X)$.

(2) $\Rightarrow$ (1) If X is not connected, we have nonempty open set Y, Z such that $X = Y \sqcup Z$. Analogously to Theorem 13.1.1, we construct map $f : X \to \mathbb{R}$, such that $f|_Y = 1$ and $f|_Z = 0$. Then $f = f^2 \in C^*(X), f \neq 0, 1$. We infer that $C^*(X)$ is not connected, which is impossible, the result follows. □

Lemma 13.1.3. *Let R be a ring. Then the following are equivalent:*

(1) *R is connected.*
(2) *$Spec(R)$ is connected.*

Proof. (1) $\Rightarrow$ (2) Suppose $SpecR$ is not connected. Then $SpecR = E(I) \sqcup E(J)$ and $I, J \neq 0$. Hence, $E(R) = E(I + J)$. If $I + J \neq R$, we have a $P \in Max(R)$ such that $I + J \subseteq P \subsetneqq R$; hence, $P \notin E(I + J)$. This implies that $P \notin SpecR$, which is impossible. Thus $I + J = R$. On the other hand, $E(I) \cap E(J) = E(I \cap J)$ implies that $I \cap J \subseteq \cap\{P | P \in SpecR\}$. So we can find $e_1 \in I, e_2 \in J$ such that $e_1 + e_2 = 1$ and $e_1 e_2 \in IJ \subseteq I \cap J \subseteq \bigcap\{P | P \in SpecR\}$.

Thus we have $s \in \mathbb{N}$ such that $(e_1 e_2)^s = 0$. Observing that $(e_1 + e_2)^{2s} = \sum_{r=0}^{2s} \binom{2s}{r} e_1^{2s-r} e_2^r = 1$, we let $f_1 = \sum_{r=0}^{s} \binom{2s}{r} e_1^{2s-r} e_2^r$ and $f_2 = \sum_{r=s+1}^{2s} \binom{2s}{r} e_1^{2s-r} e_2^r$. Then $f_1 + f_2 = 1$ and $f_1 f_2 = 0$. Furthermore, $f_1^2 = f_1(1 - f_2) = f_1, f_2^2 = f_2(1 - f_1) = f_2$, hence that f_1, f_2 are orthogonal idempotents. Since $f_1 \in I, f_2 \in J$, we have $Rf_1 \subseteq I$ and $Rf_2 \subseteq J$, and so $SpecR = E(Rf_1) \sqcup E(Rf_2)$. Moreover, $E(Rf_1) = E(I), E(Rf_2) = E(J)$. We infer that f_1, f_2 are nontrivial idempotents. Hence R is not connected, a contradiction. Therefore $SpecR$ is connected.

(2) $\Rightarrow$ (1) Suppose R is not connected. Then we have a nontrivial idempotent $e \in R$. If $P \in SpecR$ and $P \notin E(Re)$, then $e \in P$; hence, $1 - e \notin P$. That is, $P \in E(R(1 - e))$. Thus $SpecR = E(Re) \cup E(R(1 - e))$. If $P \in E(Re) \cap E(R(1 - e))$, then $e \notin P$ and $1 - e \notin P$. This implies that $e(1 - e) \notin P$, a contradiction. So $E(Re) \cap E(R(1 - e)) = \emptyset$, and then

$SpecR = E(Re) \sqcup E(R(1-e))$. Therefore $SpecR$ is not connected, which is impossible. □

Theorem 13.1.4. *Let R be a ring. Then the following are equivalent:*

(1) *R is connected.*
(2) *$C(SpecR)$ is connected.*

Proof. This is obvious by Lemma 13.1.3 and Theorem 13.1.1. □

Similarly, a commutative ring R is connected if and only if so is $C^*(SpecR)$.

Theorem 13.1.5. *Let X be a topological space. Then $C(X)$ is a pm ring.*

Proof. Let $f \in C(X)$. Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined as follows:

$$g(x) = \begin{cases} \frac{1}{x} & \text{if } |x| > \frac{1}{3} \\ 9x & \text{if } |x| \leq \frac{1}{3} \end{cases}$$

Set $a = g \circ f$ and $b = g \circ (1-f)$. We claim that $(1-af)(1-b(1-f)) = 0$.

If $|f(x)| > \frac{1}{3}$, then $(1-af)(x) = 1 - a(x)f(x) = 1 - g(f(x))f(x) = 0$.

If $|f(x)| \leq \frac{1}{3}$, then $(1-b(1-f))(x) = 1 - b(x)(1-f(x)) = 1 - g(1-f)(x)(1-f(x)) = 0$, as $|1-f(x)| > \frac{1}{3}$.

Therefore $(1-af)(1-b(1-f)) = 0$. In light of Theorem 12.3.3, $C(X)$ is a pm ring. □

Corollary 13.1.6. *Let X be a topological space. Then the following are equivalent:*

(1) *$C(X)$ is clean.*
(2) *$C(X)$ is feckly clean.*

Proof. In view of Theorem 13.1.5, $C(X)$ is a pm ring. Thus, we obtain the result by Theorem 12.4.27. □

Let $e \in C(X)$. Set $Z(e) = \{x \in X \mid e(x) = 0\}$. Then we have

Lemma 13.1.7. *Let X be a topological space. Then U is a clopen set in X if and only if $U = Z(e)$ for some idempotent $e \in C(X)$.*

Proof. $\Longrightarrow$ Since U is clopen, it is open and there exists an open set V such that $X = U \cup V$ and $U \cap V = \emptyset$. Define

$$e(x) = \begin{cases} 0 & \text{if } x \in U \\ 1 & \text{if } x \in V. \end{cases}$$

Then $e = e^2 \in C(X)$ and $U = Z(e)$, as asserted.

$\Longleftarrow$ Let $U = Z(e)$. As e is continuous, U is a open set in X. Likewise, $Z(1-e)$ is open. Clearly, $Z(e) \cap Z(1-e) = \emptyset$ and $X = Z(e) \cup Z(1-e)$. Therefore U is clopen, hence the result. □

Lemma 13.1.8. *$f \in C(X)$ is clean if and only if there exists a clopen set U in X such that $f^{-1}(1) \subseteq U \subseteq X \setminus Z(f)$.*

Proof. $\Longrightarrow$ Write $f = e + u$ with $e = e^2 \in C(X)$ and $u \in U(C(X))$. Set $U = Z(e)$. Then U is a clopen set in X, by Lemma 13.1.7. One easily checks that $f^{-1}(1) \subseteq Z(e) \subseteq X \setminus Z(f)$, as required.

$\Longleftarrow$ As U is a clopen set in X, it follows by Lemma 13.1.7 that $U = Z(e)$ for an idempotent $e \in C(X)$. Define

$$u(x) = \begin{cases} f(x) & \text{if } x \in Z(e) \\ f(x) - 1 & \text{if } x \notin Z(e). \end{cases}$$

Then $u \in U(C(X))$ and $f = e + u$, i.e., f is clean. □

Example 13.1.9. (1) Every positive power of a non-negative clean function $f \in C(X)$ is clean.

(2) Every $f \in C(X)$ with $f^{-1}(1) = \emptyset$ is clean.

(3) Every $f \in C(X)$ with $|f| < 1$ is clean.

(4) For any $f \in C(X)$, $\frac{f^2}{1+f^2}$ is clean, as $|\frac{f^2}{1+f^2}| < 1$.

Lemma 13.1.10. *Let $f \in C^*(X)$. For each $\alpha \in \mathbb{R}$, set $A_\alpha = \{x \in X \mid |f(x)| \geq \alpha\}$. If there exist $\alpha, \beta \in \mathbb{R}$ such that $0 < \alpha < \beta < 1$ and a clopen set U such that $A_\beta \subseteq U \subseteq A_\alpha$, then f is clean in $C^*(X)$.*

Proof. Let $U = Z(e)$ for an idempotent $e \in C^*(X)$. Define

$$u(x) = \begin{cases} f(x) & x \in U \\ f(x) - 1 & x \in X \setminus U. \end{cases}$$

Then u is continuous. If $x \in U$, then $x \in A_\alpha$; hence, $|u(x)| = |f(x)| \geq \alpha$. If $x \in X \setminus U$, then $|u(x)| = |f(x) - 1| \geq 1 - |f(x)| \geq 1 - \beta$. Thus $u \in U(C^*(X))$. Clearly, $e(x) = 1$ for any $x \in X \setminus U$. One easily checks that $f = e + u$, and then the result follows. □

Let A, B be two subsets of a topological space X. We say that A and B are completely separated, if there is a continuous function $f : X \to [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$. Two sets are completely separated if and only if they are contained in disjoint zero-sets. We have at our disposal all information necessary to prove the following.

Theorem 13.1.11. *Let X be a topological space. Then the following are equivalent:*

(1) *$C(X)$ is a clean ring.*
(2) *$C^*(X)$ is a clean ring.*
(3) *For any disjoint zero sets A and B in X, there exists a clopen U such that $A \subseteq U \subseteq X \setminus B$.*
(4) *For any completely separated subsets A and B in X, there exists a clopen U such that $A \subseteq U \subseteq X \setminus B$.*

Proof. (1) $\Rightarrow$ (2) Let $f \in C^*(X)$. Set $A = \{x \in X \mid |f(x)| \geq \frac{2}{3}\}$ and $B = \{x \in X \mid |f(x)| \leq \frac{1}{3}\}$. Then A and B are two disjoint zero sets. Hence they are completely separated. Thus, we have $g \in C(X)$ such that $g(A) = \{1\}$ and $g(B) = \{0\}$. Since g is clean, it follows by Lemma 13.1.8 that there exists a clopen set U in X such that $g^{-1}(1) \subseteq U \subseteq X \setminus Z(g)$. So

$$A_{\frac{2}{3}} \subseteq g^{-1}(1) \subseteq U \subseteq X \setminus B \subseteq A_{\frac{1}{3}}.$$

In light of Lemma 13.1.10, $f \in C^*(X, \mathbb{C})$ is a clean. Thus proving (2).

(2) $\Rightarrow$ (1) Let $f \in C(X)$. Then $g = (-1 \vee f) \wedge 1 \in C^*(X)$. By hypothesis, $g \in C^*(X)$ is clean. In light of Lemma 13.1.8, there exists an idempotent $e \in C(X)$ such that $g^{-1}(1) \subseteq Z(e) \subseteq X \setminus Z(g)$. Hence, $f^{-1}(1) \subseteq g^{-1}(1) \subseteq Z(e) \subseteq X \setminus Z(g) = X \setminus Z(f)$. Accordingly, $f \in C(X)$ is clean, as desired.

(1) $\Rightarrow$ (3) Let $A = Z(f_1)$ and $B = Z(f_2)$ with $A \cap B = \emptyset$. Replacing f_i by $|f_i|$ if necessary we may assume that $f_1(x), f_2(x) \geq 0$ for all $x \in X$. Hence $f_1(x) + f_2(x) > 0$ for all $x \in X$, as $A \cap B = \emptyset$. Thus, $h = \frac{f_1}{f_1 + f_2} \in C(X)$. It is easy to check that $A = h^{-1}(0), B = h^{-1}(1)$ and $0 < h(x) < 1$ for all $x \in X \setminus (A \cup B)$. Since h is clean, there exists $e^2 = e \in C(X)$ and $u \in U(C(X))$ such that $h = e + u$. Set $U = e^{-1}(1)$ and $V = e^{-1}(0)$. Then U and V are disjoint clopen sets of X with $X = U \cup V$.

If $x \in A$, then $f_1(x) = 0$; hence, $h(x) = 0$. Thus $e(x) = -u(x) \neq 0$. It follows that $e(x) = 1$, and then $x \in U$. This implies that $A \subseteq U$. If $x \in B$, then $f_2(x) = 0$, and so $h(x) = 1$. Hence $(1 - e)(x) = u(x) \neq 0$. We infer that $(1 - e)(x) = 1$; whence $e(x) = 0$. Thus $B \cap U = \emptyset$. Therefore $A \subseteq U \subseteq X \setminus B$.

(3) $\Rightarrow$ (1) Let $f \in C(X)$. The zero sets $Z(f)$ and $Z(1 - f)$ are zero sets. Hence we have a clopen set U such that $Z(1 - f) \subseteq U \subseteq X \setminus Z(f)$. In light of Lemma 13.1.8, $f \in C(X)$ is clean, as required.

(3) $\Rightarrow$ (4) Let A and B are completely separated. Then $A \subseteq Z(f), B \subseteq Z(g)$ and $Z(f) \cap Z(g) = \emptyset$. By hypothesis, we have a clopen U such that $A \subseteq Z(f) \subseteq U \subseteq X \setminus Z(g) \subseteq X \setminus B$, as needed.

(4) ⇒ (3) This is obvious. □

Corollary 13.1.12. *Let X be a topological space. Then the following are equivalent:*

(1) *$C(X)$ is clean.*
(2) *$C(X)$ has stable range 1.*

Proof. (1) ⇒ (2) In view of Theorem 1.1.8, $C(X)$ is an exchange ring. Thus proving (2), as as every abelian exchange ring has stable range 1.

(2) ⇒ (1) For any zero sets A and B in X, by [524, Theorem 5], there exists a clopen U such that $A \subseteq U \subseteq X \setminus B$. Therefore we complete the proof, by Theorem 13.1.11. □

Corollary 13.1.13. *Let X be a topological space. Then the following are equivalent:*

(1) *$C(X)$ is clean.*
(2) *The set of clean elements in $C(X)$ is a subring of $C(X)$.*
(3) *The set of clean elements in $C(X)$ is closed under sum.*

Proof. (1) ⇒ (2) ⇒ (3) These are trivial.

(3) ⇒ (1) Let A and B be two completely separated subsets of X. Then we have a continuous $g : X \to [0, 1]$ such that $g(A) = \{0\}$ and $g(B) = \{1\}$. Let $f = \frac{1}{2}g \in C(X)$. Then $|f| \leq \frac{1}{2}, f(A) = 0$ and $f(B) = \{\frac{1}{2}\}$. Since $f^{-1}(1) = \emptyset$, we see that $f \in C(X)$ is clean. By hypothesis, $2f = f + f \in C(X)$ is clean. In view of Theorem 13.1.11, there exists a clopen set U such that $(2f)^{-1}(1) \subseteq U \subseteq X \setminus Z(2f)$. That is, $B \subseteq U \subseteq X \setminus A$. Accordingly, $C(X)$ is clean, by Theorem 13.1.11. □

Theorem 13.1.14. *Let X be a topological space. Then the following are equivalent:*

(1) *$C(X)$ is a clean ring.*
(2) *Every zero-divisor in $C(X)$ is clean.*
(3) *$C(X)$ has a clean prime ideal.*

Proof. (1) ⇒ (2) This is trivial.

(2) ⇒ (3) In view of Proposition 11.1.7, every element in a minimal prime ideal of $C(X)$ is a zero-divisor. Thus, each minimal prime ideal of R is clean, as desired.

(3) ⇒ (1) Let P be a clean prime ideal of $C(X)$, and let $f \in C(X)$. Set $A = Z(f)$ and $B = Z(1 - f)$. If $A = \emptyset$ or $B = \emptyset$, then we are done by

Lemma 13.1.8. We now assume that $A, B \neq \emptyset$. Set $C = \{x \in X \mid f(x) \leq \frac{1}{3}\}$ and $D = \{x \in X \mid f(x) \geq \frac{1}{3}\}$. Clearly, C and D are both zero sets. Since $A \cap D = B \cap C = \emptyset$, A and D, B and C are completely separated. Thus, we have $g, h \in C(X)$ such that $g(A) = \{1\}, g(D) = \{0\}$ and $h(B) = \{1\}, h(C) = \{0\}$. Since $X = Z(g) \cup Z(h)$, we get $gh = 0 \in P$, and so either $g \in P$ or $h \in P$. If $g \in P$, then $g \in C(X)$ is clean. In light of Lemma 13.1.8, there exists a clopen set U such that $g^{-1}(1) \subseteq U \subseteq X \setminus Z(g)$. This implies that $A \subseteq U \subseteq X \setminus B$. If $h \in P$, analogously, we have $B \subseteq U \subseteq X \setminus A$. By virtue of Lemma 13.1.8, $f \in C(X)$ is clean, as required. □

Corollary 13.1.15. *Let X be a topological space. Then $C(X)$ is local if and only if*

(1) *X is a connected space;*
(2) *For any disjoint zero sets A and B in X, there exists a clopen U such that $A \subseteq U \subseteq X \setminus B$.*

Proof. $\Longrightarrow$ Since $C(X)$ is local, it is connected. Then X is connected by Theorem 13.1.1. On the other hand, $C(X)$ is clean, and therefore proving (2) by Theorem 13.1.11.

$\Longleftarrow$ In light of Theorem 13.1.1 and Theorem 13.1.11, $C(X)$ is a connected clean ring. For any $a \in R$, there exists an idempotent $e \in R$ such that $a - e \in U(R)$. As $e = 0$ or 1, we see that a or $1 - a$ is invertible, as desired. □

For each $M \in Max(R)$, let $O_M = \bigcap_{P \subseteq M} P$, where P ranges over all prime ideals contained in M. We use $int(A)$ to denote the interior of the subset A of a topological space.

Lemma 13.1.16. *Let R be a commutative reduced ring, and let $M \in Max(R)$. Then*

$$\begin{aligned} O_M &= \{a \in R \mid \exists\, b \in R\backslash M \text{ such that } ab = 0\} \\ &= \{a \in R \mid M \in intU(a)\}. \end{aligned}$$

Proof. Clearly, $O_M \supseteq \{a \in R \mid \exists\, b \in R\backslash M$ such that $ab = 0\}$. Let $f \in Q_M$. Suppose that $fb \neq 0$ for any $b \in R\backslash M$. Then $f^n b \neq 0$ for all $n \in \mathbb{N}$. Let $S = \{f^n g \mid n = 0, 1, 2, \ldots, g \in R\backslash M\}$. Let $\Omega = \{I \lhd R \mid I \cap S = \emptyset\}$. Then Ω is a non-empty inductive set. By using Zorn's Lemma, there exists an ideal Q which is maximal in Ω. If Q is not prime, then $ab \in Q$ while $a, b \notin Q$ for some $a, b \in R$. By the maximality, we see that $f^s g_1 \in aR + Q$ and

$f^t g_2 \in bR + Q$. It follows that $f^{s+t} g_1 g_2 \in Q$, a contradiction. This implies that Q is prime. We claim that $Q \subseteq M$; otherwise, $Q + M = R$, and so $1 = q + m$ for some $q \in Q, m \in M$. Hence $q \in R \backslash M$. But $fq \in Q \cap S$, a contradiction. We infer that $Q \subseteq M$. Accordingly, $fb = 0$ for some $b \in R \backslash M$. Thus $O_M \subseteq \{a \in R \mid \exists\ b \in R \backslash M$ such that $ab = 0\}$, and then we obtain the first equality.

We note that for any $a \in R$, $intU(a) = Max(R) \setminus U(r(a))$; hence, $M \in intU(a)$ if and only if there exists $b \in R \setminus M$ such that $ab = 0$. Therefore the proof is true. □

Theorem 13.1.17. *Let X be a topological space. Then the following are equivalent:*

(1) *$C(X)$ is a clean ring.*
(2) *$Max(C(X))$ is strongly zero-dimensional.*
(3) *$Max(C(X))$ is zero-dimensional.*
(4) *For each $M \in Max(C(X))$, O_M is generated by a set of idempotents.*

Proof. (1) ⇔ (2) ⇔ (3) In view of Theorem 13.1.5, $C(X)$ is a pm ring. These equivalences are obvious, by Lemma 12.2.1 and Theorem 12.2.3.

(3) ⇒ (4) Let $\{U_i \mid i \in I\}$ be a local base at $M \in Max(C(X))$. Then $U_i = U(e_i)$ for some idempotents $e_i \in C(X)$. We claim that O_M is generated by the set of idempotents $\{e_i \in C(X) \mid i \in I\}$. For each $e_i \in C(X)$, as $e_i(1-e_i) = 0$ with $1 - e_i \in C(X) \backslash M$, we see that $e_i \in O_M$ by Lemma 13.1.16. Let $a \in O_M$. By using Lemma 13.1.16 again, $M \in intU(a)$. As $intU(a)$ is open, we easily see that there exists some U_i such that $M \in U_i \subseteq intU(a)$. We infer that $U(e_i) \subseteq intU(a)$. This implies that $a \in e_i C(X)$, as required.

(4) ⇒ (3) Let G be an open set of $Max(C(X))$, and let $M \in G$. Then O_M is generated by the set of idempotents $\{e_i \in C(X) \mid i \in I\}$. Clearly, each $U(e_i)$ is a clopen set. We shall prove

$$\left\{ \bigcap_{i \in J} U(e_i) \;\middle|\; J \text{ is a finite subset of } I \right\}.$$

is a local base at M. Observe that $\bigcap_{i \in I} U(e_i) \subseteq G$. Otherwise, there exists $M' \in F := Max(C(X)) \setminus G$ such that each $e_i \in M'$. Hence, $O_M \subseteq M'$. In view of Theorem 13.1.5, $C(X)$ is a pm ring. If $M \neq M'$, then $M + M' = C(X)$. Write $x + y = 1$ with $x \in M$ and $y \in M'$. By virtue of Theorem 12.3.3, there exist some $s, t \in C(X)$ such that $(1+xs)(1+yt) = 0$. As $1 + xs \in R \backslash M$, it follows from Lemma 13.1.16 that $1 + yt \in O_M \subseteq M'$.

This implies that $1 \in M'$, an absurd. Hence $M = M'$, a contradiction. This implies that $F \subseteq \bigcup_{i \in I} U(1 - e_i)$. As F is a closed set of the compact space $Max(R)$, we can find a finite subset J of I such that $F \subseteq \bigcup_{i \in J} U(1 - e_i)$. Accordingly, $M \in \bigcap_{i \in J} U(e_i) \subseteq G$, and we are done. □

A ring R is clean-like if every element in R is the sum of a potent and a unit.

Theorem 13.1.18. *Let X be a topological space. Then the following are equivalent:*

(1) *$C(X)$ is clean.*
(2) *$C(X)$ is weakly clean.*
(3) *$C(X)$ is clean-like.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) These are obvious.

(3) $\Rightarrow$ (1) Let $f \in C(X)$. Set $g = 2f - 1$. By hypothesis, there exists $p = p^n \in C(X)(n \geq 2)$ such that $u := g - p \in C(X)$ is a unit. Clearly, the values of p are $-1, 0$ or 1. Let $K = \{x \in X \mid u(x) < 0\}$. Then K is a clopen set. Let χ_K be the characteristic function on K. Then $\chi_K \in C(X)$ is an idempotent. If $x \in K$, then $(f - \chi_K)(x) = f(x) - 1 \neq 0$; otherwise, $g(x) = 1$ and $u(x) > 0$, a contradiction. If $x \notin K$, then $(f - \chi_K)(x) = f(x) \neq 0$; otherwise, $g(x) = -1$ and $u(x) < 0$, a contradiction. Thus $f - \chi_K$ is nowhere zero, and so $f - \chi_K \in U(C(X))$. Therefore f is clean, as asserted. □

The Stone-Čech compactification of X is denoted by βX. We derive

Theorem 13.1.19. *Let X be a topological space. Then the following are equivalent:*

(1) *$C(X)$ is clean.*
(2) *$C(X)$ is almost clean.*
(3) *$C^*(X)$ is almost clean.*
(4) *βX is zero-dimensional.*

Proof. (1) $\Rightarrow$ (2) This is trivial.

(2) $\Rightarrow$ (1) By virtue of Theorem 13.1.5, $C(X)$ is a pm ring. Thus proving (1), by Theorem 12.2.11.

(1) $\Leftrightarrow$ (3) Since $C^*(X) = C(\beta X)$, applying (1) $\Leftrightarrow$ (2) to βX, we complete the proof, by Theorem 13.1.11.

(1) $\Leftrightarrow$ (4) As is well known, βX and $Max(C(X))$ are homomorphic in a natural way, we complete the proof, by Theorem 13.1.17. □

Corollary 13.1.20. *Let X be a topological space. If X is basically disconnected space, then $C(X)$ is clean.*

Proof. In view of Theorem 11.4.15, $C(X)$ is almost clean. Therefore $C(X)$ is clean, by Theorem 13.1.19. □

Corollary 13.1.21. *Let X be a topological space. If X is basically disconnected, then $C^*(X)$ is clean.*

Proof. This is obvious, by Corollary 13.1.20 and Theorem 13.1.11. □

Let I be an ideal of $C(X)$, and let $Z[I] = \{Z(f) \mid f \in I\}$. If $\cap Z(I) \neq \emptyset$, we call I a fixed ideal. We record the following elementary property [265, Theorem 4.6].

Lemma 13.1.22. *The fixed maximal ideals of $C(X)$ are precisely the set $M_p = \{f \in C(X) \mid f(p) = 0\}(p \in X)$. The ideals M_p are distinct for distinct p.*

Proof. Let $p \in X$, and let $\varphi : C(X) \to \mathbb{R}$ given by $f \mapsto f(p)$ for any $f \in C(X)$. Then $M_p = Ker(\varphi)$. For any $r \in \mathbb{R}$, we define $g_r : X \to \mathbb{R}, x \mapsto r$ for any $x \in X$. Then $r = \varphi(g_r)$, and so φ is surjective. Hence, $C(x)/M_p \cong \mathbb{R}$ is a field. As $p \in \bigcap Z[M_p]$, M_p is a fixed maximal ideal.

Let M be a fixed maximal ideals of $C(X)$. Then there exists $p \in \bigcap Z[M]$. Clearly, $M \subseteq M_p$, and so $M = M_p$ by the maximality of M.

Suppose that $M_p = M_q$ for some distinct $p, q \in X$. Since X is a Hausdorff space, $\{q\}$ is closed. As X is completely regular, there exists some $f \in C(X)$ such that $f(p) = 1$ and $f(q) = 0$; hence, $f \in M_q$. This implies that $f \in M_p$, and so $f(p) = 0$. This gives a contradiction. Accordingly, M_p are distinct for distinct p. □

Finally, we provide certain subclasses of topological space X such that $C(X)$ is clean. The following was proved in [250].

Theorem 13.1.23. *Let X be a topological space. Then the following are equivalent:*

(1) *$C(X)$ is semisimple.*

(2) $C(X)$ *is semilocal.*
(3) $|X| < \infty$.

Proof. (1) $\Leftrightarrow$ (2) It will suffice to prove that $J(C(X)) = 0$. If $0 \neq f \in C(X)$, then there exists some $x_0 \in X$ such that $f(x_0) \neq 0$. Let $g \in C(X)$ be defined by $x \mapsto \frac{1}{f(x_0)}$ for any $x \in\in X$. Then $(1 - fg)(x_0) = 0$; hence, $1 - fg \notin U(C(X))$. This implies that $f \notin J(C(X))$. Therefore $J(C(X)) = 0$, as desired.

(2) $\Rightarrow$ (3) For any $p \in X$, In light of [265, Theorem 4.6], $M_p = \{f \in C(X) \mid f(p) = 0\}$ is the fixed maximal ideal of $C(X)$, and that M_p are distinct for distinct p by Lemma 13.1.22. This shows that $|X| < \infty$.

(3) $\Rightarrow$ (2) Since $|X| < \infty$, X is compact. In view of [265, Theorem 4.8], every ideal I of $C(X)$ is fixed, e.g., $\bigcap_{f \in I} Z(f) \neq \emptyset$. By virtue of Lemma 13.1.22, the fixed maximal ideals M of $C(X)$ are precisely the sets $M_p = \{f \in C(X) \mid f(p) = 0\} (p \in X)$. Thus, $C(X)$ has finitely many maximal ideals. Therefore $C(X)$ is semilocal. □

Corollary 13.1.24. *Let X be a finite topological space. Then $C(X)$ is clean.*

Proof. In light of Theorem 13.1.23, $C(X)$ is semilocal. As is well known, a commutative ring is semilocal if and only if it has finitely many maximal ideals. Suppose that $M_1, \ldots, M_n$ are maximal ideals of $C(X)$. Then

$$C(X) \cong \frac{C(X)}{M_1} \times \cdots \times \frac{C(X)}{M_n}.$$

Since each $\frac{C(X)}{M_i}$ is a filed, we see that it is clean. Therefore $C(X)$ is clean. □

Recall that a singleton is a set with exactly one element.

Proposition 13.1.25. *Let X be a topological space. Then the following are equivalent:*

(1) $C(X)$ *is local.*
(2) $C(X)$ *is isomorphic to the field* $\mathbb{R}$.
(3) X *is a singleton.*

Proof. (1) $\Rightarrow$ (3) If $p, q \in X$ are distinct, then M_p and M_q are distinct maximal ideals of $C(X)$. We infer that $C(X)$ is not local, a contradiction. Hence, X is a singleton.

(3) $\Rightarrow$ (2) $\Rightarrow$ (1) These are trivial. □

A topological space X is a P-space if every prime ideal of $C(X)$ is maximal. We have

Proposition 13.1.26. *Let X be a topological space. Then the following are equivalent:*

(1) *$C(X)$ is regular.*
(2) *$M_n(C(X))$ is strongly π-regular for all $n \in \mathbb{N}$.*
(3) *Every zero set in X is clopen in X.*
(4) *X is a P-space.*

Proof. (1) $\Rightarrow$ (2) This is obvious, by Corollary 6.1.24.

(2) $\Rightarrow$ (4) By hypothesis, R is strongly π-regular. Therefore we prove (4) by Theorem 6.1.10.

(4) $\Rightarrow$ (3) Let $f \in C(X)$. Then there exists $g \in C(X)$ such that $f^k = f^k g f^k$ for some $k \in \mathbb{N}$. Set $e = f^k g$. Then $e \in C(X)$ is an idempotent. If $f(x) = 0$, then $e(x) = f^k(x)g(x) = 0$. If $e(x) = 0$, then $f^k(x) = f^k(x)g(x)f^k(x) = 0$, and so $f(x) = 0$. Accordingly, $Z(f) = Z(e)$ is clopen, in terms of Lemma 13.1.7.

(3) $\Rightarrow$ (1) Let $f \in C(X)$. Then $Z(f)$ is clopen. Define $g : X \to \mathbb{R}$ by $g(x) = 0$ if $x \in Z(f)$ and $g(x) = \frac{f(x)}{f^2(x)}$ if $x \notin Z(f)$. As the characteristic function of a subset is continuous if and only the subset is clopen. Hence $g \in C(X)$. We easily check that $f = f^2 g$, as desired. □

13.2 $C(X, \mathbb{C})$ and $C(X, A)$

The purpose of this section is to investigate the ring of continuous complex-valued functions on a Tychonoff space X, denoted $C(X, \mathbb{C})$. We also study when the rings $C(X, A)$ of all A-valued continuous functions are clean, especially when A is a dense clean subring of $\mathbb{R}$ with 1.

Theorem 13.2.1. *Let X be a topological space. Then the following are equivalent:*

(1) *$C(X)$ is a clean ring.*
(2) *$C(X, \mathbb{C})$ is a clean ring.*
(3) *$C(X, \mathbb{C})$ is weakly clean.*

Proof. (1) $\Rightarrow$ (2) Let $f \in C(X, \mathbb{C})$. Then $f = f_1 + if_2$, where $f_1, f_2 \in C(X)$. By hypothesis, we can find idempotents $e_1, e_2 \in C(X)$ and units

$u_1, u_2 \in C(X)$ such that $f_i = e_i + u_i (i = 1, 2)$. Hence, $f = e_1 + u_1 + i(e_2 + u_2) = e_1 + \big(u_1 + i(e_2 + u_2)\big)$. Clearly, $e_1 = e_1^2 \in C(X, \mathbb{C})$. As $u_1(x) \neq 0$ for any $x \in X$, we see that $u_1 + i(e_2 + u_2) \in C(X, \mathbb{C})$ is a unit. This proving (2).

(2) $\Rightarrow$ (3) This is obvious.

(3) $\Rightarrow$ (1) Let $f \in C(X)$. Then $f \in C(X, \mathbb{C})$. By hypothesis, there exists an idempotent $e = e_1 + ie_2$ and a unit $u = u_1 + iu_2$ in $C(X, \mathbb{C})$ such that $f = \pm e + u$. For any $x \in X$, we easily see that $e(x) = 0$ or 1; hence, $e(x) \in \mathbb{R}$. This implies that $e \in C(X)$. It follows that $u = f \mp e \in C(X)$. Hence, f is the sum or difference of a unit and an idempotent in $C(X)$. Thus $C(X)$ is weakly clean. In light of Theorem 13.1.18, $C(X)$ is clean, as asserted. □

Corollary 13.2.2. *Let X be a P-space. Then $C(X, \mathbb{C})$ is clean.*

Proof. In view of Proposition 13.1.26, $C(X)$ is regular, and so it is clean. Therefore $C(X, \mathbb{C})$ is clean, in terms of Theorem 13.2.1. □

Clearly, given an idempotent $e \in C(X, \mathbb{C})$, $Z(e)$ is a clopen set, and given a clopen set U in X there exists an idempotent e such that $U = Z(e)$. Analogously to Lemma 13.1.8, we have

Lemma 13.2.3. *$f \in C(X, \mathbb{C})$ is clean if and only if there exists a clopen set U in X such that $f^{-1}(1) \subseteq U \subseteq X \setminus Z(f)$.*

We use $C^*(X, \mathbb{C})$ to denote the ring of bounded complex valued continuous functions. We now derive

Lemma 13.2.4. *Let $f \in C^*(X, \mathbb{C})$. For each $\alpha \in \mathbb{R}$, set $A_\alpha = \{x \in X \mid |f(x)| \geq \alpha\}$. If there exist $\alpha, \beta \in \mathbb{R}$ such that $0 < \alpha < \beta < 1$ and a clopen set U such that $A_\beta \subseteq U \subseteq A_\alpha$. Then f is clean in $C^*(X, \mathbb{C})$.*

Proof. Let $U = Z(e)$ for an idempotent $e \in C^*(X, \mathbb{C})$. Define

$$u(x) = \begin{cases} f(x) & x \in U \\ f(x) - 1 & x \in X \backslash U. \end{cases}$$

Then u is continuous. If $x \in U$, then $x \in A_\alpha$; hence, $|u(x)| = |f(x)| \geq \alpha$. If $x \in X \backslash U$, then $|u(x)| = |f(x) - 1| \geq 1 - |f(x)| \geq 1 - \beta$. Thus $u \in U(C^*(X, \mathbb{C}))$. Clearly, $e(x) = 1$ for any $x \in X \backslash U$. One easily checks that $f = e + u$, and there the result follows. □

Following a similar route of Theorem 13.1.11, we have

Theorem 13.2.5. *Let X be a topological space. Then the following are equivalent:*

(1) *$C(X)$ is a clean ring.*
(2) *$C^*(X, \mathbb{C})$ is a clean ring.*

Proof. (1) $\Rightarrow$ (2) In view of Theorem 13.2.1, $C(X, \mathbb{C})$ is a clean ring. Let $f \in C^*(X, \mathbb{C})$. Set $A = \{x \in X \mid |f(x)| \geq \frac{2}{3}\}$ and $B = \{x \in X \mid |f(x)| \leq \frac{1}{3}\}$. Then A and B are two disjoint zero sets, and they are completely separated. Hence, we have $g \in C(X, \mathbb{C})$ such that $g(A) = \{1\}$ and $g(B) = \{0\}$. Since g is clean, it follows by Lemma 13.2.3 that there exists a clopen set U in X such that $g^{-1}(1) \subseteq U \subseteq X \setminus Z(g)$. So

$$A_{\frac{2}{3}} \subseteq g^{-1}(1) \subseteq U \subseteq X \backslash B \subseteq A_{\frac{1}{3}}.$$

In light of Lemma 13.2.4, $f \in C^*(X, \mathbb{C})$ is a clean. Thus proving (1).

(2) $\Rightarrow$ (1) Let A and B be two completely separated subsets of X. Then there exists $f \in C^*(X, \mathbb{C})$ such that $f(A) = 0, f(B) = \frac{1}{2}$ and $|f| \leq \frac{1}{2}$. By hypothesis, $2f$ is clean in $C^*(X, \mathbb{C})$. In light of Lemma 13.2.3, there exists a clopen set U in X such that $(2f)^{-1}(1) \subseteq U \subseteq X \setminus Z(2f)$. This implies that

$$B \subseteq (2f)^{-1}(1) \subseteq U \subseteq X \setminus Z(2f) \subseteq X \backslash A.$$

Therefore we complete the proof, by Theorem 13.1.11. □

Corollary 13.2.6. *Let X be a topological space. Then the following are equivalent:*

(1) *$C(X)$ is a clean ring.*
(2) *$C^*(X, \mathbb{C})$ is weakly clean.*

Proof. (1) $\Rightarrow$ (2) By virtue of Theorem 13.2.5, $C^*(X, \mathbb{C})$ is clean, and then it is weakly clean.

(2) $\Rightarrow$ (1) Let $f \in C^*(X)$. Then $f \in C^*(X, \mathbb{C})$, and so we have an idempotent $e = e_1 + ie_2$ and a unit $u = u_1 + iu_2$ in $C^*(X, \mathbb{C})$ such that $f = \pm e + u$. As in the proof of Theorem 13.2.1, we see that $e \in C^*(X)$. Hence, $u = f \mp e \in C^*(X)$. This implies that $C^*(X)$ is weakly clean. That is, $C(\beta X)$ is weakly clean. In light of Theorem 13.1.18, $C(\beta X)$ is clean, and then so is $C^*(X)$. Therefore we complete the proof, by Theorem 13.1.11. □

Lemma 13.2.7. *Let X be a topological space. Then $C(X, \mathbb{C})$ is a pm ring.*

Proof. One easily checks that there is a ring isomorphism

$$\varphi : C(X, \mathbb{C}) \to \mathbb{C}(C(X)), f + ig \mapsto \begin{pmatrix} f & g \\ -g & f \end{pmatrix}.$$

Since $C(X)$ satisfies the condition in Example 12.3.24, as a continuous function $h \in C(X)$ is invertible if and only if for any $x \in X$, $h(x) \neq 0$. Thus $1 + h^2$ is always invertible. Hence $\mathbb{C}(C(X))$ is a pm ring, in terms of Example 12.3.24. This completes the proof. □

Theorem 13.2.8. *Let X be a topological space. Then the following are equivalent:*

(1) *$C(X)$ is a clean ring.*
(2) *$C(X, \mathbb{C})$ is an almost clean ring.*
(3) *$C^*(X, \mathbb{C})$ is an almost clean ring.*

Proof. (1) $\Leftrightarrow$ (2) In view of Lemma 13.2.7, $C(X, \mathbb{C})$ is a pm ring. This equivalence is obtained by Theorem 12.2.11.

(1) $\Rightarrow$ (3) By virtue of Theorem 13.2.5, $C^*(X, \mathbb{C})$ is a clean ring. Thus. it is an almost clean ring.

(3) $\Rightarrow$ (1) As proved before, $C^*(X, \mathbb{C}) \cong \mathbb{C}(C^*(X))$, and so $\mathbb{C}(C^*(X))$ is almost clean. In view of Lemma 13.2.7, $\mathbb{C}(C^*(X))$ is a pm ring. According to Theorem 12.2.11, $\mathbb{C}(C^*(X))$ is clean. That is, $C^*(X, \mathbb{C})$ is clean. This completes the proof by Theorem 13.2.5. □

Let $\mathbb{C}_n = \{z \in \mathbb{C} \mid z^n = 1\}$.

Lemma 12.2.9. *Let X be a topological space. If $p \in C(X, \mathbb{C})$ is potent, then $p(X) \subseteq \{0\} \cup \mathbb{C}_n$ for some $n \in \mathbb{N}$. This generates a finite clopen cover of X.*

Proof. Suppose that $p = p^{n+1} (n \in \mathbb{N})$. Then p^n is an idempotent. For any $x \in X$, we see that $p(x) = 0$ or $p(x)^n = 1$. Thus, $f(X) \subseteq \{0\} \cup \mathbb{C}_n$, and so $p(X)$ is a finite set of $\mathbb{C}$, as 1 has exactly n-many roots of unity. Therefore the inverse images of the output are clopen subsets which cover X, hence the result. □

Define the map $\alpha : [0, 1] \to \mathbb{C}$ given by

$$\alpha(x) = \begin{cases} 2x & 0 \leq x \leq \frac{1}{2} \\ \cos\left(4\pi(x - \frac{1}{2})\right) + i \sin\left(4\pi(x - \frac{1}{2})\right) & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Let $\pi : \mathbb{C} \to \mathbb{R}$ be the projection map, and let $\pi_1 = -1 \vee (1 \wedge \pi)$. There is a homeomorphism $s : [-1, 1] \to [0, 1]$ which takes $\pi(R^0)$ onto $\mathbb{Q} \cap [0, 1]$. Here, $R^0 = \{0\} \cup \bigcup_{n \in \mathbb{N}} \mathbb{C}_n$. Set $\beta = s\pi_1 : \mathbb{C} \to [0, 1]$. Then α, β are continuous, where α is onto the unit skirt, $\beta : \mathbb{C} \to [0, 1]$ is surjective and maps R^0 onto

the set of rations in $[0,1]$. For $f \in C(X,[0,1])$, we set $f_\alpha = \alpha \circ f \in C(X,\mathbb{C})$. For $g \in C(X,\mathbb{C})$, we set $g_\beta = \beta \circ g \in C(X,[0,1])$.

Lemma 13.2.10. *Let X be a topological space. Then $C(X,\mathbb{C})$ is clean-like if and only if given $g \in C(X,[0,1])$ there is a finite clopen cover of X, say $K_0, K_1, \ldots, K_n$, and $q_0, q_1, \ldots, q_n \in \mathbb{Q} \cap [0,1]$ such that $g^{-1}(q_i) \cap K_i = \emptyset$ for each $i = 0, 1, \ldots, n$.*

Proof. $\Longrightarrow$ Let $g \in C(X,[0,1])$.

Case I. $g(X) \neq [0,1]$. If $0 \notin g(X)$, take $K_0 = X$ and $q_0 = 0$. If $1 \notin g(X)$, take $K_0 = X$ and $q_0 = 1$. Then we always have $g^{-1}(q_0) \cap K_0 = \emptyset$. If there exists $r \in (0,1)$ such that $r \notin g(X)$, we take $K_0 = g^{-1}([0,r))$ and $K_1 = g^{-1}((r,1])$, while $q_0 = 1$ and $q_1 = 0$. Then $g^{-1}(q_i) \cap K_i = \emptyset$ for each $i = 0, 1$.

Case II. $g(X) = [0,1]$. Then $g_\alpha = \alpha \circ g \in C(X,\mathbb{C})$. By hypothesis, there exists a potent $p \in C(X)$ such that $g_\alpha - p$ is nowhere 0. In view of Lemma 13.2.9, $p(x) \subseteq \{0\} \cup \mathbb{C}_n = \{0, \delta, \delta^2, \ldots, \delta^n\}$ for some $n \in \mathbb{N}$ and this generates a finite clopen cover X, say $K_0, K_1, \ldots, K_n$ such that $0 \notin g_\alpha(K_0)$ and $\delta^i \notin g_\alpha(K_i)$. Let $q_0, q_1, \ldots, q_n \in \mathbb{Q} \cap [0,1]$ be such that $q_0 = 0$ and $\alpha(q_i) = \delta^i$ for $i = 1, \ldots, n$. Then $g^{-1}(q_i) \cap K_i = \emptyset$, as required.

$\Longleftarrow$ Let $f \in C(X,\mathbb{C})$.

Case I. $R^0 \nsubseteq f(X)$. Then there exists $r \in R^0 \setminus f(X)$, and so $f(x) - r \in C(X)$ nowhere zero, hence invertible.

Case II. $R^0 \subseteq f(X)$. Then $f_\beta \in C(X,[0,1])$. By hypothesis, there is a finite clopen cover of X, say $K_0, K_1, \ldots, K_n$, and $q_0, q_1, \ldots, q_n \in \mathbb{Q} \cap [0,1]$ such that $f_\beta^{-1}(q_i) \cap K_i = \emptyset$ for each $i = 0, 1, \ldots, n$. Choose $r_i \in R^0$ such that $\beta(r_i) = q_i$ for each $i = 0, \ldots, n$. Let $p \in C(X,\mathbb{C})$ be the locally constant function which sends K_i to r_i. Then p is potent. Further, $f - p$ is nowhere zero, whence invertible. This completes the proof. □

The maximal connected subsets of a topological space are called the connected components of the space. A topological space X is totally disconnected if the connected components in X are the one-point sets. It is clear that every zero-dimensional space is totally disconnected. A compact Hausdorff space is zero-dimensional if and only if it is totally disconnected. The quasicomponent of a point $x \in X$ is defined as the intersection of all the clopen subsets containing x. Let Γ_x be the connected component of x in a topological space X, and Γ'_x be the quasicomponent of x. Then $\Gamma_x \subseteq \Gamma'_x$. We have

Lemma 13.2.11. *Let X be a topological space. If $C(X, \mathbb{C})$ is clean-like, then X is totally disconnected.*

Proof. Suppose that T is a quasicomponent containing distinct $x, y \in T$. Then there exists a continuous function $g : X \to [0, 1]$ such that $g(x) = 0$ and $g(y) = 1$. By the definition of a quasicomponent, $g(T) = [0, 1]$. In view of Lemma 13.2.10, there is a finite clopen cover of X, say $K_0, K_1, \ldots, K_n$, and $q_0, q_1, \ldots, q_n \in \mathbb{Q} \cap [0, 1]$ such that $g^{-1}(q_i) \cap K_i = \emptyset$ for each $i = 0, 1, \ldots, n$. It follows by the definition of a quasicomponent that $T \subseteq K_i$ for some i. Thus $f(K_i) = [0, 1]$, and so $g^{-1}(q_i) \cap K_i \neq \emptyset$, a contradiction. Therefore X is totally disconnected. □

Theorem 13.2.12. *Let X be a topological space. If X is compact, then the following are equivalent:*

(1) *$C(X)$ is clean.*
(2) *$C(X, \mathbb{C})$ is clean-like.*

Proof. (1) ⇒ (2) In view of Theorem 13.2.1, $C(X, \mathbb{C})$ is clean, and then it is clean-like.

(2) ⇒ (1) Since X is compact, $C(\beta X, \mathbb{C}) \cong C(X, \mathbb{C})$. By hypothesis, $C(\beta X, \mathbb{C})$ is clean-like. In view of Lemma 13.2.11, βX is totally disconnected, and then βX is zero-dimensional. In light of Theorem 13.1.19, $C(X)$ is clean. □

Let A be a subring with 1 and subspace of $\mathbb{R}$ in the usual topology and let $C(X, A)$ be the set of all continuous A-valued functions on X. Under point-wise operations, $C(X, A)$ is a commutative ring with identity. It is natural to ask when $C(X, A)$ is a clean ring. Roughly speaking, we seek correspondences between topological properties of X and algebraic properties of $C(X, A)$ which depend on the algebraic properties of A. A subset A of a topological space V is dense in V if for any point x in V, any neighborhood of x contains at least one point from A (i.e., A has non-empty intersection with every non-empty open subset of V). Equivalently, A is dense in V if and only if the only closed subset of V containing A is V itself. For instance, $\mathbb{Q}$ is a dense subring of $\mathbb{R}$.

Theorem 13.2.13. *Let X be a topological space and let F be a subfield of $\mathbb{R}$. Then the following are equivalent:*

(1) *$C(X)$ is clean.*
(2) *$C(X, F)$ is clean.*

Proof. By a routine adaption of the proof of [411, Theorem 6], $C(X, F)$ is clean if and only if βX is zero-dimensional. Therefore we complete the proof, by Theorem 13.1.19. □

Corollary 13.2.14. *Let X be a topological space. Then the following are equivalent:*

(1) *$C(X)$ is clean.*
(2) *$C(X, \mathbb{Q})$ is clean.*

We come now to consider the case when A is a dense clean subring of $\mathbb{R}$, which is not a field. The class of such subrings contains many familiar examples.

Example 13.2.15.

(1) For any prime p, $\mathbb{Z}_{(p)} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, (m, n) = 1, p \nmid n\}$.
(2) $\mathbb{Z}_{(2)}[\sqrt{q}]$, where $q \in \mathbb{N}$ is not a square.

Lemma 13.2.16. *Let A be a clean ring with trivial idempotents. Then $2 \notin U(A)$ if and only if there exists a unit $u \in A$ such that $u - 1$ and $u + 1$ are not units.*

Proof. $\Longrightarrow$ If not, then $1 - 1$ or $1 + 1$ is invertible. This implies that $2 \in U(A)$, an absurd.

$\Longleftarrow$ Suppose that $2 \in U(A)$. By hypothesis, $\frac{u+1}{2}, \frac{u-1}{2} \notin U(A)$. Hence, $\frac{u+1}{2} \in A$ is not clean, a contradiction. This completes the proof. □

Lemma 13.2.17. *Let X be a topological space. The the following are equivalent:*

(1) *X is a P-space.*
(2) *X is zero-dimensional and every countable union of pairwise disjoint clopen sets is closed.*
(3) *$f^{-1}(T)$ is clopen for every $f \in C(X)$ and for every $T \subseteq \mathbb{R}$.*

Proof. This is routine. □

Lemma 13.2.18. *Let A be a dense clean subring of $\mathbb{R}$, which is not a field. We use $c(A)$ to denote the set $\{a \in A \backslash U(A) \mid a - 1 \in U(A)\}$. Then $c(A)$ is dense in $\mathbb{R}$.*

Proof. Let $r \in \mathbb{R}$. Since A is clean and not a field, we may choose $b \in c(A)\backslash\{0\}$. For any $a \in A$, we define $\alpha_a = ab \in c(A)$ and find that $\alpha_a - 1 \in U(A)$. Let $\{a_n\} \subseteq A$ such that $a_n \mapsto \frac{r}{b}$. Then $\alpha_{a_n} \mapsto r$. This shows that $c(A)$ is dense in $\mathbb{R}$, as asserted. □

Theorem 13.2.19. *Let A be a dense clean subring of $\mathbb{R}$, which is not a field, and let X be a zero-dimensional space. Then the following are equivalent:*

(1) *X is a P-space.*
(2) *$C(X, A)$ is a pm ring.*
(3) *$C(X, A)$ is clean.*
(4) *$C(X, A)$ is clean-like.*

Proof. (1) ⇒ (3) Let X be a P-space and let A_d be the space A endowed with the discrete topology with identity map $i : A_d \to A$. By virtue of Lemma 13.2.17, for every $f \in C(X, A)$ there is continuous $g : X \to A_d$ such that $f = iog$. Defining $c(f) = \{x \mid f(x) \in c(A)\}$, we have $c(f) = (iog)^{-1}(c(A))$ is clopen and $f - \chi_{c(f)} \in U(C(X, A))$. Therefore, $C(X, A)$ is clean.

(3) ⇒ (2) This is proved by Theorem 12.2.3.

(2) ⇒ (1) Assume that X is not a P-space. In light of Lemma 13.2.17, let $\{K_n\}_{n=1}^{\infty}$ be pairwise disjoint clopen sets such that $\bigcup_n K_n$ is not closed. Let $\{b_n\}_{n=1}^{\infty} \subseteq c(A)$ such that $b_n \to -1$ and define $f \in C(X, A)$ such that $f|_{K_n} = b_n + 1$ and $f(x) = 0$ otherwise. Since $C(X, A)$ is a pm ring, it follows by Theorem 12.3.3 that there exist $a, a' \in C(X, A)$ such that $\big(1 - af\big)\big(1 - a'(1 - f)\big) = 0$. For $x \in K_n$, we have $1 - a'(x)(1 - f(x)) = 1 + b_n a'(x) \neq 0$, as $b_n \notin U(A)$. This implies that $a(x) = \frac{1}{f(x)} = \frac{1}{b_n+1}$ for $x \in K_n$. But a is undefined for points not in $\bigcup_n K_n$, since $b_n + 1 \to 0$. This gives a contradiction. Therefore X is a P-space.

(3) ⇒ (4) This is trivial.

(4) ⇒ (1) Let K_n be a sequence of pairwise disjoint clopen sets in X. In view of Lemma 13.2.16, we have $u \in U(A)$ such that $u + 1, u - 1$ are not units. By Lemma 13.2.18, $c(A)$ is dense in $\mathbb{R}$, let (b_n) be a sequence in $c(A)$ that converges to u. Construct $f \in C(X, A)$ such that $f\ |_{K_n} = b_n$ and $f(x) = u$ otherwise. Let U be an open set in A. If $u \in U$, then U contains all but finitely many $b_n's$. Hence, $X\backslash f^{-1}(U)$ is closed being a finite union of clopen sets. If $u \notin U$, then $f^{-1}(U)$ is a union of clopen sets; hence, it is open. Thus, $f \in C(X, A)$. By hypothesis, there exists some $p^3 = p \in C(X, A)$ such that $f - p$ is a unit. Clearly, $p(x)$ is $0, 1$ or -1.

Thus, we have disjoint clopen sets U and V such that $p = \chi_U - \chi_V$. f is already a unit outside $\bigcup_n K_n$ and adding ± 1 will make it a non-unit. In $\bigcup_n K_n$ it is a non-unit so its value needs to be changed. Hence, $\bigcup_n K_n$ has to be $U \cup V$, a clopen set. In light of Lemma 13.2.17, X is a P-space. □

Theorem 13.2.20. *Let A be a dense clean subring of $\mathbb{R}$, which is not a field, and $\frac{1}{2} \in A$. If X is a P-space, then $C^*(X, A)$ is clean-like.*

Proof. Let $f \in C^*(X, A)$. Set $c^+(f) = \{x \in X \mid f(x) \geq 0\}, c^-(f) = \{x \in X \mid f(x) < 0\}, u^+(A) = \{a \in U(A) \mid a + 1 \in U(A)\}$ and $u^-(A) = \{a \in U(A) \mid a - 1 \in U(A)\}$. Since $2 \in U(A)$, it follows by Lemma 13.2.16 that $U(A) = u^+(A) \cup u^-(A)$. Let $u^+(f) = \{x \in X \mid f(x) \in (-\frac{1}{2}, \frac{1}{2}) \cap u^+(A)\}$ and $u^-(f) = \{x \in X \mid f(x) \in (-\frac{1}{2}, \frac{1}{2}) \cap u^-(A)\}$. Looking at $f \in C(X)$, it follows by Lemma 13.2.17 that $c^+(f), c^-(f), u^+(f)$ and $u^-(f)$ are all clopen sets. One easily checks that $f - \chi_{c^-(f) \cup u^-(f)} + \chi_{c^+(f) \cup u^+(f)}$ is a unit in $C^*(X, A)$. Let $e = \chi_{c^-(f) \cup u^-(f)} - \chi_{c^+(f) \cup u^+(f)}$. Then $e = e^3 \in C^*(X, A)$, hence the result. □

Theorem 13.2.21. *Let A be a dense clean subring of $\mathbb{R}$, which is not a field. If X is an infinite zero-dimensional space, then $C^*(X, A)$ is not a weakly clean ring.*

Proof. Since X is infinite, Hausdorff and zero-dimensional, we have a sequence $\{U_n\}_{n=1}^{\infty}$ of disjoint nonempty clopen sets in X. Let $\{a_n\}_{n \in \mathbb{N}} \subseteq c(A)$ be a sequence such that $a_n \mapsto -1$ and $\{b_n\}_{n \in \mathbb{N}} \subseteq c(A)$ be a sequence such that $a_n \mapsto 1$. Let $f \in C^*(X, A)$ be such that $f \mid_{U_{2n}} = a_n + 1, f \mid_{U_{2n-1}} = b_n - 1$ and $f(x) = 0$ otherwise. If $f = u + e$, with u a unit and e an idempotent, then $u \mid_{U_{2n}} = a_n + 1$. As $a_n + 1 \mapsto 0$, this makes u^{-1} unbounded, a contradiction. Analogously, $f = u - e$ implies $u \mid_{U_{2n-1}} = b_n - 1$, a contradiction. This completes the proof. □

As every P-space is zero-dimensional, Theorem 13.2.20 and Theorem 13.2.21 provide a clean-like ring of continuous functions which is not weakly clean.

13.3 Classical Rings of Quotients

Let R be a ring and let $q(R)$ be the classical ring of quotients of R. Let $q(X) = q(C(X))$. The purpose of this section is to investigate when $q(X)$ is clean.

Theorem 13.3.1. *Let R be a ring. Then the following are equivalent:*

(1) *$q(R)$ is clean.*
(2) *For any $a, b \in R$ with $a + b \in N_D(R)$, there exist $x \in aR, y \in bR$ such that $x + y \in N_D(R)$ and $xy = 0$.*

Proof. (1) $\Rightarrow$ (2) Suppose that $a + b \in N_D(R)$ with $a, b \in R$. Then there exists some $\alpha \in q(R)$ such that $a\alpha + b\alpha = 1$. Since $q(R)$ is clean, it is an exchange ring, by Lemma 7.1.9. Then we have an idempotent $e \in q(R)$ such that $e \in b\alpha q(R) \subseteq bq(R)$ and $1 - e \in a\alpha q(R) \subseteq aq(R)$. Write $e = \frac{bs}{u}$. Set $w = bs \in bR$. Then $e = \frac{w}{u}$. Analogously, $1 - e = \frac{z}{t}$, where $z \in aR, t \in R$. Obviously, $\frac{w}{u} + \frac{z}{t} = 1$, and so $wt + zu = ut$. Choose $x = wt$ and $y = zu$. Then $x + y = ut \in N_D(R)$ and $xy = (wz)(ut) = 0$, as desired.

(2) $\Rightarrow$ (1) Suppose that $\frac{a}{s} + \frac{b}{s} = 1$ in $q(R)$. Then $a + b = s \in N_D(R)$. By hypothesis, there exist $x \in aR, y \in bR$ such that $x + y \in N_D(R), xy = 0$. Let $e = \frac{x}{x+y}$. Then $e(1 - e) = \frac{xy}{x+y} = 0$, and so $e = e^2 \in q(R)$ is an idempotent. Moreover, $e = \frac{x}{x+y} \in \left(\frac{a}{s}\right)q(R)$ and $1 - e = \frac{y}{x+y} \in \left(\frac{b}{s}\right)q(R)$. Therefore $q(R)$ is clean. □

Corollary 13.3.2. *Let R be a ring such that $q(R)$ is a finite product of indecomposable rings (e.g., if R is noetherian). Then the following are equivalent:*

(1) *$q(R)$ is clean.*
(2) *For any $P, Q \in Spec(R)$ with $P, Q \subseteq Z(R)$ and $(P + Q) \cap N_D(R) \neq \emptyset$, there exist $x \in P, y \in Q$ such that $x + y \in N_D(R)$ and $xy = 0$.*

Proof. (1) $\Rightarrow$ (2) For any $P, Q \in Spec(R)$ with $P, Q \subseteq Z(R)$ and $(P + Q) \cap N_D(R) \neq \emptyset$, we have $a \in P$ and $b \in Q$ such that $a + b \in N_D(R)$. In light of Theorem 13.3.1, there exist $x \in aR \subseteq P, y \in bR \subseteq Q$ such that $x + y \in N_D(R)$ and $xy = 0$, as required.

(2) $\Rightarrow$ (1) We may assume that $q(R)$ is indecomposable. Suppose $P', Q' \in Spec\big(q(R)\big)$ with $P' \cap N_D(q(R)) = Q' \cap N_D(q(R)) = \emptyset$ and $P' + Q' = q(R)$. Then there are $P, Q \in Spec(R)$ with $P' = S^{-1}P$

and $Q^{'} = S^{-1}Q$ where $S = N_D(R)$. Moreover, $P, Q \subseteq Z(R)$ with $(P + Q) \cap N_D(R) \neq \emptyset$. By hypothesis, there exist $x \in P$, $y \in Q$ such that $x + y \in S$ and $xy = 0$. Write $s = x + y$. Then $xs^{-1} + ys^{-1} = 1$ and then $e = xs^{-1} \in q(R)$ is a nontrivial idempotent. This gives a contradiction. Therefore $P^{'} + Q^{'} \neq q(R)$. In light of Theorem 11.4.21, $q(R)$ is almost clean, the result follows. □

Proposition 13.3.3. *Let R be a ring. Then the following are equivalent:*

(1) *$q(R)$ is clean.*
(2) *For any $a \in R, b \in N_D(R)$, there exists $e \in R$ and $x, y \in N_D(R)$ such that $ax = b(e + y)$ and $e^2 = ex$.*

Proof. (1) ⇒ (2) Let $a \in R, b \in N_D(R)$. Then we have an idempotent $\frac{e}{x} \in q(R)$ and $\frac{y}{x} \in U(q(R))$ such that $\frac{a}{b} = \frac{e}{x} + \frac{y}{x}$. Hence, $e^2x = ex^2$, and so $e^2 = ex$. Moreoverover, $ax = b(e + y)$ with $x, y \in N_D(R)$, as required.

(2) ⇒ (1) This is similar as the previous argument is completely reversible. □

The following examples were constructed by Burgess and Raphael (see [87, Example 2.3]), and Knox et al. (see [368, Example 4.5]).

Example 13.3.4. For $n \in 3\mathbb{N}$ let $R_n = \mathbb{Z}_3[x]_{(x)}$, for $n \in 3\mathbb{N} + 1$ let $R_n = \mathbb{Z}_{(3)}$ and for $n \in 3\mathbb{N} + 2$, say $n = 3k + 2$, let $R_n = \mathbb{Z}_{p_k}[x]_{(x)}$, where p_k is k th prime. Let R be the ring of sequences $(a_1, a_2, \ldots)$ with $a_n \in R_n$ for all $n \in \mathbb{N}$ with the proviso that there is some $f \in \mathbb{Z}[x]$ such that for some $m = 3k$, $a_m = \overline{f}$, $a_{m+1} = f(0)$, $a_{m+2} = \overline{f}$, with the pattern repeated for all triples beyond m. Then, R is clean but $q(R)$ is not clean.

Example 13.3.5. Let $T = \mathbb{F}[x, y]_{(x,y)}$ where $\mathbb{F}$ is a field. Then T has at least two prime ideals of height one, say P_1 and P_2. Let $M = T/P_1 \oplus T/P_2$, and let $R = T(T, M)$ be the trivial extension. Then R is local; hence, it is clean. But $q(R)$ is not clean.

Let A be a subset of X. We say that A is nowhere dense if every open set of X contains an open subset that is disjoint from A. This is equivalent to saying that A has empty interior. Clearly, every subset of a nowhere dense set is nowhere dense. We say that A is dense in X if $X \setminus A$ is nowhere dense.

Lemma 13.3.6. *Let X be a topological space, and let $f \in C(X)$. Then the following are equivalent:*

(1) $f \in N_D(X)$.
(2) $Z(f)$ *is nowhere dense.*

Proof. (1) $\Rightarrow$ (2) Let C be an open set of X, and let $B = C \cap \big(X - Z(f)\big)$. If $B \neq \emptyset$, then B is an open subset of C. In addition, $Z(f) \cap B = \emptyset$. If $B = \emptyset$, then we have $C \subseteq Z(f)$, and so $f(C) = 0$. Choose $a \in C$. Since X is a completely regular space, we can find some $g \in C(X)$ such that $g(x) = 0$ for any $x \in X - C$ and $g(a) = 1$. This implies that $fg = 0$, and so $g = 0$, an absurd.

(2) $\Rightarrow$ (1) Assume that $f\phi = 0$ for some $\phi \in C(X)$. Assume that $Z(\phi) \neq X$. By hypothesis, there exists an open subset B of $X - Z(\phi)$ such that $Z(f) \cap B = \emptyset$. Thus, we can find $x \in B$ such that $x \notin Z(f)$. This implies that $f(x), \phi(x) \neq 0$. This yields that $f\phi \neq 0$, a contradiction. Thus, $Z(\phi) = X$, and so $\phi = 0$. Accordingly, $f \in N_D(X)$. □

Theorem 13.3.7. *Let X be a topological space. Then the following are equivalent:*

(1) $q(X)$ *is clean.*
(2) *For any zero sets A and B of X such that $A \cap B$ is nowhere dense, there exist zero sets U, V such that $A \subseteq U, B \subseteq V$ such that $U \cap V$ is nowhere dense and $U \cup V = X$.*

Proof. (1) $\Rightarrow$ (2) For any zero sets A and B of X such that $A \cap B$ is nowhere dense, we can write $A = Z(f)$ and $B = Z(g)$. Since $Z(f^2 + g^2) = Z(f) \cap Z(g) = A \cap B$ is nowhere dense, it follows from Lemma 13.3.6 that $f^2 + g^2 \in N_D(X)$. In view of Lemma 13.3.1, there exist self-adjoint $h \in f^2C(X), k \in g^2C(X)$ such that $h + k \in N_D(X)$ and $hk = 0$. Let $U = Z(h)$ and $V = Z(k)$. Then $A \subseteq U, B \subseteq V$. In addition, $U \cup V = Z(h) \cup Z(k) = Z(hk) = Z(0) = X$. Further, $U \cap V = Z(h) \cap Z(k) = Z(h^2 + k^2)$. As $h^2 + k^2 = (h + k)^2$, we see that $U \cap V = Z\big((h + k)^2\big) = Z(h + k)$ is nowhere dense from Lemma 13.3.6. Since $h, k \in q(X)$ are self-adjoint, U and V are both zero sets, as required.

(2) $\Rightarrow$ (1) Let $f, g \in C(X)$ such that $f + g \in N_D(X)$. Let $A = Z(f)$ and $B = Z(g)$. Then $A \cap B = Z(f) \cap Z(g) \subseteq Z(f + g)$; hence, $A \cap B$ is nowhere dense from Lemma 13.3.6. By hypothesis, there exist zero sets U, V such that $A \subseteq U, B \subseteq V$ such that $U \cap V$ is nowhere dense and $U \cup V = X$. Thus, we can find self-adjoint $h, k \in C(X)$ such that $U = Z(h)$

and $V = Z(k)$. Set $\varphi = fh \in fC(X)$ and $\psi = gk \in gC(X)$. Then $Z(\varphi) = Z(fh) = Z(f) \cup Z(h) = Z(h)$. Likewise, $Z(\psi) = Z(k)$. Thus, $Z(\varphi^2 + \psi^2) = Z(\varphi) \cap Z(\psi) = Z(h) \cap Z(k)$ is nowhere dense, and so $\varphi^2 + \psi^2 \in N_D(X)$ from Lemma 13.3.6. As $Z(\varphi^2\psi^2) = Z(\varphi) \cup Z(\psi) = Z(h) \cup Z(k) = X$, we see that $\varphi^2\psi^2 = 0$. According to Theorem 13.3.1, $q(X)$ is strongly clean. □

The complement of a zero set is called a cozero set. The cozero set of $f \in C(X)$ is $coz(f) = \{x \in X \mid f(x) \neq 0\}$. We call U is a cozero set provided that there exists a self-adjoint $f \in C(X)$ such that $U = coz(f)$.

Corollary 13.3.8. *Let X be a topological space. Then the following are equivalent:*

(1) *$q(X)$ is clean.*
(2) *For any cozero sets A and B of X such that $A \cup B$ is dense, there exist disjoint cozero sets U, V such that $U \subseteq A, V \subseteq B$ such that $U \cup V$ is dense.*

Proof. Choose $C = X \setminus A$ and $D = X \setminus B$. Applying Theorem 13.3.7 to C and D, we complete the proof. □

Lemma 13.3.9. *Let X be a topological space. Then the following are equivalent:*

(1) *Every $f \in C(X)$ is clean in $q(X)$.*
(2) *For any $f \in C(X)$, there are disjoint cozero sets $C_1, C_2 \subseteq X$ such that $U = C_1 \cup C_2$ is a dense subset of X, $U \cap Z(f) \subseteq C_1$ and $U \cap Z(1 - f) \subseteq C_2$.*

Proof. (1) $\Rightarrow$ (2) Let $f \in C(X)$. Then there exists $e^2 = e \in q(X)$ and $u \in U(q(X))$ such that $f = e + u$. Write $u = \frac{k}{l}$. Then $k \in N_D(X)$, and so $Z(k)$ is nowhere dense. Thus, $C = coz(k)$ is dense in X. Set $C_1 = C \cap coz(e) \cap coz(u^-)$ and $C_2 = C \setminus C_1$ where $u^- = \min\{u(x), 0\}$. Clearly, $coz(e)$ and $coz(u^-)$ are clopen subsets of C. Thus C_1 and C_2 are disjoint cozero sets of X and $U = C_1 \cup C_2$ is dense in X. Furthermore, $U \cap Z(f) \subseteq C_1$ and $U \cap Z(1 - f) \subseteq C_2$, as required.

(2) $\Rightarrow$ (1) Let $f \in C(X)$. By hypothesis, there are disjoint cozero sets $C_1, C_2 \subseteq X$ such that $U = C_1 \cup C_2$ is a dense subset of X, $U \cap Z(f) \subseteq C_1$ and $U \cap Z(1 - f) \subseteq C_2$. Let $e = \chi_{C_1}$ and $u = (f - 1)\chi_{C_1} + f\chi_{C_2}$, where χ_A is the characteristic function of $A \subseteq X$. Then $u, e \in C(X) \subseteq q(X)$. One easily checks that $f = e + u$ on U. Since U is dense in X, we see that

$f = e + u$ in $q(X)$. Clearly, $e^2 = e \in q(X)$. We will suffice to prove that $u \in q(X)$ is invertible. Since $U \cap Z(f) \subseteq C_1$, $f\chi_{C_2}(x) \neq 0$ on C_2. Similarly, $(f-1)\chi_{C_1}(x) \neq 0$ on C_1. Thus $u(x) \neq 0$ for any $x \in U$. Hence $U \subseteq coz(u)$. If $x \notin U$, then $u(x) = 0$; hence, $x \notin coz(u)$. We infer that $coz(u) \subseteq U$; whence, $U = coz(u)$. As U is a dense subset of X, $Z(u)$ is nowhere dense. In light of Lemma 13.3.6, $u \in N_D(X)$. Therefore $u \in q(X)$ is invertible, hence the result. □

Let Y be a subset of X. Then restriction map $f \mapsto f|_Y$ maps $C(X)$ to $C(Y)$.

Lemma 13.2.10. *Let Y be a dense cozero subset of X. Then $q(X)$ is naturally isomorphic to $q(Y)$.*

Proof. Let $i : C(X) \to C(Y)$ be the restriction map. Then we get a natural map: $j : q(X) \to q(Y)$. Since Y is dense, j is a monomorphism. Let $f \in N_D(Y)$. Then we can regard f as an element in $C(X)$. It will suffice to show that $f \in N_D(X)$. Clearly, $Y \setminus Z(f)$ is dense in Y, and then in X. Hence $X \setminus Z(f)$ is dense in X. That is, $Z(f)$ is nowhere dense. In light of Lemma 13.3.6, $f \in N_D(X)$, as required. □

Theorem 13.3.11. *Let X be a topological space. Then the following are equivalent:*

(1) *$q(X)$ is clean.*
(2) *For any dense cozero set Y of X, every $f \in C(Y)$ is clean in $q(Y)$.*

Proof. (1) ⇒ (2) Let Y be a dense cozero set of X, and let $f \in C(Y)$. We may think $f \in q(X)$, and so there exists $e^2 = e \in q(X)$ and $u \in U(q(X))$ such that $f = e + u$. Write $u = \frac{k}{t}$. Set $Z = coz(k)$. Then Z is a dense set of X, by Lemma 13.3.6. Let $K = \{x \in Z \mid e(x) = 1\}$. Then K is a clopen subset of Z. Let $g = \chi_{K \cap Y}$, i.e., the characteristic function of $K \cap Y$ in Y. Then $g^2 = g \in q(Y)$. It is easy to verify that

$$u|_{Z\cap Y} = \begin{cases} f(x) - 1 \text{ if } x \in K \cap Y \\ f(x) \qquad \text{if } x \in \big(Z \cap Y\big) \setminus \big(K \cap Y\big). \end{cases}$$

Clearly, $u|_{Z\cap Y} \in C(Z \cap Y)$, and so we may think $u|_{Z\cap Y} \in q(Y)$ as $Z \cap Y$ is dense in Y. As $u \in U(q(X))$, we see that $u|_{Z\cap Y} \in q(Y)$ is a unit. Further, we have $f = g + u|_{Z\cap Y}$, and therefore $f \in q(Y)$ is clean.

(2) ⇒ (1) Let $f \in q(X)$. By the representation theorem for $q(X)$, we may assume that $f \in C(Y)$ for some dense cozero set $Y \subseteq X$. Then

$f \in C(Y)$ is clean in $q(Y)$. In light of Lemma 13.2.10, $q(Y)$ is naturally isomorphic to $q(X)$. Therefore $f \in q(X)$ is clean. □

Corollary 13.3.12. *Let X be a topological space. If $C(X)$ is clean, then $q(X)$ is clean.*

Proof. Suppose that $C(X)$ is clean. Let Y be any dense cozero set of X. Then $C(Y)$ is clean. Hence, every $f \in C(Y)$ is clean in $q(Y)$. According to Theorem 13.3.11, $q(X)$ is clean. □

We note that $q(\mathbb{R})$ is clean, but $C(\mathbb{R})$ is not clean, as $\mathbb{R}$ is not strongly zero-dimensional. But we indeed know that $C(X)$ is clean if and only if $q(X)$ is clean for all topological F-space (see [87, Proposition 3.3])

A commutative ring R is a PF-ring if, $ab = 0$ in R implies that $r(a) + r(b) = R$. We note that a ring R is a PF-ring if and only if every principal ideal of R is a flat R-module (see [268, Theorem 4.2.2]).

Proposition 13.3.13. *Let R be a ring. Then $q(R)$ is a PF-ring if and only if $ab = 0$ in R implies that $r(a) + r(b) \subseteq N_D(R)$.*

Proof. For any $\frac{a}{s}, \frac{b}{t} \in q(R)$, we observe that $\frac{a}{s} \cdot \frac{b}{t} = \frac{0}{1}$ in $q(R)$ if and only if $ab = 0$ in R. Moreover, $r_{q(R)}\left(\frac{a}{s}\right) = r_{q(R)}(a)$ and $r_{q(R)}\left(\frac{b}{t}\right) = r_{q(R)}(b)$. Thus, $r_{q(R)}\left(\frac{a}{s}\right) + r_{q(R)}\left(\frac{b}{t}\right) = q(R)$ if and only if $r(a) + r(b) \subseteq N_D(R)$, as desired. □

Theorem 13.3.14. *Let X be a topological space. Then the following are equivalent:*

(1) *$q(X)$ is a clean PF-ring.*
(2) *For any $f \in q(X)$, $r(f)$ is generated by idempotents in $q(X)$.*
(3) *For any $f, g \in q(X)$, $fg = 0 \Longrightarrow \exists$ an idempotent $e \in q(X)$ such that $f \in eq(X)$ and $g \in (1-e)q(X)$.*
(4) *For any $f, g \in q(X)$, $fg = 0 \Longrightarrow \exists$ finitely generated ideals I, J of $q(X)$ such that $f \in I, g \in J, IJ = 0$ and $(I + J) \cap N_D(q(X)) \neq \emptyset$.*
(5) *For any $a, b \in C(X)$, $ab = 0 \Longrightarrow \exists$ finitely generated ideals I, J of $C(X)$ such that $a \in I, b \in J, IJ = 0$ and $(I + J) \cap N_D(X) \neq \emptyset$.*

Proof. (1) ⇒ (3) Let $fg = 0$ with $f = \frac{a}{s}, g = \frac{b}{s} \in q(X)$. Since $q(X)$ is a PF-ring, we can find $x \in r(\frac{a}{s}), y \in r(\frac{b}{s})$ such that $x + y = 1$ in $q(X)$. Since $q(X)$ is clean, it is an exchange ring by Lemma 7.1.9. Thus, we can find an idempotent $e \in q(X)$ such that $e \in yq(X)$ and $1 - e \in xq(X)$.

Write $e = yk$ and $1 - e = xl$ for some $k, l \in q(X)$. As $xa = 0$, we get $f = \big((1-x)a\big) \cdot \frac{1}{s} = (ya) \cdot \frac{1}{s} = \big((1-(1-e))ya\big) \cdot \frac{1}{s} \in eq(X)$. Likewise, $g \in (1-e)q(X)$, thus proving (3).

(3) $\Rightarrow$ (1) Suppose that $fg = 0$ for some $f, g \in q(X)$. Then we have an idempotent $e \in q(X)$ such that $f \in eq(X)$ and $g \in (1-e)q(X)$. Clearly, $1 - e \in r(f)$ and $e \in r(g)$. It follows from $1 = (1-e) + e$ that $r(f) + r(g) = q(R)$. This shows that $q(X)$ is a PF-ring.

Let M and N be distinct maximal ideals of $q(X)$. Then $M + N = q(X)$. Write $p + q = 1$ with $p \in M, q \in N$. We note that $q(X)$ is a pm ring. In view of Theorem 12.3.3, there exist $s, t \in q(X)$ such that $(1 - ps)(1 - qt) = 1$. Set $a = 1 - ps$ and $b = 1 - qt$. Then $ab = 0$ in $q(X)$. By hypothesis, there exists an idempotent $e \in q(X)$ such that $a \in eq(X)$ and $b \in (1-e)q(X)$. Clearly, $e \notin M$, and so $1 - e \in M$. Similarly, $e \in N$. Accordingly, $q(X)$ is clean, by Theorem 12.3.8.

(2) $\Rightarrow$ (3) For any $f, g \in q(X)$, $fg = 0$ implies that $g \in r(f)$. By hypothesis, there exists an idempotent $e \in q(X)$ such that $g \in (1-e)q(X) \subseteq r(f)$. This shows that $f(1-e) = 0$, and so $f = ef \in eq(X)$, as desired.

(3) $\Rightarrow$ (2) Let $f \in q(X)$ and $g \in r(f)$. Then $fg = 0$. Hence, we have an idempotent $e \in q(X)$ such that $f \in eq(X)$ and $g \in (1-e)q(X)$. This implies that $g \in (1-e)q(X) \subseteq r(f)$, as required.

(3) $\Rightarrow$ (4) Let $f, g \in q(X)$ be such that $fg = 0$. Then there exists an idempotent $e \in q(X)$ such that $f \in eq(X)$ and $g \in (1-e)q(X)$. Set $I = eq(X)$ and $J = (1-e)q(X)$. Then $f \in I, g \in J, IJ = 0$. As $1 \in I + J$, we see that $(I + J) \cap N_D(q(X)) \neq \emptyset$, as required.

(4) $\Rightarrow$ (5) Let $a, b \in C(X)$ be such that $ab = 0$. Then there exist finitely generated ideals K, L of $q(X)$ such that $a \in K, b \in L, KL = 0$ and $(K + L) \cap N_D(q(X)) \neq \emptyset$. As $N_D(q(X)) = U(q(X))$, we see that $K + L = q(X)$. Write $\frac{x}{s} + \frac{y}{s} = \frac{1}{1}$ where $\frac{x}{s} \in K$ and $\frac{y}{t} \in L$. Then $K = xq(X)$ and $L = yq(X)$. Set $I = aC(X) + xC(X)$ and $J = bC(X) + yC(X)$. Then I and J are finitely generated ideals of $C(X)$. Further, $a \in I, b \in J, IJ = 0$ and $(I + J) \cap N_D(X) \neq \emptyset$. Thus proving (4).

(5) $\Rightarrow$ (1) Let $f = \frac{a}{s}, g = \frac{b}{t} \in q(X)$ be such that $fg = 0$. Then $ab = 0$. By hypothesis, there exist two finitely generated ideals I, J of $C(X)$ such that $a \in I, b \in J, IJ = 0$ and $(I + J) \cap N_D(X) \neq \emptyset$. Say $r = x + y \in N_D(X)$ where $x \in I, y \in J$. Then $\frac{x}{r} + \frac{y}{r} = \frac{1}{1}$. As $xy = 0$, we see that $e = \frac{x}{r} \in q(R)$ is an idempotent, and that $1 - e = \frac{y}{r}$. Since $ay = 0$, we have $f = \frac{a}{s} = \frac{a}{s}\big(\frac{x}{r} + \frac{y}{r}\big) \in eq(X)$. Likewise, $g \in (1-e)q(X)$, as desired. □

13.4 Feebly Clean Rings

An element a in a ring R is feebly clean if a can be written as $a = e_1 - e_2 + u$ where u is a unit and e_1, e_2 are orthogonal idempotents, that is, $e_1^2 = e_1, e_2^2 = e_2$ and $e_1 e_2 = 0 = e_2 e_1$. A ring R is feebly clean if every element in R is feebly clean. We investigate, in this section, feebly cleanness for the rings of complex-valued continuous functions. We begin with elementary properties of feebly clean rings.

Proposition 13.4.1. *Let R be a ring. Then the following are equivalent:*

(1) *R is feebly clean.*
(2) *For any $a \in R$, there exist idempotents e_1, e_2 and unit $u \in R$ such that $a = e_1 + e_2 + u$.*
(3) *For any $a \in R$, there exist idempotents e_1, e_2 and unit $u \in R$ such that $a = e_1 - e_2 + u$.*

Proof. (1) $\Rightarrow$ (3) This is obvious.

(3) $\Rightarrow$ (2) Let $a \in R$. Then there exist idempotents e_1, e_2 and unit $u \in R$ such that $1 - a = e_1 - e_2 + u$. Hence $a = (1 - e_1) + e_2 - u$, as desired.

(2) $\Rightarrow$ (1) Let $a \in R$. Then there exist idempotents e_1, e_2 and unit $u \in R$ such that $1 - a = e_1 + e_2 + u$. Thus $a = (1 - e_1) - e_2 - u = (1 - e_1)(1 - e_2) - e_2 e_1 - u$, as required. □

Proposition 13.4.2. (1) *Every weakly clean ring is feebly clean.*

(2) *Every 2-nil-clean ring is feebly clean.*

(3) *Every homomorphic image of feebly clean rings is feebly clean.*

(4) *A direct product $R = \prod_{i \in I} R_i$ of rings R_i is feebly clean if and only if each R_i is feebly clean.*

Proof. Straightforward. □

Proposition 13.4.3. *Let R be a ring. Then the following are equivalent:*

(1) *R is feebly clean.*
(2) *$R[[x]]$ is feebly clean.*

Proof. (1) $\Rightarrow$ (2) Let $\psi(x) \in R[[x]]$. Then there exist two idempotents $e, f \in R$ and a unit $u \in R$ such that $\psi(0) = e + f + u$. Hence $\psi(x) = e + f + (u + \psi(x) - \psi(0))$ with $u + \psi(x) - \psi(0) \in U(R[[x]])$, as desired.

(2) $\Rightarrow$ (1) Since R is a homomorphic image of $R[[x]]$, we obtain the result by Proposition 4.2. □

Example 13.4.4. The polynomial ring $R[x]$ is not feebly clean.

Proof. Assume $x = e_1 - e_2 + u$ where e_1, e_2 are orthogonal idempotents in $R[[x]]$ and $u \in U(R[[x]])$. Then $e_1, e_2 \in R$, and so $(e_2 - e_1) + x \in R[x]$ is invertible, an absurd. □

Proposition 13.4.5. *Let I be a nil ideal of a ring R. If R/I is feebly clean, then R is feebly clean.*

Proof. Let $a \in R$. In view of Proposition 3.4.1, there exists two idempotents $\overline{e_1}, \overline{e_2}$ in R/I and a unit $\overline{u} \in R/I$ such that $\overline{a} = \overline{e_1 + e_2 + u}$. Since every idempotent and every unit lift modulo I, we may assume that $e_1, e_2 \in R$ are idempotents and $u \in U(R)$. Hence $a = e_1 + e_2 + (u + r)$ for some $r \in I$. As $u + r \in U(R)$, we obtain the result. □

Theorem 13.4.6. *Let R be a ring. If $\frac{1}{2} \in R$, then the following are equivalent:*

(1) *R is feebly clean.*
(2) *For any $a \in R$, there exists $e^3 = e$ and $u \in U(R)$ such that $a = e + u$.*
(3) *For any $a \in R$, there exist $e, f \in R$ such that $a = e - f + u, e^2 = f^2 = 1$ and $(1 + e)(1 + f) = 0$.*

Proof. (1) ⇒ (2) Let $a \in R$. Then there exist idempotents e_1, e_2 and unit $u \in R$ such that $a = e_1 - e_2 + u$. Clearly, $(e_1 - e_2)^3 = e_1 - e_2$, as desired.

(2) ⇒ (1) Let $a \in R$. Then there exists $e^3 = e$ and $u \in U(R)$ such that $a = e + u$. Let $b = \frac{e^2+e}{2}$ and $c = \frac{e^2-e}{2}$. Then $b^2 = b$ and $c^2 = c$. Moreover, $a = b - c + u$. Therefore R is feebly clean by Proposition 13.4.1.

(1) ⇒ (3) Let $a \in R$. We have two orthogonal idempotents e_1, e_2 and unit $u \in R$ such that $\frac{a}{2} = e_1 - e_2 + u$. Thus $a = (2e_1 - 1) - (2e_2 - 1) + 2u$. Set $e = 2e_1 - 1, f = 2e_2 - 1$. Then $e^2 = f^2 = 1$ and $(1 + e)(1 + f) = 4e_1e_2 = 0$, as desired.

(3) ⇒ (1) Let $a \in R$. By hypothesis, there exist $e, f \in R$ such that $2a = e - f + u, e^2 = f^2 = 1$ and $(1 + e)(1 + f) = 0$. Then

$$a = \frac{1+e}{2} - \frac{1+f}{2} + \frac{1}{2},$$

where $\frac{1+e}{2}, \frac{1+f}{2} \in R$ are orthogonal idempotents. Therefore R is feebly clean. □

Example 13.4.7. Let $R = \mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)}$. Then $R \times R$ is a feebly clean ring, while it is not weakly clean.

Proof. In view of Theorem 9.1.7, R is weakly clean, and so it is feebly clean. By virtue of Proposition 13.4.2, $R \times R$ is feebly clean. But $(\frac{3}{2}, \frac{5}{2}) \in R \times R$ is not a weakly clean element. □

Theorem 13.4.8. *Let X be a topological space and $f \in C(X)$. Then f is feebly clean in $C(X)$ if and only if there exist disjoint clopen sets U_1, U_2 in X, such that $Z(f) \subseteq U_1 \cup U_2, Z(1-f) \subseteq X \setminus U_1$ and $Z(1+f) \subseteq X \setminus U_2$.*

Proof. $\Longrightarrow$ Since f is feebly clean, there exist two orthogonal idempotents e_1, e_2 and a unit u in $C(X)$ such that $f = e_1 - e_2 + u$. Set $e_1 = \chi_{U_1}, e_2 = \chi_{U_2}$. Then $U_1 \cap U_2 = \emptyset$.

If $x \in Z(f)$, then $f(x) = 0$, then $e_1(x) \neq e_2(x)$, and so $x \in U_1 \cup U_2$. Hence $Z(f) \subseteq U_1 \cup U_2$.

If $x \in Z(1-f)$, then $f(x) = 1$. If $x \in U_1$, then $e_1(x) = 1$, and so $e_2(x) = u(x) \neq 0$. Hence $e_2(x) = 1$. This implies that $e_1 e_2 \neq 0$, a contradiction. Thus $Z(1-f) \subseteq X \setminus U_1$.

If $x \in Z(1+f)$, then $f(x) = -1$. If $x \in U_2$, then $e_2(x) = 1$, and so $e_1(x) = -u(x) \neq 0$. Hence $e_1(x) = 1$. This implies that $e_1 e_2 \neq 0$, a contradiction. Thus $Z(1+f) \subseteq X \setminus U_2$.

$\Longleftarrow$ Let $e_1 = \chi_{U_1}, e_2 = \chi_{U_2}$. Then e_1 and e_2 are orthogonal idempotents in $C(X)$. Since $f - e_1 + e_2$ is nowhere zero, we see that $f - e_1 + e_2$ is a unit in $C(X)$, as required. □

Furthermore, we have

Theorem 13.4.9. *Let R be a ring, and let $a \in R$. Then the following are equivalent:*

(1) *a is feebly clean.*
(2) *There exist idempotents e_1 and e_2 with $U(e_1) \cap U(e_2) = \emptyset$, such that $V(a) \subseteq U(e_1) \cup U(e_2), V(1-a) \subseteq V(e_1)$ and $V(1+a) \subseteq V(e_2)$.*
(3) *There exist orthogonal idempotents $e_1, e_2 \in R$ and $c, u, v \in R$ such that $ac = (1-e_1)(1-e_2), (1-a)u = e_1$ and $(1+a)v = e_2$.*

Proof. (1) $\Rightarrow$ (3) By hypothesis, there exist two orthogonal idempotents $e_1, e_2 \in R$ and a unit $u \in R$ such that $a = e_1 - e_2 + u$. Then $au^{-1}(1 - e_1)(1-e_2) = (1-e_1)(1-e_2)$, $(1-a)e_1(-u^{-1}) = e_1$ and $(1+a)e_2u^{-1} = e_2$.

(3) $\Rightarrow$ (2) By hypothesis, there exist orthogonal idempotents $e_1, e_2 \in R$ and $c, u, v \in R$ such that $ac = (1-e_1)(1-e_2), (1-a)u = e_1$ and $(1+a)v = e_2$.

Step 1. $U(e_1) \cap U(e_2) = \emptyset$. If $M \in U(e_1) \cap U(e_2)$, then $e_1, e_2 \notin M$, and so $0 = e_1 e_2 \notin M$. This is an absurd.

Step 2. $V(a) \subseteq U(e_1) \cup U(e_2)$. If $M \in V(a)$, then $a \in M$, and so $(1-e_1)(1-e_2) \in M$; hence, $1-e_1$ or $1-e_2$ in M. Hence, $e_1 \notin M$ or $e_2 \notin M$. That is, $M \in U(e_1) \cup U(e_2)$, as claimed.

Step 3. $V(1-a) \subseteq V(e_1)$. This is obvious, as $(1-a)u = e_1$.

Step 4. $V(1+a) \subseteq V(e_2)$. This is obvious, as $(1+a)v = e_2$.

Thus proving (2).

(2) $\Rightarrow$ (1) We have idempotents $e_1, e_2 \in R$ as in (2). Let $u = a - e_1 + e_2$. If $uR \neq R$, by using Zorn's Lemma, we have a maximal ideal M such that $uR \subseteq M \subsetneqq R$. If $a \in M$, then $e_1 - e_2 = a - u \in M$. As $e_1e_2 = 0$, we see that $e_1, e_2 \in M$. This contradicts $V(a) \subseteq U(e_1) \cup U(e_2)$.

We now assume that $a \notin M$. Since $e_1(1-e_1) \in M$, we have $e_1 \in M$ or $1-e_1 \in M$. Thus $M \in V(e_1)$ or $M \in V(1-e_1)$. Likewise, $M \in V(e_2)$ or $M \in V(1-e_2)$.

If $M \in V(e_1)$ and $M \in V(e_2)$, then $e_1, e_2 \in M$, and so $a = e_1 - e_2 + u \in M$, an absurd.

If $M \in V(e_1)$ and $M \in V(1-e_2)$, then $e_1, 1-e_2 \in M$, and so $1+a = e_1 + (1-e_2) + u \in M$. Hence $M \in V(1+a) \subseteq V(e_2)$, i.e., $e_2 \in M$. Thus $1 = e_2 + (1-e_2) \in M$, an absurd.

If $M \in V(e_2)$, analogously, we give a contradiction. Therefore $u \in U(R)$, hence the result. □

By virtue of Theorem 13.1.18, we easily see that the ring $C(X)$ is clean if and only if it is feebly clean. We are ready to prove:

Theorem 13.4.10. *The ring $C(X, \mathbb{C})$ is clean if and only if it is feebly clean.*

Proof. $\Longrightarrow$ This is trivial.

$\Longleftarrow$ Let $f \in C(X, \mathbb{C})$ be feebly clean. Then $f = f_1 + if_2$, where $f_1, f_2 \in C(X)$. Let $g = 2f_1 - 1 \in C(X) \subseteq C(X, \mathbb{C})$. Then there exist orthogonal idempotents $e_1, e_2 \in C(X, \mathbb{C})$ such that $u := g - e_1 + e_2 \in C(X, \mathbb{C})$ is a unit. Since e_1 and e_2 take values $0, 1$, we see that $u \in C(X)$. Clearly, $u < 0$ on $Z(f_1)$ and $u_1 > 0$ on $Z(1-f_1)$. Let $K = \{x \in X \mid u(x) < 0\}$. Then K is a clopen set, and χ_K is the characteristic function on K. Then $f_1 - \chi_K$ is nowhere zero; hence, $v := f_1 - \chi_K \in U(C(X))$. Accordingly, $f = \chi_K + (v + if_2)$ is clean, as $v + if_2$ is a unit. □

Lemma 13.4.11. *Let A be a dense feebly clean subring of $\mathbb{C}$, which is not a field. Then the set of all non-units in A is also dense in $\mathbb{C}$.*

Proof. Let $b \in A$ be a nonzero non-unit. For any $a \in A$, we define $\alpha_a = ab$ which is also a non-unit. For any $c \in \mathbb{C}$, let $\{a_n\} \subseteq A$ such that $a_n \mapsto \frac{c}{b}$. Then $\alpha_{a_n} \mapsto c$, and therefore the set of all non-units is dense in $\mathbb{C}$. □

Lemma 13.4.12. *Let A be a feebly clean ring with trivial idempotents. If A is not clean, then there exists a unit $u \in A$ such that $u-1$ and $u+1$ are not units.*

Proof. Since A is not clean, there exists $a \in A$ such that a and $a-1$ are both non-units. Hence, $2a$ and $2a-2$ are both non-units. Then $2a-1$, being feebly clean, is the required unit. □

With the aid of these lemmas, we now prove:

Theorem 13.4.13. *Let A be a dense feebly clean subring of $\mathbb{C}$, which is not a field, and let X be a zero-dimensional space. Then*

(1) *If X is a P-space, then $C(X, A)$ is feebly clean.*
(2) *If A is not clean and $C(X, A)$ is feebly clean, then X is a P-space.*
(3) *$C(X, A)$ is clean if and only if X is a P-space and A is clean.*

Proof. (1) Let $f \in C(X, A)$, then $f \in C(X, \mathbb{C})$. Let $u(f) = \{x \in A \mid f(x) \in U(A)\}, u^+(f) = \{x \in A \mid f(x) \notin U(A), f(x)+1 \in U(A)\}, u^-(f) = \{x \in A \mid f(x) \notin U(A), f(x)-1 \in U(A)\}$. As A is feebly clean, we have $X = u(f) \cup u^+(f) \cup u^-(f)$. Since X is a P-space, it follows by Lemma 13.2.17 that $u^+(f)$ and $u^-(f)$ are clopen sets. We easily see that $f + \chi_{u^+(f)} - \chi_{u^-(f) \setminus u^+(f)}$ is a unit, hence the result.

(2) Since A is not clean, by virtue of Lemma 13.4.12, there exists a unit $u \in A$ such that $u-1$ and $u+1$ are not units. Let $\{K_n\}_{n=1}^{\infty}$ be pairwise disjoint clopen sets in X. Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of non-units in A that converges to u. Define $f \in C(X, A)$ such that $f \mid_{K_n} = b_n$ and $f(x) = u$ otherwise. Since $C(X, A)$ is feebly clean, we have disjoint clopen sets U and V such that $f - \chi_U + \chi_V$ is a unit of $C(X, A)$. f is already a unit outside $\bigcup_n K_n$ and adding ± 1 will make it a non-unit. In $\bigcup_n K_n$ it is a non-unit, so its value needs to be changed. Hence $\bigcup_n K_n$ has to be $U \cup V$, a clopen set. According to Lemma 13.2.17, X is a P-space, as asserted.

(3) $\Longrightarrow$ Since A is a homomorphic image of $C(X, A)$, A is clean. We note that Theorem 13.2.19 is easily generalized to the case, when A is a dense clean subring of $\mathbb{C}$ with 1. Thus X is a P-space.

$\Longleftarrow$ This is obvious by the note above. □

Notes. Theorem 13.1.5 can be seen in [205] and [265]. Theorem 13.1.11 and Theorem 13.1.14 are due to Azarpanah [40, Theorem 2.5] and McGovern [411, Theorem 13]. Theorem 13.1.17 was proved by McGovern [411, Theorem 6] and Samei [462, Theorem 2.7]. Theorem 13.1.18 was obtained in [26, Theorem 4.1]. Theorem 13.1.19 is due to McGovern [411, Theorem 13]. Theorem 13.2.1 and Theorem 13.2.12 were observed by Arora and Kundu [26, Theorem 5.1]. Theorem 13.2.19 was also proved by Arora and Kundu [26, Theorem 4.4]. Theorem 13.2.20 and Theorem 13.2.21 are in [25, Theorem 4.5 and Theorem 4.6]. The main results in Section 13.3 are due to Knox et al. [368]. Theorem 13.4.8, Theorem 13.4.9, Theorem 13.4.10 and Theorem 13.4.13 are contained in [27, Theorem 4.1, Corollary 4.5, Theorem 4.7 and Theorem 4.13].

Chapter 14

Rings with Involutions

An involution of a ring R is just an anti-automorphism whose square is the identity map 1_R. Thus an involution of a ring R is an operation $* : R \to R$ such that $(x+y)^* = x^* + y^*$, $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x, y \in R$. A ring R with involution $*$ is called a *-ring. The class of *-ring is very large. For instances, all C^*-algebra, all Rickart *-rings, all Baer *-rings, etc. Moreover, every commutative ring can be seen as a *-ring with the identity involution *. For general theory of a *-ring, we refer the reader to [68]. The purpose of this section is to explore various strong cleanness of *-rings. In Section 14.1, we investigate strongly *-clean rings, and discuss the several classes of such rings. In Section 14.2, we consider the uniqueness of strongly *-clean rings, and find the criteria for a *-ring to be uniquely strongly *-clean. In Section 14.3, we introduce a subclass of uniquely strongly *-clean rings, and present many examples of strongly nil-*-clean rings. In the final section we discuss topological spaces with strong *-cleanness, and determine when a continuous function ring is strongly *-clean.

14.1 Strongly *-Clean Rings

An element e in a *-ring R is called a projection if $e^2 = e = e^*$. A *-ring R is strongly *-clean provided that for any $a \in R$ there exists a projection $e \in R$ such that $a - e \in U(R)$ and $ae = ea$.

Theorem 14.1.1. *A *-ring R is strongly *-clean if and only if*

(1) *R is strongly clean;*
(2) *Every idempotent in R is a projection.*

Proof. Suppose R is strongly *-clean. Then it is strongly clean. Let $e \in R$ be an idempotent. There exists a projection $f \in R$ and a unit $u \in R$ such that $e = f+u$ and $ef = fe$. Thus, $(e-f)^3 = e-f$, and so $(e-f)^2 = 1$. This implies that $e(1-2f) = 1-f$, and so $e = (1-f)(1-2f)$. As $f^* = f$, we see that $e^* = e$. Thus $e \in R$ is a projection. Accordingly, every idempotent in R is a projection. The converse is obvious. □

Corollary 14.1.2. *Every strongly *-clean ring is abelian.*

Proof. Let R be strongly *-clean and $e^2 = e, x \in R$. Then $e+ex(1-e) \in R$ is an idempotent. In view of Theorem 14.1.1, we get $e + ex(1-e) = \big(e + ex(1-e)\big)^* = e^* + (1-e^*)x^*e^* = e + (1-e)x^*e$. Multiplying by e from the left sides, we have $e + ex(1-e) = e$, and so $ex = exe$. Likewise, $xe = exe$. Hence $ex = exe = xe$. Therefore R is abelian. □

Corollary 14.1.3. *A *-ring R is strongly *-clean if and only if for any $a \in R$ there exists a projection $e \in aR$ such that $1-e \in (1-a)R$.*

Proof. Suppose R is strongly *-clean. Then R is strongly clean. Let $a \in R$. In view of Theorem 1.1.13, there exists an idempotent $e \in aR$ such that $1-e \in (1-a)R$. By Theorem 14.1.1, $e \in R$ is a projection, as required.

Conversely, letting $e \in R$ be an idempotent. Then there exists a projection $f \in eR$ such that $1-f \in (1-e)R$. Write $f = er$ and $1-f = (1-e)s$ with $r, s \in R$. Then $f-e = (1-e)f - e(1-f) = (1-e)er - e(1-e)s = 0$, and so $e = f$. That is, every idempotent in R is a projection. According to Theorem 14.1.1, R is strongly *-clean. □

Let $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then the map $* : R \to R, (a,b)^* = (b,a)$ is an involution. Obviously, R is an strongly clean ring. Furthermore, R is a commutative *-ring. But R is not strongly *-clean. As the idempotent $e = (1,0) \in R$ is not a projection, we are though by Theorem 14.1.1.

Let R be a *-ring. An element $a \in R$ is called a partial isometry provided that $a = aa^*a$. An element $u \in R$ is called a unitary element provided that $uu^* = u^*u = 1$.

Proposition 14.1.4. *Let R be a strongly *-clean ring. Then*

(1) *Every partial isometry in R is the product of a projection and a unitary element.*
(2) *Every partial isometry in R is the sum of a projection and a unitary element.*

Proof. (1) In view of Corollary 14.1.3, R is abelian. Let $a \in R$ be a partial isometry. Then $a = aa^*a$, and so $a^* = \big(aa^*a\big)^* = a^*aa^*$. Set $u = 1 - aa^* + a$. Then $uu^* = \big(1 - aa^* + a\big)\big(1 - aa^* + a\big)^* = 1 - aa^* + (1 - aa^*)a^* + a(1 - aa^*) + aa^* = 1$. Likewise, $u^*u = 1$. Hence, $u \in R$ is a unitary element. Further, we see that $a = aa^*a = aa^*\big(1 - aa^* + a\big) = eu$, where $e = aa^*$. It is easy to check that $e^2 = e = e^*$, and so a is the product of a projection e and a unitary element u.

(2) Let $a \in R$ be a partial isometry. By (1), there exists a projection $f \in R$ and a unitary element $w \in R$ such that $a = fw = wf$. Set $e = 1 - f$ and $u = a - e$. It is easy to verify that $uu^* = (fw - e)(fw - e)^* = (fw - e)(w^*f - e) = fww^* + e = f + e = 1$. Likewise, $u^*u = 1$. Therefore $a = e + u$, and we are done. □

Two projections e and f in a *-ring R are said to be equivalent, written $e \sim f$, in case there exists $w \in R$ such that $e = ww^*$ and $f = w^*w$. One can suppose, without loss of generality, that $w \in eRf$.

Corollary 14.1.5. *Let R be a strongly *-clean ring, and let $e, f \in R$ be projections. If $e \sim f$, then $e = f$.*

Proof. Given $e \sim f$, then there exists $w \in eRf$ such that $e = ww^*$ and $f = w^*w$. This implies that $ww^*w = ew = w$, and so $w \in R$ is a partial isometry in R. By virtue of Proposition 14.1.4, we can find a projection $f \in R$ and a unitary element $v \in R$ such that $w = ev$. Set $u = (1 - e - vw^*)v(1 - f - w^*v)$. Then $uu^* = 1$. Likewise, $u^*u = 1$. Further, we check that $eu = -evw^*v(1 - f - w^*v) = -w(1 - f - w^*v) = ww^*v = w$, and $uf = -(1 - e - vw^*)vw^*vf = -(1 - e - vw^*)vw^*vw^*w = -(1 - e - vw^*)vv^*evv^*ew = -(1 - e - vw^*)ew = vw^*(ww^*)w = vw^*w = vv^*ew = ew = w$. Thus $eu = uf$, and so $e = (eu)u^* = (uf)u^* = f$. This establishes the proof. □

Lemma 14.1.6. *Let R be a *-ring, let $I \subseteq J(R)$, and let $e \in R$ be an idempotent. If $e - e^* \in I$, then there exists a projection $f \in R$ such that $eR = fR$ and $e - f \in I$.*

Proof. Let $z = 1 + (e^* - e)^*(e^* - e)$. Then $z \in U(R)$ and $z^* = z$. Let $t = z^{-1}$. Then $t^* = t$. We check that $ez = e(1 - e - e^* + ee^* + e^*e) = ee^*e = (1 - e - e^* + ee^* + e^*e)e = ze$; whence, $et = te$ and $e^*t = te^*$. Let $f = ee^*t$. Then $f^* = f$, and $f^2 = ee^*tee^*t = tee^*ee^*t = (ee^*e)te^*t = (ez)te^*t = ee^*t = f$. Hence, $f \in R$ is a projection. Obviously, $fR \subseteq eR$, and from $fe = ee^*te = ee^*et = ezt = e$ one has $eR \subseteq fR$. Therefore $eR = fR$. Further, $e - f = e(ez - ee^*)t = e(ee^*e - ee^*)t = ee^*(e - e^*)t \in I$, as asserted. □

We say that an ideal I of a *-ring R is a *-ideal in case $I^* \subseteq I$. If I is a *-ideal of a *-ring, it is easy to check that R/I is also a *-ring.

Theorem 14.1.7. *Let I be a *-ideal of a *-ring R. If $I \subseteq J(R)$, then R is strongly *-clean if and only if*

(1) *R/I is strongly *-clean*;
(2) *Every projection of R is abelian*;
(3) *Every idempotent lifts modulo I.*

Proof. One direction is obvious from Corollary 14.1.2 and Theorem 1.1.11. It will suffice to prove the converse. For any idempotent $e \in R$, $\overline{e} \in R/I$ is an idempotent. By (1), $\overline{e} \in R/I$ is a projection. Thus, $e - e^* \in I \subseteq J(R)$. In view of Lemma 14.1.6, there exists a projection $f \in R$ such that $eR = fR$. We infer that $e = fe$ and $f = ef$. By (2), we get $e = f$. Therefore every idempotent of R is a projection, and R is abelian. By (1) and (3), R is an exchange ring, and then it is strongly clean. The theorem is therefore established from Theorem 14.1.1. □

As a consequence, we can improve [394, Corollary 2.11] as follows.

Corollary 14.1.8. *A *-ring R is strongly *-clean if and only if*

(1) *$R/J(R)$ is strongly *-clean*;
(2) *Every projection of R is abelian*;
(3) *Every idempotent lifts modulo $J(R)$.*

Proof. Since $J(R)$ is a *-ideal of R, it is immediate from Theorem 14.1.7. □

Let R be a *-ring, and let $R[i] = \{x + yi \mid x, y \in R, i^2 = -1\}$. Then $R[i]$ is also a *-ring by defining $* : x + yi \mapsto x^* + y^*i$.

Corollary 14.1.9. *Let R be a commutative *-ring. If $2 \in J(R)$, then $R[i]$ is strongly *-clean if and only if so is R.*

Proof. $\Longrightarrow$ Let $a \in R$. Then we have a projection $e + fi \in R[i]$ and a $u + vi \in U\big(R[i]\big)$ such that $a = (e + fi) + (u + vi)$. Thus, $a = e + u$. It follows from $(e + fi)^* = e + fi$ that $e^* = e$. Since $e + fi = (e + fi)^2$, we get $e^2 - f^2 = e$ and $ef + fe = f$. This implies that $(2e - 1)f = 0$, and then $f = 0$ as $2 \in J(R)$. Hence, $e \in R$ is a projection. As $f + v = 0$, we get $v = 0$. Hence, $u \in U(R)$, and therefore R is strongly *-clean.

$\Longleftarrow$ Let $I = \{x+yi \mid x+y \in J(R)\}$. For any $a+bi \in I, c+di \in R[i]$, we see that $(a+bi)(c+di) = ac-bd+(ad+bc)i$. Hence, $(ac-bd)+(ad+bc) = (a+b)(c+d) - 2bd \in J(R)$, and then I is an ideal of $R[i]$. As in the proof of Theorem 4.4.31, $1-(a+bi)(c+di) \in U\big(R[i]\big)$; whence, $a+bi \in J\big(R[i]\big)$. Thus $I \subseteq J\big(R[i]\big)$.

Set $S = R[i]/I$. For any $\overline{a+bi} \in S$, we see that $\overline{a+bi} = \overline{a+b}$. As R is strongly *-clean, we have a projection $g \in R$ and $u \in U(R)$ such that $a+b = g+u$. This implies that $\overline{a+bi} = \overline{g}+\overline{u}$. Therefore $R[i]/I$ is strongly *-clean.

Suppose $(e+fi)-(e+fi)^2 \in I$. Then $(e-e^2+f^2)+(f-ef-fe)i \in I$. This implies that $(e-e^2+f^2)+(f-ef-fe) \in J(R)$. Hence, $(e+f)^2 - (e+f) = -(e-e^2+f^2)-(f-ef-fe)+2f^2 \in J(R)$. Since R is strongly *-clean, it is an exchange ring, and so we have an idempotent $g \in R$ such that $e+f = g+r$ for a $r \in J(R)$. Therefore $(e+fi)-g \equiv e+f-g(mod\ I)$. That is, idempotents lift modulo I. Accordingly, $R[i]$ is strongly *-clean by Theorem 14.1.7. □

Let R be a *-ring and G be a group. Then * induces an involution of the group ring RG, denoted by *, where $(\sum_g a_g g)^* = \sum_g a_g^* g^{-1}$.

Corollary 14.1.10. *Let R be a commutative *-ring, and let G be an abelian p-group. If $p \in J(R)$, then RG is strongly *-clean if and only if so is R.*

Proof. For any $a \in R$, there exists a projection $e \in RG$ and a unit $u \in RG$ such that $a = e + u$ and $ae = ea$. Let $\varepsilon : RG \to R, \sum\limits_{g\in G} r_g g \to \sum\limits_{g\in G} r_g$. Then $a = \varepsilon(a) = \varepsilon(e) + \varepsilon(u)$ and $a\varepsilon(e) = \varepsilon(e)a$. As $uv = 1$ for a $v \in RG$, we see that $\varepsilon(u)\varepsilon(v) = 1$, and so $\varepsilon(u) \in U(R)$. Obviously, $\varepsilon(e) \in R$ is an idempotent. Write $e = \sum\limits_{g\in G} s_g g$. Then $e^* = \sum\limits_{g\in G} s_g^* g^{-1}$, and so $\varepsilon(e)^* = \left(\sum\limits_{g\in G} s_g\right)^* = \sum\limits_{g\in G} s_g^* = \varepsilon(e^*) = \varepsilon(e)$. Therefore $\varepsilon(e) \in R$ is a projection, and so R is strongly *-clean.

Conversely, assume that R is strongly *-clean. Set $\Delta(RG) = Ker\varepsilon$. Then $RG/\Delta(RG) \cong R$, and so $RG/\Delta(RG)$ is strongly *-clean. If $\sum_{g\in G} r_g g \in \Delta(RG)$, then $\sum_{g\in G} r_g = 0$, and so $\sum_{g\in G} r_g g = \sum_{g\in G} r_g(1-g)$. As G is a p-group, we can find some positive integers p^{n_g} such that $g^{p^{n_g}} = 1$, and so $(1-g)^{p^{n_g}} \equiv 0(mod\ J(R)G)$ as $p \in J(R)$. Thus, we can find a larger $m \in \mathbb{N}$ such that $\left(\sum_{g\in G} r_g g\right)^m \in J(R)G \subseteq J(RG)$. This implies that $\sum_{g\in G} r_g g \in \Delta(RG)$. Thus, $\Delta(RG) \subseteq J(RG)$. Clearly, RG is abelian. If $\overline{a} = \overline{a}^2 \in RG/\Delta(RG)$, then $\varepsilon(a) = \varepsilon(a)^2 \in R$. Further, $a - \varepsilon(a) \in \Delta(RG)$. Therefore idempotents lift modulo $\Delta(RG)$. This completes the proof by Theorem 14.1.7. □

Let R be a *-ring, and let $T(R,R)$ be the trivial extension of R by R. Define $* : T(R,R) \to T(R,R)$ given by $(x,y) \to (x^*, y^*)$. Then $T(R,R)$ is a *-ring.

Corollary 14.1.11. *Let R be a *-ring. Then $T(R,R)$ is strongly *-clean if and only if so is R.*

Proof. One direction is obvious. Conversely, assume that R is strongly *-clean. Clearly, $J\big(T(R,R)\big) = \{(x,y) \mid x \in J(R), y \in R\}$, and so $T(R,R)/J\big(T(R,R)\big) \cong R$. Thus, $T(R,R)/J\big(T(R,R)\big)$ is strongly *-clean. Further, every idempotent in $T(R,R)$ lifts modulo $J\big(T(R,R)\big)$. Let $(e,f) \in T(R,R)$ be an idempotent. Then $e = e^2 \in R$; hence, $e \in R$ is central. In addition, $ef + fe = f$, and so $(2e-1)f = 0$. Clearly, $(2e-1)^2 = 1$, and then $f = 0$. Therefore $(e,f) \in T(R,R)$ is central, i.e., $T(R,R)$ is abelian. According to Corollary 14.1.8, $T(R,R)$ is strongly *-clean, as asserted. □

Example 14.1.12. Let $R = T_2\big(\mathbb{Z}_2\big)$. Define $* : R \to R$ by $\begin{pmatrix} x & z \\ 0 & y \end{pmatrix} \mapsto \begin{pmatrix} y & z \\ 0 & x \end{pmatrix}$. Then R is a strongly clean ring by Corollary 3.1.5. But R is not strongly *-clean as R is not abelian.

Example 14.1.13. Let $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Define $\sigma : R \to R$ by $\sigma(x,y) = (y,x)$. Consider the ring $T_2(R,\sigma) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \;\middle|\; a,b \in R \right\}$ with the following

operations:

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} + \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ 0 & a+c \end{pmatrix}, \quad \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \cdot \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} = \begin{pmatrix} ac & ad+b\sigma(c) \\ 0 & ac \end{pmatrix}.$$

Then $J(T_2(R,\sigma)) = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \;\middle|\; b \in R \right\}$. Define $* : T_2(R,\sigma) \to T_2(R,\sigma)$ by $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}^* = \begin{pmatrix} a & \sigma(b) \\ 0 & a \end{pmatrix}$. Clearly, $\sigma^2 = 1_R$, and so * is an involution of $T_2(R,\sigma)$.

For any $\overline{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}} - \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in T_2(R,\sigma)/J\big(T_2(R,\sigma)\big)$, we see that

$$\overline{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}} = \overline{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}} = \overline{\begin{pmatrix} a-1 & 0 \\ 0 & a-1 \end{pmatrix}} + \overline{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}};$$

hence, $T_2(R,\sigma)/J\big(T_2(R,\sigma)\big)$ is strongly *-clean. Also we easily check that idempotents lifts modulo $J\big(T_2(R,\sigma)\big)$. But $T_2(R,\sigma)$ is not strongly *-clean since $\begin{pmatrix} (0,1) & (0,0) \\ (0,0) & (0,1) \end{pmatrix}$ is not central.

Lemma 14.1.14. *Let R be a *-ring. Then $P(R)$ is a nil *-ideal of R.*

Proof. Let $a \in P(R)$. For every sequence $a_0, a_1, \ldots, a_i, \ldots$ such that $a_0 = a^*$ and $a_{i+1} \in a_i R a_i$, we get a sequence $a_0^*, a_1^*, \ldots, a_i^*, \ldots$ such that $a_0^* = a$ and $a_{i+1}^* \in a_i^* R a_i^*$. As $a \in R$ is strongly nilpotent, we can find some n such that $a_n^* = 0$, and so $a_n = 0$. This implies that a^* is strongly nilpotent; hence, $a^* \in P(R)$, as desired. □

Theorem 14.1.15. *A *-ring R is strongly *-clean if and only if*

(1) *Every projection of R is central;*
(2) *$R/P(R)$ is strongly *-clean.*

Proof. One direction is obvious from Corollary 14.1.2. Suppose (1) and (2) hold. As $P(R)$ is nil, every idempotent modulo $P(R)$. Further, $P(R) \subseteq J(R)$. This completes the proof by Theorem 14.1.7. □

The involution of a *-ring is said to be proper if $x^*x = 0$ implies $x = 0$. Clearly, the involution of a *-ring is proper if and only if $\ell(x^*x) = \ell(x)$, if and only if x^*x is a projection implies that x is a partial isometry. A *-regular ring is a regular ring with proper involution.

Lemma 14.1.16. *Let R be a *-ring. Then the following are equivalent:*

(1) *R is *-regular.*
(2) *For each $a \in R$, there exists a projection $e \in R$ such that $aR = eR$.*

Proof. (1) $\Rightarrow$ (2) Clearly, R is regular. Let $a \in R$. Then $a = aba$ for some $b \in R$. Setting $x = ab$, we see that $x \in R$ is an idempotent, and $aR = xR$. Write $x^*x = x^*xyx^*x$. Then $x^*x(1 - yx^*x) = 0$, and so $\big(x(1 - yx^*x)\big)^*\big(x(1 - yx^*x)\big) = (1 - yx^*x)^*x^*x(1 - yx^*x) = 0$. Since the involution is proper, we get $x(1 - yx^*x) = 0$. This implies that $x = xyx^*x$. Setting $e = xyx^*$, we have $e^2 = (xyx^*x)yx^* = xyx^* = e$, and $ee^* = (xyx^*)(xyx^*)^* = (xyx^*x)y^*x^* = xy^*x^* = e^*$. Thus, $e = (e^*)^* = (ee^*)^* = ee^* = e^*$. This means that $e \in R$ is a projection. Obviously, $eR \subseteq xR$. Moreover, $x = xyx^*x = ex$ shows that $xR \subseteq eR$. Thus, we have $xR = eR$, as required.

(2) $\Rightarrow$ (1) Let $a \in R$. Then there exists a projection $e \in R$ such that $aR = eR$. Write $a = ex$ and $e = ab$. Then $aba = ea = e(ex) = ex = a$. Thus, R is regular. If $a^*a = 0$, it follows from $e^* = e$ that $a = (ab)a = (b^*a^*)a = 0$, as required. □

Proposition 14.1.17. *Every abelian *-regular ring is strongly *-clean.*

Proof. Let R be an abelian *-regular ring. Then R is an abelian regular ring. In view of Proposition 1.1.17 and Corollary 1.1.14, R is strongly clean. Let $e \in R$ be an idempotent. Then $eR = fR$ for a projection f by Lemma 14.1.14. Hence, $e = fe$ and $f = ef$. As R is abelian, we get $e = f$. That is, every idempotent in R is a projection. The proof is therefore established by Theorem 14.1.1. □

Example 14.1.18. Let $R = M_n(\mathbb{R})(n \geq 2)$. Then R is a *-regular ring with transpose as the involution, but it is not strongly *-clean.

Proof. As is well known, every matrix ring over a regular ring is regular (cf. [273, Theorem 1.7]). Then $M_n(\mathbb{R})$ is regular. For any $(a_{ij}) \in M_n(R)$, if $(a_{ij})^*(a_{ij}) = 0$, then

$$\sum_{i=1}^{n} a_{ij}^2 = 0, j = 1, 2, \ldots, n.$$

Hence, $a_{ij} = 0, 1 \leq i, j \leq n$. That is, $(a_{ij}) = 0$. Therefore R is *-regular. Clearly, R is not abelian, hence the result. □

14.2 The Uniqueness

The aim of this section is to explore the uniqueness for strongly *-clean rings. We say that a *-ring R is uniquely strongly *-clean provided that for any $a \in R$ there exists a uniquely projection $e \in R$ such that $a - e \in U(R)$ and $ae = ea$. We start by the relationship among uniquely strong *-cleanness, strong J-cleanness and strong *-cleanness.

Theorem 14.2.1. *Let R be a *-ring. Then the following are equivalent:*

(1) *R is uniquely strongly *-clean.*
(2) *R is strongly J-clean and strongly *-clean.*
(3) *R is a uniquely clean ring in which every idempotent is a projection.*

Proof. (1) $\Rightarrow$ (3) Since R is strongly *-clean, it follows from Theorem 14.1.1 that every idempotent is a projection. This implies that R is uniquely clean.

(3) $\Rightarrow$ (2) In view of Corollary 4.4.10, R is strongly J-clean, and so R is strongly clean. According to Theorem 14.1.1, R is strongly *-clean.

(2) $\Rightarrow$ (1) For any $a \in R$, there exists an idempotent $e \in R$ such that $u := a - e \in J(R)$ and $ae = ea$. Then $a = (1 - e) + (2e - 1 + u)$. In view of Theorem 14.1.1, $1 - e \in R$ is a projection. As $(2e - 1)^2 = 1$, we see that $2e - 1 + u \in U(R)$. By virtue of Corollary 14.1.2, R is abelian. If there exists a projection $f \in R$ such that $v := a - f \in U(R)$, then $e-f = (a-f)-(a-e) \in U(R)$. As $ef = fe$, we see that $(e-f)(e+f) = e-f$, and so $e = 1 - f$. This completes the proof. □

Let R be a commutative *-ring. We claim that $R[i]$ is uniquely strongly *-clean if and only if so is R. This is easily proved by Theorem 4.4.31, Corollary 14.1.9, and Theorem 14.2.1.

Corollary 14.2.2. *A *-ring R is uniquely strongly *-clean if and only if*

(1) *R is strongly *-clean;*
(2) *$R/M \cong \mathbb{Z}_2$ for all $M \in Max(R)$.*

Proof. $\Longleftarrow$ By virtue of Theorem 14.2.1, R is strongly *-clean and uniquely clean. According to Theorem Theorem 4.4.6, $R/M \cong \mathbb{Z}_2$ for all maximal ideals M of R.

$\Longrightarrow$ By Theorem 14.1.1, every idempotent in R is a projection. Further, R is an abelian exchange ring. According to Theorem 4.4.6, R is uniquely clean. Therefore R is uniquely strongly *-clean by Theorem 14.2.1. □

Corollary 14.2.3. *A *-ring R is uniquely strongly *-clean if and only if*

(1) *R is strongly *-clean;*
(2) *For all $M \in Max(R)$, $\overline{1} \neq \overline{u} + \overline{v}$ for any $\overline{u}, \overline{v} \in U\big(R/M\big)$.*

Proof. One direction is obvious from Corollary 14.2.2. Conversely, assume that (1) and (2) hold. In view of Theorem 14.1.1, every idempotent in R is a projection. Further, R is an abelian exchange ring from Corollary 14.1.2. According to Theorem 4.4.6, R is uniquely clean. Therefore we complete the proof by Theorem 14.2.1. □

Example 14.2.4. Construct $R = \left\{ \begin{pmatrix} x & y \\ z & x \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{Z}_2 \right\}$ as a subring of $M_{(0)}(\mathbb{Z}_2)$. Define $* : R \to R$ by $\begin{pmatrix} x & y \\ z & x \end{pmatrix}^* = \begin{pmatrix} x & z \\ y & x \end{pmatrix}$. Then R is a *-ring and the idempotents of R are zero and identity; hence, every idempotent in R is a projection. In view of Example 5.2.24, R is Boolean-like, and so it is strongly nil-clean. This implies that R is uniquely nil-clean by Theorem 5.3.3. According to Lemma 5.3.7, R is uniquely clean. Therefore R is uniquely strongly *-clean by Theorem 14.2.1.

Proposition 14.2.5. *A *-ring R is uniquely strongly *-clean if and only if*

(1) *R is strongly *-clean;*
(2) *$R/J(R)$ is Boolean.*

Proof. Suppose R is uniquely strongly *-clean. In view of Theorem 14.2.1, R is uniquely clean and R is strongly *-clean. By Theorem 4.4.6, $R/J(R)$ is Boolean.

Conversely, assume that (1) and (2) hold. Then R is an abelian exchange ring. According to Theorem 4.4.6, R is uniquely clean. In light of Theorem 14.1.1, every idempotent in R is a projection. Therefore R is uniquely strongly *-clean by Theorem 14.2.1. □

Corollary 14.2.6. *A *-ring R is uniquely strongly *-clean if and only if*

(1) *R is strongly *-clean;*
(2) *Every nonzero idempotent in R is not the sum of two units.*

Proof. $\Longleftarrow$ In view of Proposition 14.2.5, R is strongly *-clean and $R/J(R)$ is Boolean. Let $0 \neq e \in R$ be an idempotent. If $e = u + v$ for some $u, v \in U(R)$, then $\overline{e} = \overline{u} + \overline{v}$ in $R/J(R)$. Since $\overline{u} = \overline{v} = \overline{1}$, $2 - e \in J(R)$. On the other hand, R is strongly J-clean by Theorem 14.2.1. This implies that $2 \in J(R)$; hence, $e \in J(R)$. Thus $e = 0$, a contradiction. Therefore every nonzero idempotent in R is not the sum of two units.

$\Longrightarrow$ Clearly, R is an abelian exchange ring. Suppose $R/J(R)$ is not Boolean. Then there exists some $a \in R$ such that $a - a^2 \notin J(R)$. By Lemma 4.4.2, we have an idempotent $0 \neq e \in (a-a^2)R$. Write $e = (a-a^2)r$. Then $ea(1-a)r = e$, and so $ea \in U(eRe)$. Further, $e(1-a) \in U(eRe)$. Clearly, $e = \big(ea+1-e\big) + \big(e(1-a)-1+e\big)$, and $ea+1-e, e(1-a)-1+e \in U(R)$. This gives a contradiction. Thus $R/J(R)$ is Boolean. The corollary is therefore established by Proposition 14.2.5. □

Corollary 14.2.7. *Let R be a local *-ring. Then the following are equivalent:*

(1) *R is uniquely strongly *-clean.*
(2) *R is strongly J-clean.*
(3) *R is uniquely clean.*
(4) *$R/J(R) \cong \mathbb{Z}_2$.*
(5) *1 is not the sum of two units in R.*

Proof. Since R is a local *-ring, it is strongly *-clean. (1), (2), (3) and (4) are equivalent by Proposition 14.2.5 and Theorem 14.2.1. (1) and (4) are equivalent from Corollary 14.2.6. □

Let $R = \mathbb{Z}_{(3)}$ be the localization, and $* = 1_R$, the identical automorphism of R. Since R is local, it is strongly *-clean but not uniquely strongly *-clean because $R/J(R) \ncong \mathbb{Z}_2$ and $J(R) = 3R$.

Corollary 14.2.8. *Let R be a *-regular ring. Then R is uniquely strongly *-clean if and only if R is Boolean.*

Proof. Suppose R is Boolean. In view of Proposition 14.1.17, R is strongly *-clean. Clearly, R is strongly J-clean. According to Theorem 14.2.1, R is uniquely strongly *-clean.

Conversely, assume that R is uniquely strongly *-clean. Then $R/J(R)$ is Boolean by Proposition 14.2.5. As R is regular, $J(R) = 0$, and therefore R is Boolean. □

Lemma 14.2.9. *A *-ring R is uniquely strongly *-clean if and only if*

(1) *R is uniquely clean;*
(2) *For any $a \in R$, $a + a^* \in J(R)$.*

Proof. $\Longleftarrow$ In view of Theorem 14.2.1, R is uniquely clean and strongly J-clean. For any $a \in R$, there exists a projection $e \in R$ and an element $u \in J(R)$ such that $a = e + u$ and $ae = ea$. Thus $a^* = e^* + u^*$. It follows from $\big(J(R)\big)^* \subseteq J(R)$ that $a - a^* = (e - e^*) + (u - u^*) \in J(R)$. By virtue of Theorem 4.1.9, $2 \in J(R)$. Therefore $a + a^* = (a - a^*) + 2a^* \in J(R)$.

$\Longrightarrow$ For any idempotent $e \in R$, $e + e^* \in J(R)$. Thus $(e - e^*)(e + e^*)^2 = e - e^*$; hence, $(e - e^*)\big(1 - (e + e^*)^2\big) = 0$. This implies that $e = e^*$, i.e., every idempotent is a projection. According to Theorem 14.2.1, we complete the proof. □

We are now ready to prove:

Theorem 14.2.10. *Let R be a *-ring. Then the following are equivalent:*

(1) *R is uniquely strongly *-clean.*
(2) *For any $a \in R$, there exists a central projection $e \in R$ such that $a - e \in J(R)$.*
(3) *For any $a \in R$, there exists a unique projection $e \in R$ such that $a - e \in J(R)$.*

Proof. (1) $\Rightarrow$ (2) In view of Theorem 14.2.1, R is a strongly J-clean ring in which every idempotent is a projection, and then (2) holds from Theorem 4.4.13

(2) $\Rightarrow$ (3) This is obvious by Theorem 14.2.1 and Theorem 4.4.13.

(3) $\Rightarrow$ (1) For any idempotent e in R, there exists a projection $g \in R$ such that $e - g \in J(R)$. Then $e^* - g \in J(R)$, and so $e - e^* = (e - g) - (e^* - g) \in J(R)$. Clearly, there exists a projection $h \in R$ such that $2 = h + w$ where $w \in J(R)$. Thus, $1 - h = -1 + w \in U(R)$. Hence $1 - h = 1$, and so $h = 0$. This implies that $2 \in J(R)$. Therefore $e + e^* = (e - e^*) + 2e^* \in J(R)$.

Set $z = 1 + (e - e^*)^*(e - e^*)$. Write $t = z^{-1}$. Since $z^* = z$, we have $t^* = t$. Also $e^*z = e^*ee^* = ze^*$, and so $e^*t = te^*$, and $et = te$. Set $f = e^*et = te^*e$. Then $f^* = f, f^2 = e^*ete^*et = e^*ee^*(tet) = e^*ztet = e^*et = f, fe = f$ and $ef = ee^*et = ezt = e$. Now $e = f + (e - f)$ and $e - f = e - e^*et = ee^*et - e^*et = (e - e^*)e^*et \in J(R)$. Here $f = f^* = f^2$. In addition, $f = e^*e\big(1 + (e^* - e)(e - e^*)\big)^{-1}$.

Set $z' = 1 + (e^* - e)^*(e^* - e)$. Write $t' = (z')^{-1}$. Since $(z')^* = z'$, $(t')^* = t'$. Also $ez' = ee^*e = z'e$. Set $f' = ee^*t' = t'ee^*$. As in the

preceding proof, we see that $f' = (f')^2 = (f')^*$ and $ef' = f'$, $f'e = e$. In addition,

$$e - f' = f'e - f' = t'ee^*(e - e^*) \in J(R),$$

where $f' = \big(1 + (e - e^*)(e^* - e)\big)^{-1} ee^*$.

Thus $e = f + (e - f) = f' + (e - f')$ with $e - f, e - f' \in J(R)$, f and f' are projections. By the uniqueness, we get

$$e^*e\big(1 + (e^* - e)(e - e^*)\big)^{-1} = \big(1 + (e - e^*)(e^* - e)\big)^{-1} ee^*.$$

This shows that

$$\big(1 + (e - e^*)(e^* - e)\big)e^*e = ee^*\big(1 + (e^* - e)(e - e^*)\big).$$

Obviously, $(e - e^*)(e^* - e)e^*e = -e^*e + e^*ee^*e$ and $ee^*(e^* - e)(e - e^*) = -ee^* + ee^*ee^*$. Consequently, $e^*ee^*e = ee^*ee^*$. One easily checks that

$$(e - e^*)^3 - (e - e^*) = -ee^*e + e^*ee^*;$$
$$\big((e - e^*)^3 - (e - e^*)\big)(e + e^*) = (e - e^*)^3 - (e - e^*).$$

Then $(e - e^*)\big((e - e^*)^2 - 1\big)\big((e + e^*) - 1\big) = 0$. Since $e - e^*$, $e + e^* \in J(R)$, we get $e = e^*$. This shows that every idempotent is a projection. Clearly, R is strongly *-clean, and so it is abelian by Corollary 14.1.2, hence the result follows. □

Corollary 14.2.11. *Let R be a *-ring. Then R is uniquely strongly *-clean if and only if for any $a \in R$, there exists a unique projection $e \in R$ such that $a - e \in U(R)$ and $ae = ea$.*

Proof. One direction is obvious from Theorem 14.2.1. Conversely, letting $a \in R$, we have a unique projection $e \in R$ such that $a - e \in U(R)$ and $ae = ea$. Hence R is strongly *-clean, and then every idempotent in R is a projection. Thus R is uniquely clean. Therefore the proof is true by Theorem 14.2.1. □

We note that the unique projection in the preceding corollary cannot be replaced by the unique idempotent even for a commutative *-ring as the following shows.

Example 14.2.12. Let $R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}$ where $0, 1 \in \mathbb{Z}_2$. Define $* : R \to R$, $\begin{pmatrix} x & y \\ s & t \end{pmatrix} \mapsto \begin{pmatrix} x + y & y \\ x + y + s + t & y + t \end{pmatrix}$. Then R is

a commutative *-ring with the usual matrix addition and multiplication. In fact, R is Boolean, and so, for any $a \in R$, there exists a unique idempotent $e \in R$ such that $a - e \in U(R)$ and $ae = ea$. But R is not uniquely strongly *-clean, even not a strongly *-clean ring.

Theorem 14.2.13. *A *-ring R is uniquely strongly *-clean if and only if*

(1) *R is strongly *-clean*;
(2) $J(R) = \{x \in R \mid 1 - x \in U(R)\}$.

Proof. One direction is clear from Theorem 14.2.1 and Theorem 4.1.9. Conversely, assume that (1) and (2) hold. Then R is strongly J-clean by Theorem 4.1.9, and so the proof is complete from Theorem 14.2.1. □

Corollary 14.2.14. *A *-ring R is uniquely strongly *-clean if and only if*

(1) *R is strongly *-clean*;
(2) *For any $a \in R$, $a + a^* \in J(R)$*;
(3) $J(R) = \{x \in R \mid 1 + xx^* \in U(R)\}$.

Proof. Suppose R is uniquely strongly *-clean. Then (1) and (2) hold by Theorem 14.2.1 and Lemma 14.2.9. For any $x \in R$, we have $x + x^* \in J(R)$. If $1 + xx^* \in U(R)$, then $(1 + x)(1 + x^*) = 1 + x + x^* + xx^* \in U(R)$. This implies that $1 + x \in R$ is right invertible. Similarly, $1 + x \in R$ is left invertible. In view of Theorem 14.2.13, $-x \in J(R)$, and so $x \in J(R)$. Then $J(R) = \{x \in R \mid 1 + xx^* \in U(R)\}$.

Conversely, assume that (1), (2) and (3) hold. If $1 + x \in U(R)$, then $1+x^* \in U(R)$; hence, $(1+x)(1+x^*) \in U(R)$. As $x+x^* \in J(R)$, we see that $1 + xx^* \in U(R)$. This implies that $x \in J(R)$, and so $-x \in J(R)$. Hence, $J(R) = \{x \in R \mid 1 - x \in U(R)\}$. By Theorem 4.1.9 and Theorem 14.2.1, we complete the proof. □

Corollary 14.2.15. *A *-ring R is uniquely strongly *-clean if and only if*

(1) $2 \in J(R)$;
(2) *R is strongly *-clean*;
(3) *$U\big(R/J(R)\big)$ is torsion.*

Proof. $\Longrightarrow$ Clearly, R is strongly *-clean and $2 \in J(R)$. Additionally, $R/J(R)$ is Boolean. Thus $U\big(R/J(R)\big) = \{\overline{1}\}$ is torsion.

$\Longleftarrow$ Given $1 - x \in U(R)$, then $\overline{1-x} \in U\big(R/J(R)\big)$. By hypothesis, there exists some $n \in \mathbb{N}$ such that $(\overline{1-x})^n = \overline{1}$, and so $(\overline{1-x})^{2n} = \overline{1}$. As

$2 \in J(R)$, we see that $x^{2n} \in J(R)$. Clearly, R is an abelian exchange ring, and then so is $R/J(R)$. This implies that $R/J(R)$ is reduced, i.e. it has no nonzero nilpotents, and so $x \in J(R)$. Hence, $J(R) = \{x \in R \mid 1 - x \in U(R)\}$. Accordingly, R is uniquely strongly *-clean by Theorem 14.2.13. □

Theorem 14.2.16. *Let I be a *-ideal of a *-ring R. If $I \subseteq J(R)$, then R is uniquely strongly *-clean if and only if*

(1) *R is abelian*;
(2) *Every idempotent lifts modulo I*;
(3) *R/I is uniquely strongly *-clean.*

Proof. $\Longrightarrow$ By Theorem 14.1.7, R/I is strongly *-clean, R is abelian and every idempotent lifts modulo I. In view of Theorem 14.2.1, R is strongly J-clean, and then so is R/I. By using Theorem 14.2.1 again, R/I is uniquely strongly *-clean.

$\Longleftarrow$ It follows by Theorem 14.1.7 that R is strongly *-clean. For any $a \in R$, by Theorem 14.2.10, we have a projection $\overline{e} \in R/I$ such that $\overline{a} - \overline{e} \in J(R/I)$. As $e - e^2 \in I$, by hypothesis, there exists an idempotent $f \in R$ such that $\overline{e} = \overline{f}$. Since $I \subseteq J(R)$, it follows from $\overline{a - f} \in J(R/I)$ that $a - f \in J(R)$. This means that R is strongly J-clean. So the theorem is true by Theorem 14.2.1. □

A *-ring R is *-Boolean in case every element in R is a projection.

Corollary 14.2.17. *Let R be a *-ring. Then R is uniquely strongly *-clean if and only if*

(1) *R is abelian*;
(2) *Every idempotent lifts modulo $J(R)$*;
(3) *$R/J(R)$ is *-Boolean.*

Proof. One easily checks that $J(R)$ is a *-ideal of R, and the result follows by Theorem 14.2.16. □

Corollary 14.2.18. *A *-ring R is uniquely strongly *-clean if and only if R is abelian and $R/P(R)$ is uniquely strongly *-clean.*

Proof. Setting $I = P(R)$ in Theorem 14.2.16, we obtain the result. □

Let R be a *-ring, and let $C(R) = \{a \in R \mid ax = xa \text{ for any } x \in R\}$. Then $C(R)$ is a subring of R. It is easy to check that $* : C(R) \to C(R)$ is also an anti-automorphism. Thus $C(R)$ is a *-ring.

Proposition 14.2.19. *A *-ring R is uniquely strongly *-clean if and only if*

(1) *$C(R)$ is uniquely strongly *-clean;*
(2) *$R = C(R) + J(R)$.*

Proof. Suppose R is uniquely strongly *-clean. For any $a \in R$, it follows from Theorem 14.2.10 that there exists a central projection $e \in R$ and an element $u \in J(R)$ such that $a = e+u$. Hence, $e \in C(R)$ is a projection. This implies that $R \subseteq C(R) + J(R)$, and so (2) holds. In addition, $a \in C(R)$ implies that $u \in J(R) \cap C(R) \subseteq J\big(C(R)\big)$. Therefore $C(R)$ is uniquely strongly *-clean from Theorem 14.2.10.

Conversely, assume that (1) and (2) hold. For any idempotent $e \in R$, there exist $b \in C(R), c \in J(R)$ such that $e = b + c$. As $C(R)$ is uniquely strongly *-clean, by virtue of Theorem 14.2.10, we have a projection $f \in C(R)$ and a $w \in J\big(C(R)\big)$ such that $b = f + w$. Hence, $e = f + (w + c)$, and so $e - f = w + c$. Clearly, $(e-f)\big(1-(e-f)^2\big) = 0$. But $(e-f)^2 \in J\big(C(R)\big) + J(R)$, and so $1 - (e - f)^2 \in U(R)$. Therefore $e = f$, i.e., every idempotent in R is a projection. For any $a \in R$, write $a = s+t, s \in C(R), t \in J(R)$. We have a projection $g \in R$ such that $s = g + v$ with $v \in J\big(C(R)\big)$. This shows that $a = g + (t + v)$, and so R is uniquely strongly *-clean from Theorem 14.2.10. □

Let R be a *-ring. Then * induces an involution of the power series ring $R[[x]]$, denoted by *, where $\left(\sum\limits_{i=0}^{\infty} a_i x^i\right)^* = \sum\limits_{i=0}^{\infty} a_i^* x^i$. Likewise, * induces an involution of $R[[x]]/(x^n)$ $(n \geq 1)$, denoted by *, where $\left(\sum\limits_{i=0}^{n-1} a_i x^i\right)^* = \sum\limits_{i=0}^{n-1} a_i^* x^i$.

Proposition 14.2.20. *Let R be a *-ring. Then the following are equivalent:*

(1) *R is uniquely strongly *-clean.*
(2) *$R[[x]]$ is uniquely strongly *-clean.*
(3) *$R[[x]]/(x^2)$ is uniquely strongly *-clean.*
(4) *$R[[x]]/(x^n)$ is uniquely strongly *-clean for all $n \geq 2$.*

Proof. (1) ⇒ (2) Since R is uniquely strongly *-clean, it is uniquely clean and every idempotent in R is a projection. Then R is abelian. In view

of Example 4.4.24, $R[[x]]$ is uniquely clean. Let $\sum_{i=0}^{\infty} e_i x^i \in R[[x]]$ be an idempotent. Then

$$e_0^2 = e_0, e_0e_1 + e_1e_0 = e_1, \ldots, e_0e_n + e_1e_{n-1} + \cdots + e_{n-1}e_1 + e_ne_0 = e_n, \ldots.$$

As $e_0 \in R$ is a central projection, we see that $(2e_0 - 1)e_1 = 0$, and so $e_1 = 0$ as $(2e_0 - 1)^2 = 1$. Since $e_0e_2 + e_2e_0 = e_2$, we derive that $(2e_0 - 1)e_2 = 0$; hence, $e_2 = 0$. Likewise, $e_3 = \cdots = 0$. Therefore $\sum_{i=0}^{\infty} e_i x^i \in R[[x]]$ is a projection. Hence $R[[x]]$ is a uniquely strongly *-clean ring, by Theorem 14.2.1.

(2) $\Rightarrow$ (4) In view of Theorem 14.2.1, $R[[x]]$ is uniquely clean, and then so is $R[[x]]/(x^n)$ by Proposition 4.4.25. Obviously, $S := R[[x]]/(x^n) = \left\{ \sum_{i=0}^{n-1} a_i x^i \mid a_0, \ldots, a_{n-1} \in R, x^n = 0 \right\}$. For any $f \in S$, it follows from Lemma 14.2.9 that $f(0) + \big(f(0)\big)^* \in J(R)$, and so $f + f^* = f(0) + (f(0))^* + \sum_{i=1}^{n-1} b_i x^i \in J(S)$. By using Lemma 14.2.9 again, $R[[x]]/(x^n)$ is uniquely strongly *-clean

(4) $\Rightarrow$ (3) This is trivial.

(3) $\Rightarrow$ (1) Let $S = R[[x]]/(x^2)$. Then $S = \{a + bx \mid a, b \in R, x^2 = 0\}$. For any $a \in R$, it follows from Theorem 14.2.10 that there exists a central projection $e + fx \in S$ such that $a - (e + fx) \in J(S)$. This implies that $e \in R$ is a central projection and $a - e \in J(R)$. By using Theorem 14.2.10 again, R is uniquely strongly *-clean. □

A *-ring R is uniquely strongly *-clean if and only if so is $R[x]/(x^2)$. Obviously, $R[x]/(x^2) \cong R[[x]]/(x^2)$, and we are through.

Proposition 14.2.21. *Let R be a *-ring, and let G be a locally finite group. Then RG is uniquely strongly *-clean if and only if R is uniquely strongly *-clean and G is a 2-group.*

Proof. Suppose RG is uniquely strongly *-clean. Then it is uniquely clean. By virtue of Proposition 4.4.34, R is uniquely clean and G is a 2-group. In view of Lemma 4.4.3, $2 \in J(RG)$, and so $2 \in J(R)$. In addition, every idempotent in RG is a projection. This implies that every idempotent in R is a projection. In light of Lemma 14.2.1, R is uniquely strongly *-clean.

Conversely, assume that R is a uniquely strongly *-clean ring and G is a 2-group. By virtue of Proposition 4.4.34, RG is uniquely clean. As R is

uniquely strongly *-clean, we see that $2 \in J(R)$. Further, every idempotent in R is a projection. According to Lemma 4.4.33, every idempotent in RG is in R. Therefore every idempotent in RG is a projection. By using Theorem 14.2.1, RG is uniquely strongly *-clean. □

Corollary 14.2.22. *Let R be a *-ring, and let G be a solvable group. Then RG is a uniquely strongly *-clean ring if and only if R is a strongly J-*-clean ring and G is a 2-group.*

Proof. Suppose RG is uniquely strongly *-clean. Then it is uniquely clean. In light of Proposition 4.4.34, R is uniquely clean and G is a 2-group. As in the preceding discussion, R is uniquely strongly *-clean.

Conversely, assume that R is a uniquely strongly *-clean ring and G is a 2-group. As in the proof of Corollary 4.4.37, G is locally finite, thus completing the proof by Proposition 14.2.21. □

Example 14.2.23. Let R be a Boolean ring and let G be a locally finite 2-group which is not abelian. Let * be the identical automorphism of R. In view of Proposition 14.2.21 RG is a uniquely strongly *-clean ring which is not commutative.

14.3 Strongly Nil-*-clean Rings

The purpose of this section is to investigate strongly nil cleanness for any *-ring. We call a *-ring R strongly nil-*-clean if for any $a \in R$ there exists a projection $e \in R$ such that $a - e \in N(R)$ and $ae = ea$. We observe that the behave of such rings is very different from that of strongly nil-clean rings. Many necessary and sufficient conditions under which a *-ring is strongly nil-*-clean will be given. Further, we consider strong nil-*-cleanness for several hypercomplex rings over commutative *-rings.

Theorem 14.3.1. *A *-ring R is strongly nil-*-clean if and only if*

(1) *R is strongly nil-clean;*
(2) *R is strongly *-clean.*

Proof. Suppose R is strongly nil-*-clean. Then it is strongly nil-clean. For any $a \in R$, there exists a projection $e \in R$ such that $w := a - e \in N(R)$ and $ae = ea$. Thus, $a = (1 - e) + (2e - 1 + w)$. Clearly, $ew = we$, and so $2e - 1 + w \in U(R)$. This implies that R is strongly *-clean.

Conversely, assume that (1) and (2) hold. By virtue of Theorem 14.1.1, every idempotent in R is a projection, hence the result follows. □

Corollary 14.3.2. *A *-ring R is strongly nil-*-clean if and only if*

(1) *R is uniquely nil-clean;*
(2) *Every idempotent in R is a projection.*

Proof. Suppose that R is strongly nil-*-clean. In view of Theorem 14.3.1, R is strongly *-clean. Thus, every idempotent in R is a projection from Theorem 14.1.1. In addition, R is strongly nil-clean. According to Corollary 14.1.2, R is abelian. Therefore R is uniquely nil clean by Theorem 5.3.3. The converse is obvious. □

Corollary 14.3.3. *Let R be a *-ring. Then the following are equivalent:*

(1) *R is strongly nil-*-clean.*
(2) *For any $a \in R$, there exists a unique projection $e \in \mathbb{Z}[a]$ such that $a - e \in R$ is nilpotent.*
(3) *For any $a \in R$, there exists a unique projection $e \in R$ such that $a-e \in R$ is nilpotent and $ea = ae$.*

Proof. (1) $\Rightarrow$ (2) In view of Theorem 14.3.1, R is strongly nil-clean. For any $a \in R$, it follows from Corollary 5.1.2 that there exists a unique idempotent $e \in \mathbb{Z}[a]$ such that $a - e \in R$. But every idempotent in R is a projection, as required.

(2) $\Rightarrow$ (3) For any $a \in R$, there exists a unique projection $e \in \mathbb{Z}[a]$ such that $a - e \in N(R)$. Obviously, $ae = ea$. We easily prove that R is strongly *-clean; hence, R is abelian. Assume that there exists a projection $f \in R$ such that $a - f \in R$ is nilpotent and $fa = af$. Then $e - f = (a - f) - (a - e) \in N(R)$. Since $(e - f)^3 = e - f$, we get $e = f$, as desired.

(3) $\Rightarrow$ (1) This is trivial. □

Example 14.3.4. Construct a *-ring R as in Example 14.2.12. Then R is a commutative *-ring with the usual matrix addition and multiplication. In fact, R is Boolean. Thus for any $a \in R$, there exists a unique idempotent $e \in R$ such that $a - e \in R$ is nilpotent. But it is not strongly nil-*-clean.

Theorem 14.3.5. *A *-ring R is strongly nil-*-clean if and only if*

(1) *R is uniquely strongly *-clean;*
(2) *$J(R)$ is nil.*

Proof. Suppose that R is strongly nil-*-clean. Then R is strongly nil-clean and strongly *-clean. In view of Corollary 14.1.2, R is abelian. It follows from Theorem 5.3.3 that R is uniquely nil-clean. According to Lemma 5.3.7 and Corollary 4.4.10, R is strongly J-clean and $J(R)$ is nil. Therefore R is uniquely strongly *-clean by Theorem 14.2.1.

Conversely, assume that (1) and (2) hold. By virtue of Theorem 14.2.1, R is a uniquely clean ring in which every idempotent in R is a projection. In addition, R is abelian. Let $a \in R$. According to Theorem 4.4.9, there exists a projection $e \in R$ such that $a - e \in J(R) \subseteq N(R)$, and we therefore complete the proof. □

Form the preceding observation, we see that {strongly nil *-clean rings} $\subsetneqq$ {uniquely strongly *-clean rings} $\subsetneqq$ {strongly *-clean rings}.

Corollary 14.3.6. *A *-ring R is strongly nil-*-clean if and only if*

(1) *Every idempotent in R is a projection*;
(2) *$J(R)$ is nil*;
(3) *$R/J(R)$ is Boolean.*

Proof. Suppose that R is strongly nil-*-clean. In view of Corollary 14.3.2 and Theorem 14.3.5, (1) and (2) hold. By virtue of Corollary 14.3.2, R is uniquely nil-clean. According to Corollary 5.3.6, $R/J(R)$ is Boolean.

Conversely, assume that (1), (2) and (3) hold. For any $a \in R$, $a - a^2 \in J(R)$. As $J(R)$ is nil, we see that $a - a^2 \in N(R)$. In view of Theorem 5.1.1, R is strongly nil-clean. Since every idempotent in R is a projection, the result follows. □

Corollary 14.3.7. *A *-ring R is strongly nil-*-clean if and only if*

(1) *Every idempotent in R is a projection*;
(2) *R is periodic*;
(3) *$R/J(R)$ is Boolean.*

Proof. Suppose that R is strongly nil-*-clean. (1) and (3) are clear. In view of Theorem 14.3.1, R is strongly nil-clean. Therefore R is periodic by Theorem 5.1.12. Conversely, $J(R)$ is nil as R is periodic, thus completing the proof by Corollary 14.3.6. □

Theorem 14.3.8. *A *-ring R is strongly nil-*-clean if and only if*

(1) *Every idempotent in R is a projection*;

(2) $N(R)$ *forms an ideal*;
(3) $R/N(R)$ *is Boolean.*

Proof. Suppose R is strongly nil-*-clean. By virtue of Theorem 14.3.1, R is strongly nil-clean and strongly *-clean. Thus, every idempotent in R is a projection. Thus, (2) and (3) hold from Theorem 5.1.5.

Conversely, assume that (1), (2) and (3) hold. For any $a \in R$, we see that $\overline{a} = \overline{a^2}$ in $R/N(R)$; hence, $a - a^2 \in N(R)$. Thus R is strongly nil-clean. By (1), R is strongly nil-*-clean, as asserted. □

Corollary 14.3.9. *A *-ring R is strongly nil-*-clean if and only if for any $a \in R$, there exists a unique projection $e \in R$ such that $R(a-e)R$ is nil.*

Proof. Suppose that R is strongly nil-*-clean. For any $a \in R$, there exists a projection $e \in R$ such that $a - e \in N(R)$. In view of Theorem 14.3.8, $N(R)$ is an ideal of R. Thus $R(a-e)R$ is nil. Assume that there exists a projection $f \in R$ such that $R(a-f)R$ is nil. Clearly, R is abelian. Hence $e - f = (a-f) - (a-e) \in N(R)$. As $(e-f)^3 = e - f$, we see that $e = f$, and so the uniqueness is proved.

Conversely, for any $a \in R$, there exists a unique projection $e \in R$ such that $R(a-e)R$ is nil; hence, $a - e \in J(R)$. If $x \in J(R)$, then we can find a projection $f \in R$ such that $x - f \in J(R)$. This yields that $f \in J(R)$, and then $f = 0$. Hence $J(R)$ is nil. If there exists a projection $g \in R$ such that $a - g \in J(R)$, then $R(a-g)R \subseteq J(R)$. Hence, $R(a-g)R$ is nil. By hypothesis, $e = g$. According to Theorem 14.2.10, R is uniquely strongly *-clean. In light of Theorem 14.3.5, R is strongly nil-*-clean. □

Proposition 14.3.10. *A *-ring R is strongly nil-*-clean if and only if*

(1) R *is strongly *-clean*;
(2) $N(R) = \{x \in R \mid 1 - x \in U(R)\}$.

Proof. Suppose that R is strongly nil-*-clean. Then R is strongly nil-clean and strongly *-clean by Theorem 14.3.1. We conclude that $N(R) = \{x \in R \mid 1 - x \in U(R)\}$ by Proposition 5.1.4.

Conversely, assume that (1) and (2) hold. Then R is strongly clean. It follows from Proposition 5.1.4 that R is strongly nil clean. Therefore the proof is true by Theorem 14.3.1. □

Corollary 14.3.11. *A *-ring R is strongly nil-*-clean if and only if so is $R[x]/(x^n)$ $(n \in \mathbb{N})$.*

Proof. One direction is obvious. Conversely, assume that R is strongly nil-*-clean. It follows by Theorem 14.3.5 that R is uniquely strongly *-clean In light of Proposition 14.2.20, $R[x]/(x^n)$ is uniquely strongly *-clean, and then it is strongly *-clean.

Clearly, $N\big(R[x]/(x^n)\big) = \{a_0 + a_1x + \cdots + a_{n-1}x^{n-1} \mid a_0 \in N(R), a_1, \ldots, a_{n-1} \in R\}$. In view of Proposition 14.3.10, $N(R) = \{x \in R \mid 1-x \in U(R)\}$. Thus, $N\big(R[x]/(x^n)\big) = \{a_0+a_1x+\cdots+a_{n-1}x^{n-1} \mid 1-a_0 \in U(R), a_1, \ldots, a_{n-1} \in R\}$. By using Proposition 14.3.10 again, we complete the proof. □

Let $R[\alpha] = \{x + y\alpha \mid x, y \in R, \alpha^2 = \mu\alpha + \eta \text{ for some } \mu, \eta \in R\}$. Let R be a *-ring. Then $R[\alpha]$ is a *-ring, where the involution $* : R[\alpha] \to R[\alpha]$, $x + y\alpha \mapsto x^* + y^*\alpha$. We now explore the algebraic extensions of a strongly nil-*-clean ring.

Theorem 14.3.12. *Let R be a commutative *-ring. Then $R[\alpha]$ is strongly nil-*-clean if and only if*

(1) *R is strongly nil-*-clean;*
(2) *$\mu\eta$ is nilpotent.*

Proof. Suppose that $R[\alpha]$ is strongly nil-*-clean. Then every idempotent in R is a projection. Since R is commutative, $N(R)$ forms an ideal. For any $a \in R$, we see that $a - a^2 \in N\big(R[\alpha]\big)$, and so $a - a^2 \in N(R)$. This implies $R/N(R)$ is Boolean. Therefore R is strongly nil-*-clean by Theorem 14.3.8. As $R[\alpha]/N\big(R[\alpha]\big)$ is Boolean, $\alpha - \alpha^2 \in N\big(R[\alpha]\big)$. This shows that $\eta + (\mu - 1)\alpha \in N\big(R[\alpha]\big)$, and so $\mu\eta + (\mu^2 - \mu)\alpha \in N\big(R[\alpha]\big)$. As $\mu^2 - \mu \in N(R)$, we see that $\mu\eta \in N\big(R[\alpha]\big)$. That is, $\mu\eta$ is nilpotent.

Conversely, assume that (1), (2) hold. Since R is commutative, $N\big(R[\alpha]\big)$ forms an ideal of $R[i]$. Let $a+b\alpha \in R[i]$ be an idempotent. Then we can find projections $e, f \in R$ and nilpotents $u, v \in R$ such that $a = e+u$, $b = f+w$. Then $a-a^*, b-b^* \in N(R)$. This shows that $(a+b\alpha)-(a+b\alpha)^* = (a-a^*)+(b-b^*)\alpha \in N\big(R[\alpha]\big)$. Clearly, $\big((a+b\alpha)-(a+b\alpha)^*\big)^3 = (a+b\alpha)-(a+b\alpha)^*$. Therefore $(a + b\alpha) - (a + b\alpha)^* = 0$. That is, $a + b\alpha \in R[\alpha]$ is a projection.

Since R is strongly nil *-clean, it follows from Theorem 5.1.1 that $2-2^2 \in N(R)$, and so $2 \in N(R)$. For any $a + b\alpha \in R[\alpha]$, it is easy to verify that

$$\begin{aligned}(a + b\alpha) - (a + b\alpha)^2 &= (a - a^2) - 2ab\alpha + b\alpha - b^2\alpha^2 \\ &\equiv -b^2\eta + (b - b^2\mu)\alpha \\ &\equiv -b\eta + b(1 - \mu)\alpha \ \big(mod\ N\big(R[i]\big)\big).\end{aligned}$$

This shows that

$$\begin{aligned}\big((a+b\alpha)-(a+b\alpha)^2\big)^2 &\equiv b^2\eta^2-2b^2\eta(1-\mu)\alpha+b^2(1-\mu)^2\alpha^2\\ &\equiv b\eta^2+b(1-\mu)^2(\mu\alpha+\eta)\\ &\equiv b\eta+b(1-\mu)(\mu\alpha+\eta)\\ &\equiv 2b\eta-b\mu\eta+b(\mu-\mu^2)\alpha\\ &\equiv -b\mu\eta\\ &\equiv 0\ \big(mod\ N\big(R[\alpha]\big)\big).\end{aligned}$$

Hence $\overline{(a+b\alpha)}=\overline{(a+b\alpha)^2}$. That is, $R[\alpha]/N\big(R[\alpha]\big)$ is Boolean. According to Theorem 14.3.8, we complete the proof. □

As an immediate consequence, we deduce that a commutative *-ring R is strongly nil-*-clean if and only if so is $R[i]$ where $i^2=-1$.

Example 14.3.13. The ring $\mathbb{Z}_4[x]/(x^2-x+2)$ is strongly nil-*-clean, where * is the identity involution. Clearly $\mathbb{Z}_4[x]/(x^2-x+2)\cong\mathbb{Z}_4[\alpha]$, where $\alpha^2=\alpha+2$. Since $2\in N(\mathbb{Z}_4)$ and $\mathbb{Z}_4$ is strongly nil *-clean, where the involution is the identity, we are done by Theorem 14.3.12.

Let the ring $R[\beta]=\{x+y\beta+z\beta^2\,|\,x,y,z\in R,\ \beta^3=\mu\beta^2+\eta\beta+\lambda,\ \text{for some}\,\mu,\eta,\lambda\in R\}$.

Theorem 14.3.14. *Let R be a commutative *-ring. If $\lambda+\mu\eta\in N(R)$, then $R[\beta]$ is strongly nil-*-clean if and only if*

(1) *R is strongly nil-*-clean;*
(2) *$\lambda(\mu+\eta)$ is nilpotent.*

Proof. Suppose that $R[\beta]$ is strongly nil-*-clean. Then every idempotent in R is a projection. In addition, $N(R)$ forms an ideal of R. For any $a\in R$, we see that $a-a^2\in N\big(R[\beta]\big)$, and so $R/N(R)$ is Boolean. Therefore R is strongly nil-*-clean by Theorem 14.3.8. We show that $\beta-\beta^2,\beta-\beta^3\in N\big(R[\beta]\big)$. This yields that $(\mu+\eta-1)\beta+\lambda\in N\big(R[\beta]\big)$. As $(\mu+\eta)(\mu+\eta-1)\in N(R)$, we get $\lambda(\mu+\eta)\in N\big(R[\beta]\big)$, and so $\lambda(\mu+\eta)\in N(R)$, as required.

Conversely, assume that (1) and (2) hold. Clearly, $N\big(R[\beta]\big)$ forms an ideal of $R[\beta]$ and every idempotent in $R[\beta]$ is a projection. In addition, $2\in N(R)$. Let $a+b\beta+c\beta^2\in R[\beta]$. As $a-a^2,b-b^2,c-c^2\in N(R)$, we see that $(a+b\beta+c\beta^2)^2\equiv a+b\beta^2+c\beta^4\ \big(mod\ N\big(R[\beta]\big)\big)$. It follows from $\beta^3=\mu\beta^2+\eta\beta+\lambda$ that $\beta^4=(\mu^2+\eta)\beta^2+(\mu\eta+\lambda)\beta+\mu\lambda\equiv(\mu+\eta)\beta^2+(\lambda+\mu\eta)\beta-\lambda\eta\ \big(mod\ N\big(R[\beta]\big)\big)$. Hence, $(\beta-\beta^2)^2\equiv\beta^2-\beta^4\equiv(\mu+\eta-1)\beta^2+(\lambda+\mu\eta)\beta+\mu\lambda\ \big(mod\ N\big(R[\beta]\big)\big)$. This shows that

$(\beta-\beta^2)^4 \equiv (\lambda+\mu\eta)\beta^2+(\lambda+\mu\eta)(\mu+\eta-1)\beta+\mu\lambda(\mu+\eta) \equiv (\lambda+\mu\eta)(\beta-\beta^2)\ \big(mod\ N\big(R[\beta]\big)\big)$. Since $\lambda+\mu\eta \in N(R)$, $\beta-\beta^2 \in R[\beta]$ is nilpotent. Clearly, $(a+b\beta+c\beta^2)-(a+b\beta+c\beta^2)^2 \equiv (b+c)(\beta-\beta^2)\ \big(mod\ N\big(R[\beta]\big)\big)$. Therefore $(a+b\beta+c\beta^2)-(a+b\beta+c\beta^2)^2 \in N\big(R[\beta]\big)$, and so $R[i]/N\big(R[\beta]\big)$ is Boolean. Hence $R[\beta]$ is strongly nil-*-clean, by Theorem 14.3.8. □

Example 14.3.15. The ring $\mathbb{Z}_4[x]/(x^3+x^2-x+1)$ is strongly nil-*-clean, where * is the identity involution. Obviously, $\mathbb{Z}_4[x]/(x^3+x^2-x+1) \cong \mathbb{Z}_4[\beta]$, where $\beta^3 = -\beta^2+\beta-1$. Clearly, $2 \in N(\mathbb{Z}_4)$ and $\mathbb{Z}_4$ is strongly nil *-clean, where the involution is the identity, the result follows from Theorem 14.3.14.

14.4 *-Spaces

Let X be a topological space. We say that X is a *-space provided that $C(X)$ is a *-ring. The aim of this section is concerned with the strongly *-clean rings in terms of topological properties of *-spaces.

Lemma 14.4.1. *Let R be a *-ring. Then $Spec(R)$ is a *-space.*

Proof. Let P be a prime ideal of R, and let $P^* = \{x \in R \mid x^* \in P\}$. Then P^* is an ideal of R. If $aRb \in P^*$, then $b^*Ra^* \in P$. Since P is prime, we have either $a \in P^*$ or $b \in P^*$, and so $P^* \in Spec(R)$. Construct a map $* : C\big(Spec(R)\big) \to C\big(Spec(R)\big)$ given by $f \mapsto f^*$, where $f^*(P) = P^*$ for any $P \in Spec(R)$. Then f^* is continuous for any $f \in C(X)$; hence, this map is well defined. Further, * is a ring morphism. If $f^* = 0$, then for any $P \in Spec(R)$, $f^*(P^*) = 0$, and so $f(P) = 0$. This shows that $f = 0$; whence, * is injective. For any $g \in C\big(Spec(R)\big)$, we have $f^* = g$ where $f = g^*$. Accordingly, * is an anti-automorphism, as required. □

Let X be a *-space. We say that a subset U of a *-space X is *-clopen provided that there exists a projection $e \in C(X)$ such that $e(x) = 1$ for any $x \in U$ and $e(x) = 0$ for any $x \in X - U$. A *-space X is strongly *-zero-dimensional provided that for any disjoint closed sets A and B there exists a *-clopen set U of X such that $A \subseteq U$ and $B \subseteq X - U$.

Theorem 14.4.2. *A *-ring R is strongly *-clean if and only if*

(1) *R is an exchange ring;*

(2) *Every projection in R is abelian*;
(3) *$Spec(R)$ is strongly *-zero-dimensional.*

Proof. $\Longrightarrow$ In view of Theorem 1.1.8 and Corollary 14.1.2, R is an abelian exchange ring. Thus (1) and (2) hold. Let A and B be disjoint closed sets of $Spec(R)$. As in the proof of Theorem 12.1.3, there exist two ideals I and J such that $A = V(I)$ and $B = V(J)$ with $I + J = R$. Write $a + b = 1$ where $a \in I$ and $b \in J$. By using Lemma 12.1.2, there exists a projection $e \in R$ such that

$$V(a-1) \subseteq V(1-e) \subseteq Spec(R) - V(a).$$

It is easy to check that

$$\begin{aligned} B = V(J) &\subseteq V(b) \\ &= V(a-1) \subseteq V(1-e) \subseteq Spec(R) - V(a) \\ &\subseteq Spec(R) - V(I) = Spec(R) - A. \end{aligned}$$

Clearly, $B \subseteq V(1-e)$. As $V(1-e) \subseteq Spec(R) - A$, we see that $A \subseteq V(e)$. Obviously, $Spec(R) = V(e) \sqcup V(1-e)$. Define $f : Spec(R) \to \mathbb{R}$ given by $f(P) = 1$ for any $P \in V(e)$ and $f(P) = 0$ for any $P \in V(1-e)$. Then $f = f^2 \in C\big(Spec(R)\big)$. For any $P \in V(e)$, we see that $e \in P$, and so $e \in P^*$. This implies that $P^* \in V(e)$. Thus $f^*(P) = f(P^*) = f(P) = 1$. Likewise, $f^*(P) = f(P) = 0$ for any $P \in V(1-e)$. Therefore $f = f^*$, and so $V(e)$ is a *-clopen set. Therefore $Spec(R)$ is strongly *-zero-dimensional.

$\Longleftarrow$ For any $a \in R$, we have $V(a) \cap V(1-a) = \emptyset$, and then there exists a *-clopen U such that $V(a-1) \subseteq U \subseteq Spec(R) - V(a)$. Thus, we have a projection $f \in C\big(Spec(R)\big)$ such that $f(P) = 1$ for any $P \in U$ and $f(P) = 0$ for any $P \in Spec(R) - U$. As U is clopen and $P(R)$ is nil, analogously to Lemma 12.1.6, we can find an idempotent $e \in R$ such that $U = V(e)$.

For any $P \in V(e)$, we have $f(P) = 1$, and so $f(P^*) = f^*(P) = f(P) = 1$. Thus, $P^* \in U = V(e)$; hence, $e \in P^*$. This implies that $e^* \in P$, and so $P \in V(e^*)$. It follows that $V(e) \subseteq V(e^*)$. For any $P \in V(1-e)$, we have $f(P) = 0$, and so $f(P^*) = f^*(P) = f(P) = 0$. Hence, $P^* \subseteq V(1-e)$. We infer that $1 - e \in P^*$, and so $1 - e^* \in P$. It follows that $P \subseteq V(1-e^*)$. This proves that $V(1-e) \subseteq V(1-e^*)$. As $Spec(R) = V(e) \sqcup V(1-e) = V(e^*) \sqcup V(1-e^*)$, we get $V(e) = V(e^*)$ and $V(1-e) = V(1-e^*)$. For any $P \in Spec(R)$, if $P \in V(e)$, then $P \in V(e^*)$, and so $e, e^* \in P$. Thus, $e - e^* \in P$. If $P \in V(1-e)$, then $P \in V(1-e^*)$. Hence $1-e, 1-e^* \in P$. This implies that $e - e^* = (1-e^*) - (1-e) \in P$. Therefore $e - e^* \in P(R)$. But $P(R)$ is nil, and so $e - e^* \in J(R)$. In view of Lemma 14.1.6, we can

find a projection $g \in R$ such that $eR = gR$. Thus $e = ge$ and $g = eg$. By hypothesis, $g \in R$ is central, and so $e = g$ is a projection. Accordingly, R is strongly *-clean, by Theorem 14.1.1. □

Let X be a topological space. Two sets A and B of X is said to be completely separated if there exists some $f \in C(X)$ such that $0 \leq f \leq 1$, $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$. A subset A of X is a zero set in X provided that there exists some $f \in C(X)$ such that $A = \{x \in X \mid f(x) = 0\}$. Every zero set is a closed set, but the converse does not always hold.

Lemma 14.4.3. *Let X be a *-space. Then X is strongly *-zero-dimensional if and only if*

(1) *Any two disjoint closed sets of X are completely separated*;
(2) *$C(X)$ is strongly *-clean.*

Proof. Suppose X is strongly *-zero-dimensional. Then any disjoint closed sets of X are completely separated. Let $f \in C(X), A = f^{-1}(0)$ and $B = f^{-1}(1)$. Since every zero set of X is closed, A and B are disjoint closed sets of X. By hypothesis, there exists a *-clopen set U of X such that $A \subseteq U$ and $B \subseteq X - U$. Hence, we have a projection $e \in C(X)$ such that $e(x) = 1$ for any $x \in U$ and $e(x) = 0$ for any $x \in X - U$. Let $u = f - e$. For any $x \in U$, $e(x) = 1$. If $f(x) = 1$, then $x \in B$, and so $x \in X - U$, a contradiction. Thus, $f(x) \neq 1$. This implies that $u(x) \neq 0$ for any $x \in U$. For any $x \in X - U$, $e(x) = 0$. If $f(x) = 0$, then $x \in A \subseteq U$, a contradiction, and so $f(x) \neq 0$. This implies that $u(x) \neq 0$ for any $x \in X - U$. Therefore $u(x) \neq 0$ for any $x \in X$. Hence $u^{-1}(x) := \frac{1}{u(x)}$ for any $x \in X$. That is, $u \in C(X)$ is invertible. Accordingly, $f = e + u \in C(X)$ is strongly *-clean.

Conversely, assume that (1), (2) and (3) hold. Let A and B be disjoint closed sets. Then A and B are completely separated. In light of [265, Theorem 1.15], A and B are contained in disjoint zero sets. Thus, we can find some $f_1, f_2 \in C(X)$ such that $A \subseteq Z(f_1), B \subseteq Z(f_2)$ and $Z(f_1) \cap Z(f_2) = \emptyset$. This shows that $|f_1| + |f_2| > 0$ for all $x \in X$. Choose $h = \frac{|f_1|}{|f_1|+|f_2|} \in C(X)$. Since $C(X)$ is strongly *-clean, there exists a projection $e \in C(X)$ and a unit $u \in C(X)$ such that $h = e + u$. For any $x \in X$, $e(x) \cdot e(x) = e(x)$, and so $e(x) = 0$ or $e(x) = 1$. Set $U = \{x \in X \mid e(x) = 0\}$ and $V = \{x \in X \mid e(x) = 1\}$. Then $X = U \sqcup V$. As U and V are closed, V is clopen. Further, V is *-clopen. Since $u \in C(X)$ is a unit, $u(x) \neq 0$ for all $x \in X$. For any $x \in A$, we get $f_1(x) = 0$, and then $h(x) = 0$.

Thus, $e(x) \neq 0$ as $u(x) \neq 0$; hence, $x \in V$. This shows that $A \subseteq V$. For any $x \in B$, $f_2(x) = 0$, and so $h(x) = 1$. We infer that $e(x) = 0$; whence, $x \in X - V$. This shows that $B \subseteq X - V$, hence the result. □

A topological space is perfectly normal provided that every closed set is a zero set. A topological space X is perfectly normal if and only if every two disjoint non-empty closed sets E and F can be precisely separated by a continuous function $f \in C(X)$: the preimages of 0 and 1 under f are, respectively, E and F. Let X be a perfectly normal *-space. We claim that X be strongly *-zero-dimensional if and only if $C(X)$ is strongly *-clean. One direction is clear by Lemma 14.4.3. Conversely, let A and B be disjoint closed sets of X. Then A and B are zero sets of X. According to [265, Theorem 1.15], A and B are completely separated, and then we are done by Lemma 14.4.3.

Proposition 14.4.4. *A *-ring R is strongly *-clean if and only if*

(1) *R is an exchange ring*;
(2) *Every projection in R is abelian*;
(3) *$C\big(Spec(R)\big)$ is strongly *-clean.*

Proof. One direction is clear from Theorem 14.4.2 and Lemma 14.4.3.

Conversely, assume that (1), (2) and (3) hold. In view of Theorem 12.1.7, $Spec(R)$ is strongly zero dimensional. Thus, for any disjoint closed sets A and B of $Spec(R)$, there exists a clopen U such that $A \subseteq U$ and $B \subseteq X - U$. It follows from Urysohn's Lemma, there exists a continuous function on $[0, 1]$ such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$. Thus, A and B are completely separated. By Lemma 14.4.3, $Spec(R)$ is strongly *-zero-dimensional. Therefore we obtain the result by Theorem 14.4.2. □

Clearly, the Jacobson radical $J(R)$ is semiprime, and so $J(R)$ is the intersection of some prime ideals. Thus, $J(R) = \bigcap_{P \in J\text{-}spec(R)} P$. Let I be an ideal of R, and let $F(I) = \{P \in J\text{-}spec(R) \mid I \not\subseteq P\}$. Then $F(R) = J\text{-}spec(R), F(0) = \emptyset, F(I) \cap F(J) = F\big(IJ\big)$ and $\bigcup_i F(I_i) = F\Big(\sum_i I_i\Big)$. So J-$spec(R)$ is a topological subspace of $Spec(R)$, where $\{F(I) \mid I \trianglelefteq R\}$ is the collection of its open sets. Let $W(I) = J\text{-}spec(R) - F(I)$. Then $W(I) = \{P \in J\text{-}spec(R) \mid I \subseteq P\}$ is the collection of its closed sets. Let R be a *-ring. As in the preceding discussion, J-$spec(R)$ is a *-space.

Lemma 14.4.5. *A *-ring R is strongly *-clean if and only if*

(1) *R is an exchange ring;*
(2) *Every projection in R is abelian;*
(3) *$C\big(J\text{-}spec(R)\big)$ is strongly *-clean.*

Proof. Construct a map $\varphi : J\text{-}spec(R) \to Spec\big(R/J(R)\big)$ given by $P \mapsto \overline{P}$ for any $P \in J\text{-}spec(R)$. Then φ is a continuous map. If $\varphi(P) = \varphi(Q)$, then $\overline{P} = \overline{Q}$. For any $p \in P$, write $p + J(R) = q + J(R)$ for a $q \in Q$. Then $p \in q + J(R) \subseteq Q$, and so $P \subseteq Q$. Likewise, $Q \subseteq P$. This shows that $P = Q$; hence, φ is injective. For any $\overline{P} \in Spec\big(R/J(R)\big)$, then $P \in J\text{-}spec(R)$, and so φ is surjective. That is, φ is bijective. Further, one easily checks that φ is a homeomorphism. Construct a map $\phi : C\big(J\text{-}spec(R)\big) \to C\big(Spec(R/J(R))\big)$ given by $f \mapsto f\varphi^{-1}$ for any $f \in C\big(J\text{-}spec(R)\big)$. Clearly, $\varphi(f^*) = \big(\varphi(f)\big)^*$. Therefore $C\big(J\text{-}spec(R)\big)$ and $C\big(Spec(R/J(R))\big)$ are *-isomorphism.

If R is strongly *-clean, then (1) and (2) holds. Further, $R/J(R)$ is strongly *-clean. It follows from Theorem 14.4.2, $C\big(Spec(R/J(R))\big)$ is strongly *-clean, and therefore so is $C\big(J\text{-}spec(R)\big)$. Conversely, assume that (1), (2) and (3) hold. Then $R/J(R)$ is an exchange ring and $C\big(Spec(R/J(R))\big)$ is strongly *-clean. Let $\overline{e} \in R/J(R)$ be a projection. We may assume that $e = e^2 \in R$. Further, $e - e^* \in J(R)$. In view of Lemma 14.1.6, we have a projection $f \in R$ such that $eR = fR$. By hypothesis, $f \in R$ is central, and so $e = f$. Hence $e \in R$ is a projection, and then it is central. Therefore $\overline{e} \in R/J(R)$ is central. In light of Theorem 14.4.2, $R/J(R)$ is strongly *-clean. Clearly, every idempotent of R lifts modulo $J(R)$. Therefore R is strongly *-clean by Corollary 14.1.8. □

Theorem 14.4.6. *A *-ring R is strongly *-clean if and only if*

(1) *R is an exchange ring;*
(2) *Every projection in R is abelian;*
(3) *$J\text{-}spec(R)$ is strongly *-zero-dimensional.*

Proof. $\Longrightarrow$ By Lemma 14.4.5, $C\big(J\text{-}spec(R)\big)$ is strongly *-clean. Analogously to Proposition 14.4.4, any two disjoint closed sets of $J\text{-}spec(R)$ are completely separated. Therefore $J\text{-}spec(R)$ is strongly *-zero-dimensional from Lemma 14.4.3, as desired.

$\Longleftarrow$ In view of Lemma 14.4.3, $C\big(J\text{-}spec(R)\big)$ is strongly *-clean. Hence the result follows by Lemma 14.4.5. □

Let R be a *-ring. Clearly, $M \in Max(R)$ if and only if $M^* \in Max(R)$. Construct a map $* : C\big(Max(R)\big) \to C\big(Max(R)\big)$ given by $f \mapsto f^*$, where

$f^*(M) = f(M^*)$ for any $M \in Max(R)$. As in the proof of Lemma 14.4.1, * is an anti-automorphism of $C\big(Max(R)\big)$. Therefore $Max(R)$ is a *-space.

Theorem 14.4.7. *A *-ring R is strongly *-clean if and only if*

(1) *R is an abelian exchange ring;*
(2) *$Max(R)$ is strongly *-zero-dimensional.*

Proof. Suppose R is strongly *-clean. Then R is an abelian exchange ring. Let A and B be disjoint closed sets of $Max(R)$. Analogously to the discussion in Theorem 14.4.2, there exists a projection $e \in R$ such that $A \subseteq V(e)$ and $B \subseteq V(1-e)$. Define $f : Max(R) \to \mathbb{R}$ given by $f(M) = 1$ for any $M \in V(e)$ and $f(M) = 0$ for any $M \in V(1-e)$. Then $f \in C\big(Max(R)\big)$. Similarly to the consideration in Theorem 14.4.2, $V(e)$ is a *-clopen set. We conclude that $Max(R)$ is strongly *-zero-dimensional.

Conversely, assume that (1) and (2) hold. Then R is abelian clean. In view of Theorem 12.1.13, R is a pm ring. Thus, there exists a map $\varphi : Spec(R) \to Max(R)$, $\varphi(P) = M$, where M is the unique maximal ideal such that $P \subseteq M$. For any $I \trianglelefteq R$, we let $V_S(I) := \{P \in Spec(R) \mid I \subseteq P\}$ and $V_M(I) := \{P \in Max(R) \mid I \subseteq P\}$. Then $\varphi\big(V_S(I)\big) = V_M(I)$, and so φ is continuous. For any disjoint closed sets $A, B \subseteq Spec(R)$, we have two ideals I and J of R such that $A = V_S(I)$ and $B = V_S(J)$. Thus, $\varphi(A)$ and $\varphi(B)$ are both closed. It follows from $V_S(I) \cap V_S(J) = \emptyset$ that $V_S(I+J) = \emptyset$; whence, $I + J = R$. Hence, $V_M(I) \cap V_M(J) = V_M(I+J) = V_M(R) = \emptyset$. This implies that $\varphi(A)$ and $\varphi(B)$ are disjoint closed sets of $Max(R)$. By hypothesis, we have disjoint *-clopen sets $U, V \subseteq Max(R)$ such that $V_M(I) \subseteq U, V_M(J) \subseteq V$. Obviously, $A \subseteq \varphi^{\leftarrow}\varphi(A) \subseteq \varphi^{\leftarrow}(U)$ and $B \subseteq \varphi^{\leftarrow}\varphi(B) \subseteq \varphi^{\leftarrow}(V)$. As is well known, φ is continuous; hence, $\varphi^{\leftarrow}(U)$ and $\varphi^{\leftarrow}(V)$ are clopen. For any $P \in \varphi^{\leftarrow}(U) \cap \varphi^{\leftarrow}(V)$, there exists a unique $M \in Max(R)$ such that $P \subseteq M$. Thus $M \in U \cap V$, a contradiction. Therefore $\varphi^{\leftarrow}(U) \cap \varphi^{\leftarrow}(V) = \emptyset$.

Since U is a *-clopen set of $Max(R)$, we have a projection $e \in C\big(Max(R)\big)$ such that $e(x) = 1$ for any $x \in U$ and $e(x) = 0$ for any $x \in X - U$. Construct a function $f : Spec(R) \to \mathbb{R}$ given by $P \mapsto e\varphi(P)$ for any $P \in Spec(R)$. Then $f \in C\big(Spec(R)\big)$ is a projection. Moreover, $f(y) = e\varphi(y) = 1$ for any $y \in \varphi^{\leftarrow}(U)$ and $f(y) = e\varphi(y) = 0$ for any $y \in Spec(R) - \varphi^{\leftarrow}(U)$. This shows that $\varphi^{\leftarrow}(U)$ is *-clopen. Likewise, $\varphi^{\leftarrow}(V)$ is *-clopen. Therefore $Spec(R)$ is strongly *-zero-dimensional, and we complete the proof by Theorem 14.4.2. □

Corollary 14.4.8. *A *-ring R is strongly *-clean if and only if*

(1) *R is an abelian exchange ring;*
(2) *$C\big(Max(R)\big)$ is strongly *-clean.*

Proof. $\Longrightarrow$ This is obvious by Theorem 14.4.7 and Lemma 14.4.3.

$\Longleftarrow$ Clearly, R is clean. By Theorem 12.1.13, $Max(R)$ is strongly zero-dimensional. As in the proof of Proposition 14.4.4, by Urysohn's Lemma, any two disjoint closed sets of $Max(R)$ are completely separated. According to Lemma 14.4.3, $Max(R)$ is strongly *-zero-dimensional, as required. □

A *-space X of a *-ring R is a spectrum space provided that it is $Spec(R), J\text{-}spec(R)$ or $Max(R)$.

Proposition 14.4.9. *Let X be a spectrum space of a *-ring R. Then R is uniquely strongly *-clean if and only if R is uniquely clean and X is strongly *-zero dimensional.*

Proof. $\Longrightarrow$ This is obvious by Theorem 14.2.1, Theorem 14.4.2, Theorem 14.4.6 or Theorem 14.4.7.

$\Longleftarrow$ In view of Theorem 4.4.6, R is an abelian exchange ring. Thus, R is strongly *-clean by Theorem 14.4.2, Theorem 14.4.6 or Theorem 14.4.7. So the proposition is true from Theorem 14.2.1. □

Proposition 14.4.10. *Let X be a spectrum space of a *-ring R. Then R is strongly nil-*-clean if and only if R is uniquely nil-clean and X is strongly *-zero dimensional.*

Let R be a commutative *-ring with an identity, and let $q(R)$ be the classical ring of quotients of R. We say that $x \in R$ is self-adjoint provided that $x^* = x$. Let $N_D(R)$ denote the set of all non-zero-divisors of R, and let $N_D(X) = NZ\big(C(X)\big)$. Construct a ring morphism $* : q(R) \to q(R), \frac{r}{s} \mapsto \frac{r^*}{s^*}$. Then φ is also an anti-automorphism of $q(R)$. Thus $q(R)$ is a *-ring. Let X be a *-space. Then $C(X)$ is a *-ring. We denote $q\big(C(X)\big)$ by $q(X)$, and so $q(X)$ is a *-ring. In what follows, we investigate when $q(X)$ is strongly *-clean.

Lemma 14.4.11. *Let R be a commutative *-ring. If $e \in q(R)$ is self-adjoint, then there exist self-adjoint $a, b \in R$ such that $e = \frac{a}{b}$.*

Proof. Write $e = \frac{a}{b}$. Since $e \in q(R)$ is self-adjoint, $e^* = \big(\frac{c}{d}\big)^* = \frac{c^*}{d^*} = \frac{c}{d}$. Thus $c^*d = d^*c$. Clearly, $d, d^* \in N_D(R)$; hence, $e = \frac{cd^*}{dd^*}$. Set $a = cd^*$ and

$b = dd^*$. Then $a^* = dc^* = a$ and $b^* = b$. That is, $a, b \in R$ are self-adjoint. In addition, $e = \frac{a}{b}$, as required. □

Lemma 14.4.12. *Let R be a commutative *-ring. Then the following are equivalent:*

(1) *$q(R)$ is strongly *-clean.*
(2) *For any $a, b \in R$ with $a + b \in N_D(R)$, there exist self-adjoint $x \in aR, y \in bR$ such that $x + y \in N_D(R)$ and $xy = 0$.*
(3) *For any $a, b \in R$ with $a + b \in N_D(R)$, there exist $x \in aR, y \in bR$ such that $x + y \in N_D(R), xy = 0$ and x^*y is self-adjoint.*

Proof. (1) ⇒ (2) Suppose that $a + b \in N_D(R)$ with $a, b \in R$. Then there exists some $\alpha \in q(R)$ such that $a\alpha + b\alpha = 1$. Since $q(R)$ is strongly *-clean, we can find a projection $e \in q(R)$ such that $e \in b\alpha q(R) \subseteq bq(R)$ and $1 - e \in a\alpha q(R) \subseteq aq(R)$. Write $e = \frac{bs}{t} = \frac{bst^*}{tt^*}$. Set $w = bst^*$ and $u = tt^*$. Then $e = \frac{w}{u}$, where $w, u \in R$ are self-adjoint and $w \in bR$. Analogously, $1-e = \frac{z}{t}$, where $z, t \in R$ are self-adjoint and $z \in aR$. Obviously, $\frac{w}{u} + \frac{z}{t} = 1$, and so $wt + zu = ut$. Choose $x = wt$ and $y = zu$. Then $x+y = ut \in N_D(R)$. Further, $xy = (wz)(ut) = 0$ and $x, y \in R$ are self-adjoint.

(2) ⇒ (3) This is trivial.

(3) ⇒ (1) Suppose that $\frac{a}{s} + \frac{b}{s} = 1$ in $q(R)$. Then $a + b = s \in N_D(R)$. By hypothesis, there exist $x \in aR, y \in bR$ such that $x + y \in N_D(R), xy = 0$ and x^*y is self-adjoint. Let $e = \frac{x}{x+y}$. Then $e(1 - e) = \frac{xy}{x+y} = 0$, and so $e = e^2 \in q(R)$ is an idempotent. Since $x^*y \in R$ is self-adjoint, we see that $(x^*y)^* = x^*y = xy^*$, and so $e^* = \frac{x^*}{x^*+y^*} = \frac{x}{x+y} = e$; hence, $e \in q(R)$ is a projection. Moreover, $e = \frac{x}{x+y} \in \left(\frac{a}{s}\right)q(R)$ and $1 - e = \frac{y}{x+y} \in \left(\frac{b}{s}\right)q(R)$. Therefore $q(R)$ is strongly *-clean. □

We say that U is a *-zero set of X provided that there exists a self-adjoint $f \in C(X)$ such that $A = Z(f)$.

Theorem 14.4.13. *Let X be a completely regular *-space. Then the following are equivalent:*

(1) *$q(X)$ is strongly *-clean.*
(2) *For any zero sets A and B of X such that $A \cap B$ is nowhere dense, there exist *-zero sets U, V such that $A \subseteq U, B \subseteq V$ such that $U \cap V$ is nowhere dense and $U \cup V = X$.*

Proof. (1) ⇒ (2) For any zero sets A and B of X such that $A \cap B$ is nowhere dense, we can write $A = Z(f)$ and $B = Z(g)$.

Since $Z(f^2+g^2) = Z(f) \cap Z(g) = U \cap V$ is nowhere dense, it follows from Lemma 13.3.6 that $f^2+g^2 \in N_D(X)$. In view of Lemma 14.4.12, there exist self-adjoint $h \in f^2C(X), k \in g^2C(X)$ such that $h+k \in N_D(X)$ and $hk = 0$. Let $U = Z(h)$ and $V = Z(k)$. Then $A \subseteq U, B \subseteq V$. In addition, $U \cup V = Z(h) \cup Z(k) = Z(hk) = Z(0) = X$. Further, $U \cap V = Z(h) \cap Z(k) = Z(h^2+k^2)$. As $h^2+k^2 = (h+k)^2$, we see that $U \cap V = Z\big((h+k)^2\big) = Z(h+k)$ is nowhere dense from Lemma 13.3.6. Since $h, k \in q(X)$ are self-adjoint, U and V are both *-zero sets, as required.

(2) $\Rightarrow$ (1) Let $f, g \in C(X)$ such that $f+g \in N_D(X)$. Let $A = Z(f)$ and $B = Z(g)$. Then $A \cap B = Z(f) \cap Z(g) \subseteq Z(f+g)$; hence, $A \cap B$ is nowhere dense from Lemma 13.3.6. By hypothesis, there exist *-zero sets U, V such that $A \subseteq U, B \subseteq V$ such that $U \cap V$ is nowhere dense and $U \cup V = X$. Thus, we can find self-adjoint $h, k \in C(X)$ such that $U = Z(h)$ and $V = Z(k)$. Set $\varphi = fh \in fC(X)$ and $\psi = gk \in gC(X)$. Then $Z(\varphi) = Z(fh) = Z(f) \cup Z(h) = Z(h)$. Likewise, $Z(\psi) = Z(k)$. Thus, $Z(\varphi^2+\psi^2) = Z(\varphi) \cap Z(\psi) = Z(h) \cap Z(k)$ is nowhere dense, and so $\varphi^2+\psi^2 \in N_D(X)$ from Lemma 13.3.6. As $Z(\varphi^2\psi^2) = Z(\varphi) \cup Z(\psi) = Z(h) \cup Z(k) = X$, we see that $\varphi^2\psi^2 = 0$. In addition, it follows from $Z(hk) = Z(h) \cup Z(k) = X$ that $hk = 0$. Therefore $\big(\varphi^2\big)^*\psi^2 = (fg)^2hk(hk) = 0$. According to Lemma 14.4.12, $q(X)$ is strongly *-clean. □

The complement of a zero set is called a cozero set. The cozero set of $f \in C(X)$ is $coz(f) = \{x \in X \mid f(x) \neq 0\}$. We call U is a *-cozero set provided that there exists a self-adjoint $f \in C(X)$ such that $U = coz(f)$.

Corollary 14.4.14. *Let X be a completely regular *-space. Then the following are equivalent:*

(1) *$q(X)$ is strongly *-clean.*
(2) *For any cozero sets A and B of X such that $A \cup B$ is dense, there exist disjoint *-cozero sets U, V such that $U \subseteq A, V \subseteq B$ such that $U \cup V$ is dense.*

Proof. Choose $C = X - A$ and $D = X - B$. Applying Theorem 14.4.13 to C and D, we complete the proof. □

Lemma 14.4.15. *Let X be a completely regular *-space. Then $C(X)$ is strongly *-clean if and only if*

(1) *X is strongly zero-dimensional*;
(2) *$q(X)$ is strongly *-clean.*

Proof. Suppose that $C(X)$ is strongly *-clean. Then $q(X)$ is strongly *-clean. By [40, Theorem 2.5], X is strongly zero-dimensional, as desired.

Conversely, assume that (1) and (2) hold. Let A and B be disjoint closed sets. Since X is strongly zero-dimensional, there exists a clopen set U such that $A \subseteq U$ and $B \subseteq V$. Thus, there exists $e \in C(X)$ such that $e(x) = 1$ for any $x \in U$ and $e(x) = 0$ for any $x \in X - U$. Clearly, $e = e^2 \in C(X)$. By hypothesis, we have a projection $f \in q(X)$ and a unit $u \in q(X)$ such that $e = f + u$. By virtue of Lemma 14.4.11, write $f = \frac{a}{b}$ with self-adjoint $a, b \in R$. Since $e, f \in q(X)$ are idempotents, we have $(e - f)^3 = e - f$, and so $u^2 = 1$. That is, $(e - f)^2 = 1$. Then $e(1 - 2f) = 1 - f$; hence, $e = (1-2f)(1-f)$. Thus, $\frac{e}{1} = \frac{(b-2a)(b-a)}{b^2}$, and so $eb^2 = (b-2a)(b-a)$. As $a, b \in R$ are self-adjoint, we get $e^*b^2 = \big((b-2a)(b-a)\big)^* = (b-2a)(b-a) = eb^2$. But $b \in N_D(R)$, and so $e = e^* = e^2$. Thus, U is a *-clopen, and so X is strongly *-zero-dimensional. This completes the proof by Lemma 14.4.3. □

Theorem 14.4.16. *Let X be a spectrum space of a *-ring R. Then R is a strongly *-clean ring if and only if*

(1) *R is an abelian exchange ring*;
(2) *$q(X)$ is strongly *-clean.*

Proof. As is well known, every locally compact Hausdorff space is completely regular. Hence, the spectrum space X of R is always completely regular.

If R is a strongly *-clean ring, then R is an abelian exchange ring. By virtue of Lemma 14.4.3, $C(X)$ is strongly *-clean. In light of Lemma 14.4.15, $q(X)$ is strongly *-clean.

Conversely, assume that (1) and (2) hold. Then X is strongly zero-dimensional. In view of Lemma 14.4.3, $C(X)$ is strongly *-clean. Accordingly, R is strongly *-clean by Theorems Proposition 14.4.4, Theorem 14.4.6 and Corollary 14.4.8. □

Notes. Theorem 14.1.1 is due to Li and Zhou [394, Theorem 2.2]. Theorem 14.2.10 and Theorem 14.2.16 were contained in [133, Theorem 3.2 and Theorem 3.7]. Proposition 14.2.21 is due to X.L. Wang and C. Li [529, Theorem 3.1]. Theorem 14.3.5, Corollary 14.3.6 and Corollary 14.3.7 are due to [303, Theorem 2.7]. Theorem 14.4.2 and Theorem 14.4.13 were proved by Chen and Harmanci [132, Theorem 1 and Theorem 7].

This page intentionally left blank

Chapter 15

The Generalized Drazin Inverse

Let $\mathcal{A}$ be a complex Banach algebra with an identity 1. The commutant of $a \in \mathcal{A}$ is defined by $comm(a) = \{x \in \mathcal{A} \mid xa = ax\}$. The double commutant of $a \in \mathcal{A}$ is defined by $comm^2(a) = \{x \in \mathcal{A} \mid xy = yx \text{ for all } y \in comm(a)\}$. An element a in $\mathcal{A}$ has g-Drazin inverse, i.e., a is GD-invertible, if and only if there exists $b \in comm(a)$ such that $b = bab$ and $a - a^2b \in \mathcal{A}^{qnil}$. Such b, if exists, is unique, and is denoted by a^d. We call a^d the g-Drazin inverse of a. The aim of this chapter is to explore the g-Drazin inverse in a Banach algebra. In Section 15.1, we establish various elementary properties of the g-Drazin inverse in Banach algebras. In Section 15.2, we investigate when the sum of two g-Drazin invertible elements has g-Drazin inverse. In Section 15.3, we explore the g-Drzin invertibility for anti-triangular block matrices. Finally, in Section 15.4, we are concerned with the g-Drazin inverse of block operator matrices over a Banach space.

15.1 The g-Drazin Inverse

Let $\mathcal{A}^{qnil}$ denote the set of all quasinilpotents of the ring R, i.e., $\mathcal{A}^{qnil} = \{a \in R \mid 1 + ax \in R \text{ invertible for all } x \in comm(a)\}$. Evidently,

$$a \in A^{qnil} \Leftrightarrow \lim_{n\to\infty} \| a^n \|^{\frac{1}{n}} = 0 \Leftrightarrow 1 + \lambda a \in \mathcal{A}^{-1} \text{ for any } \lambda \in \mathbb{C}.$$

We start by recalling the following elementary property.

Lemma 15.1.1. *Let $\mathcal{A}$ be a Banach algebra, $a, b \in \mathcal{A}$ and $ab = ba$.*

(1) *If $a, b \in \mathcal{A}^{qnil}$, then $a + b \in \mathcal{A}^{qnil}$.*
(2) *If a or $b \in \mathcal{A}^{qnil}$, then $ab \in \mathcal{A}^{qnil}$.*

Proof. (1) Let $\lambda \in \mathbb{C}$. Since $a \in \mathcal{A}^{qnil}$, $1 + \lambda a \in \mathcal{A}^{-1}$. It follows from $ab = ba$ that $\lambda(1 + \lambda a)^{-1}b = b\lambda(1 + \lambda a)^{-1}b$. As $b \in \mathcal{A}^{qnil}$, we have $1 + \lambda(1 + \lambda a)^{-1}b \in \mathcal{A}^{-1}$. Therefore

$$1 + \lambda(a + b) = (1 + \lambda a)[1 + \lambda(1 + \lambda a)^{-1}b] \in \mathcal{A}^{-1}.$$

Then $a + b \in \mathcal{A}^{qnil}$, as required.

(2) We may assume that $a \in \mathcal{A}^{qnil}$. Let $\lambda \in \mathbb{C}$. Then $(\lambda b)a = a(\lambda b)$. Hence,

$$1 + \lambda(ab) = 1 + (\lambda b)a \in \mathcal{A}^{-1}.$$

This implies that $ab \in \mathcal{A}^{qnil}$, as desired. □

Lemma 15.1.2. *Let $\mathcal{A}$ be a Banach algebra, and let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) *There exists an idempotent $e \in comm^2(a)$ such that $a - e \in \mathcal{A}^{qnil}$.*
(2) *$a - a^2 \in \mathcal{A}^{qnil}$.*

Proof. (1) $\Rightarrow$ (2) Write $a = e + w$ with $e^2 = e \in comm^2(a), w \in \mathcal{A}^{qnil}$. Then $a - a^2 = (1 - 2e - w)w \in \mathcal{A}^{qnil}$, by Lemma 15.1.1.

(2) $\Rightarrow$ (1) Let $q := a - a^2$. Then $4q(4q-1)^{-1} \in \mathcal{A}^{qnil}$ by Lemma 15.1.1. Consider the infinite series

$$-\frac{1}{2}\sum_{k=1}^{\infty}\binom{\frac{1}{2}}{k}\left(4q(4q-1)^{-1}\right)^k,$$

where the coefficients are binomial coefficients. Then the series converges absolutely to an element z. Moreover, we have the formal relation

$$1 - \sqrt{1 - 4q(4q-1)^{-1}} = 2z.$$

This implies that

$$z^2 - z = q(4q-1)^{-1}.$$

Here, z commutes with every element of $\mathcal{A}$ which commutes with $q(4q - 1)^{-1}$. That is, there exists $z \in \mathcal{A}$ such that $z^2 - z = q(4q-1)^{-1}$ and $z \in comm^2(q(4q-1)^{-1})$ (see [455, Lemma 2.3.8]).

Let $0 \neq \lambda \in \mathbb{C}$. Set $y = 2\lambda z$. Then $y^2 - 2\lambda y = 4\lambda^2 q(4q-1)^{-1} \in \mathcal{A}^{qnil}$. Hence,

$$\begin{aligned}(y-\lambda)^2 &= y^2 - 2\lambda y + \lambda^2 \\ &= \lambda^2 + 4\lambda^2 q(4q-1)^{-1} \\ &\in \mathcal{A}^{-1}.\end{aligned}$$

It follows that $y - \lambda \in \mathcal{A}^{-1}$, and so $y \in \mathcal{A}^{qnil}$. This implies that $z = \frac{1}{2}\lambda^{-1} y \in \mathcal{A}^{qnil}$.

Since $a \in comm(q(4q-1)^{-1})$, we have $az = za$. Let $f = a - (2a-1)z$. Then $a - f = (2a-1)z \in \mathcal{A}^{qnil}$ by Lemma 15.1.1. We easily see that $(1-2a)^2(1-4q)^{-1} = 1$; hence,

$$\begin{aligned}&(a-(2a-1)z)(1-a+(2a-1)z) \\ &= (a-a^2) + (-(2a-1)(1-a) + a(2a-1))z - (2a-1)^2 z^2 \\ &= (a-a^2) + (2a-1)^2(z-z^2) \\ &= q - (2a-1)^2 q(1-4q)^{-1} \\ &= 0,\end{aligned}$$

and so $(a-(2a-1)z)^2 = a-(2a-1)z$. That is, $f^2 = f \in comm^2(a)$, as desired. □

Theorem 15.1.3. *Let $\mathcal{A}$ be a Banach algebra, and let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^d$.
(2) *There exists some $x \in comm(a)$ such that $a - a^2 x \in \mathcal{A}^{qnil}$.*

Proof. (1) $\Rightarrow$ (2) This is obvious by choosing $x = a^d$.

(2) $\Rightarrow$ (1) By hypothesis, there exists some $x \in comm(a)$ such that $a - a^2 x \in \mathcal{A}^{qnil}$. Set $z = xax$. Then $z \in comm(a)$. We check that

$$\begin{aligned}a - a^2 z &= a - axaxa \\ &= (1+ax)(a-a^2x) \\ &\in \mathcal{A}^{qnil}, \\ z - z^2 a &= xax - xaxaxax \\ &= x(a-a^2x)x + xax(a-a^2x)x \\ &\in \mathcal{A}^{qnil}.\end{aligned}$$

$$az - (az)^2 = (a-a^2z)z \in \mathcal{A}^{qnil}.$$

In view of Lemma 15.1.2, we have an idempotent $e \in comm^2(az)$ such that $az - e \in \mathcal{A}^{qnil}$. We easily check that

$$(a+1-az)((z+1-az) = 1 + (a-a^2z)(1-z) + (z-z^2a).$$

Hence,

$$\begin{aligned} a+1-e &= (a+1-az)+(az-e) \in \mathcal{A}^{-1}, \\ a(1-e) &= (a-a^2z)+a(az-e) \in \mathcal{A}^{qnil}. \end{aligned}$$

Since $a \in comm(az)$, we have $ea = ae$. That is, $a \in \mathcal{A}$ is quasipolar. Therefore $a \in \mathcal{A}^d$, by Theorem 6.4.8. □

Corollary 15.1.4. *Let $\mathcal{A}$ be a Banach algebra, and let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^d$.
(2) *There exists a invertible $u \in comm(a)$ such that $a - a^2u \in \mathcal{A}^{qnil}$.*

Proof. (1) $\Rightarrow$ (2) In view of Theorem 6.4.8, there exists an idempotent $e \in comm(a)$ such that $u := a + e \in \mathcal{A}^{-1}$ and $ae \in \mathcal{A}^{qnil}$. Hence, $ae = a(u-a) \in \mathcal{A}^{qnil}$. Then $a - a^2u^{-1} \in \mathcal{A}^{qnil}$, as desired.

(2) $\Rightarrow$ (1) This is obvious by Theorem 15.1.3. □

Corollary 15.1.5. *Let $\mathcal{A}$ be a Banach algebra, and let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^d$.
(2) *There exists some $p \in comm(a)$ such that $a+p \in \mathcal{A}^{-1}$ and $ap \in \mathcal{A}^{qnil}$.*

Proof. (1) $\Rightarrow$ (2) This is clear by Theorem 6.4.8.

(2) $\Rightarrow$ (1) Set $b = (a+p)^{-1}(1-p)$. Then $b \in comm(a)$ and

$$\begin{aligned} ab &= a(a+p)^{-1}(1-p) \\ &= (a+p)(a+p)^{-1}(1-p) - p(a+p)^{-1}(1-p) \\ &= 1-p-p(a+p)^{-1}(1-p). \end{aligned}$$

In view of Lemma 15.1.9, we have

$$\begin{aligned} a-a^2b &= a(1-ab) \\ &= ap[1+(a+p)^{-1}(1-p)] \\ &\in \mathcal{A}^{qnil}. \end{aligned}$$

This completes the proof by Theorem 15.1.3. □

Proposition 15.1.6. *Let $\mathcal{A}$ be a Banach algebra, and let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^d$.
(2) *There exist an idempotent e, a unit u and a quasinilpotent w that commute such that $a = eu + w$.*

Proof. (1) $\Rightarrow$ (2) By hypothesis, there exists a invertible $u \in comm(a)$ such that $a - a^2u \in \mathcal{A}^{qnil}$. Then $(u^{-1}a)^2 - u^{-1}a \in \mathcal{A}^{qnil}$. In light of Lemma 15.1.2, there exists $e^2 = e \in comm^2(u^{-1}a)$ such that $w := u^{-1}a - e \in \mathcal{A}^{qnil}$. Then $a = ue + uw$. Clearly, $eu = ue$ and $ea = ae$; hence, $uw = wu, (ue)(uw) = (uw)(ue)$ and $uw \in \mathcal{A}^{qnil}$, as required.

(2) $\Rightarrow$ (1) Write $a = eu + w$ for an idempotent e, an invertible u and a quasinilpotent w that commute. Then $(u^{-1}a)^2 - u^{-1}a \in \mathcal{A}^{qnil}$. Then $a - a^2u^{-1} \in \mathcal{A}^{qnil}$. This completes the proof by Corollary 15.1.4. □

Corollary 15.1.7. *Let $a \in \mathcal{A}^d$. Then a is the sum of two units in $\mathcal{A}$.*

Proof. Since $a \in \mathcal{A}^d$, it follows by Lemma 15.1.1 that $\frac{a}{2} \in \mathcal{A}^d$. In view of Proposition 15.1.6, there exist an idempotent e, a unit u and a quasinilpotent w that commute such that $\frac{a}{2} = eu + w$. Hence,

$$a = 2eu + 2w = (2e - 1)u + u + 2w = (2e - 1)u + u(1 + 2u^{-1}w).$$

Since $(2e - 1)^2 = 1$ and $1 + 2u^{-1}w \in \mathcal{A}^{-1}$, a is the sum of two units, as asserted. □

The g-Drazin inverse can be characterized by quasipolar property. We next characterize it in terms of tripotents in a Banach algebra.

Lemma 15.1.8. *Let $\mathcal{A}$ be a Banach algebra, and let $a \in \mathcal{A}, p^3 = p \in comm(a)$. If $ap \in \mathcal{A}^{qnil}$, then $ap^2 \in \mathcal{A}^{qnil}$.*

Proof. Let $\lambda \in \mathbb{C}$. Since $ap \in \mathcal{A}^{qnil}$, we see that $1+\lambda(ap^2) = 1+(\lambda p)(ap) \in \mathcal{A}^{-1}$ as $\lambda p \in comm(ap)$. Therefore $ap^2 \in \mathcal{A}^{qnil}$, as asserted. □

Theorem 15.1.9. *Let $\mathcal{A}$ be a Banach algebra, and let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^d$.
(2) *There exists $p^3 = p \in comm^2(a)$ such that $a+p \in \mathcal{A}^{-1}$ and $ap \in \mathcal{A}^{qnil}$.*
(3) *There exists $p^3 = p \in comm(a)$ such that $a + p \in \mathcal{A}^{-1}$ and $ap \in \mathcal{A}^{qnil}$.*

Proof. (1) $\Rightarrow$ (2) This implication follows by Theorem 6.4.8.

(2) $\Rightarrow$ (3) This is trivial.

(3) $\Rightarrow$ (1) By hypothesis, there exists $p^3 = p \in comm(a)$ such that $a+p \in \mathcal{A}^{-1}$ and $ap \in \mathcal{A}^{qnil}$. Set $b = (1 - p^2)(a+p)^{-1}$. Then $b \in comm(a)$

and $ba = (1-p^2)(a+p)^{-1}a = (1-p^2)(a+p)^{-1}(a+p) = 1-p^2$. Thus,

$$\begin{aligned} b^2a - b &= -bp^2 \\ &= -(1-p^2)(a+p)^{-1}p^2 \\ &= (p^3-p)p(a+p)^{-1} \\ &= 0; \end{aligned}$$

whence, $b^2a = b$. Moreover, we have

$$\begin{aligned} a - a^2b &= a - a^2(1-p^2)(a+p)^{-1} \\ &= a(a+p)^{-1}[(a+p) - a(1-p^2)] \\ &= a(a+p)^{-1}(ap^2+p) \\ &= a(a+p)^{-1}(a+p)p^2 \\ &= ap^2. \end{aligned}$$

In light of Lemma 15.1.8, $a - a^2b \in \mathcal{A}^{qnil}$. Therefore $a \in R$ has g-Drazin inverse. □

Corollary 15.1.10. *Let $\mathcal{A}$ be a Banach algebra, let $a \in \mathcal{A}$ and let $x \in comm(a) \cap \mathcal{A}^{-1}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^d$.
(2) *There exists $p = p^3 \in comm^2(a)$ such that $a+xp \in \mathcal{A}^{-1}$ and $ap \in \mathcal{A}^{qnil}$.*

Proof. (1) $\Rightarrow$ (2) Since $x \in comm(a) \cap \mathcal{A}^{-1}$, we see that $x^{-1}a \in \mathcal{A}^d$ and $[x^{-1}a]^d = xa^d$. In view of Theorem 15.1.9, there exists $p = p^3 \in comm^2(x^{-1}a)$ such that $x^{-1}a + p \in \mathcal{A}^{-1}$ and $(x^{-1}a)p \in \mathcal{A}^{qnil}$. Therefore $p = p^3 \in comm^2(a), a + xp \in \mathcal{A}^{-1}$ and $ap \in \mathcal{A}^{qnil}$.

(2) $\Rightarrow$ (1) By hypothesis, there exists $p = p^3 \in comm^2(a)$ such that $a + xp \in \mathcal{A}^{-1}$ and $ap \in \mathcal{A}^{qnil}$. Hence, $p = p^3 \in comm^2(x^{-1}a), x^{-1}a + p \in \mathcal{A}^{-1}$ and $(x^{-1}a)p \in \mathcal{A}^{qnil}$. In light of Theorem 15.1.9, $x^{-1}a \in \mathcal{A}^d$. Accordingly, $a \in \mathcal{A}^d$, as asserted. □

As an immediate of Corollary 15.1.10, we prove that $a \in \mathcal{A}^d$ if and only if there exists $p = p^3 \in comm^2(a)$ such that $a+p$ or $a-p$ is invertible and $ap \in \mathcal{A}^{qnil}$.

Corollary 15.1.11. *Let $\mathcal{A}$ be a Banach algebra, and let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in A^d$.
(2) *There exist idempotents $e, f \in comm(a)$ such that $a + e - f \in \mathcal{A}^{-1}$ and $ae, af \in \mathcal{A}^{qnil}$.*

Proof. (1) $\Leftrightarrow$ (2) This is obvious by Theorem 6.4.8.

(2) $\Rightarrow$ (1) By hypothesis, there exist idempotents $e, f \in comm(a)$ such that $a + e - f \in \mathcal{A}^{-1}$ and $ae, af \in \mathcal{A}^{qnil}$. Clearly, $e - f \in comm(a)$ and $(e - f)^3 = e - f$. Since $ae, af \in \mathcal{A}^{qnil}$, by Lemma 51.1.1, $a(e - f) = ae - af \in A^{qnil}$. According to Theorem 15.1.9, we complete the proof. □

For future use, we now record the following.

Lemma 15.1.12. *Let $a, b \in \mathcal{A}^d$. If $ab = ba$, then $ab \in \mathcal{A}^d$ and $(ab)^d = a^d b^d$.*

Proof. Since $ab = ba$, we have $ab^d = b^d a$ and $ba^d = a^d b$. One easily checks that $(ab)(a^d b^d) = (a^d b^d)(ab)$ and $a^d b^d = (a^d b^d)(ab)(a^d b^d)$. By virtue of Lemma 15.1.1, we have

$$\begin{aligned} ab - (ab)^2(a^d b^d) &= ab - a^2 a^d b + a^2 a^d b - a^2 b^2 a^d b^d \\ &= (a - a^2 a^d)b + a^2 a^d (b - b^2 b^d) \\ &\in \mathcal{A}^{qnil}. \end{aligned}$$

Therefore $(ab)^d = a^d b^d$, the result follows. □

Lemma 15.1.13. *Let $a \in \mathcal{A}$ and $n \in \mathbb{N}$. Then $a^n \in \mathcal{A}^d$ if and only if $a \in \mathcal{A}^d$.*

Proof. $\Longrightarrow$ In view of Lemma 15.1.12, $-a^n \in \mathcal{A}^d$. Then there exists $e^2 = e \in comm^2(-a^n)$ such that

$$-a^n + e \in \mathcal{A}^{-1}, (-a^n)e \in \mathcal{A}^{qnil}.$$

Hence we have

$$e^2 = e \in comm(-a), a^n - e \in \mathcal{A}^{-1}, (-a)^n e \in \mathcal{A}^{qnil}.$$

Since $a^n - e = (a-e)(a^{n-1} + a^{n-2}e + \cdots + e) \in \mathcal{A}^{-1}$, we see that $(-a) + e = -(a - e) \in \mathcal{A}^{-1}$. Obviously, $(-a)e \in \mathcal{A}^{qnil}$. By virtue of Theorem 6.4.8, $-a \in \mathcal{A}^d$. By using Lemma 15.1.12 again, $a \in \mathcal{A}^d$.

$\Longleftarrow$ This is obvious by Lemma 15.1.12 and induction. □

We now turn to consider the generalized Cline's formula in a Banach algebra. We observe that the condition "$b(ac)a = b(db)a, c(ac)d = c(db)d$" in Theorem 6.4.11 can be dropped in a Banach algebra.

Theorem 15.1.14. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b, c, d \in \mathcal{A}$ satisfying*

$$\begin{aligned} (ac)^2 &= (db)(ac), \\ (db)^2 &= (ac)(db). \end{aligned}$$

Then $ac \in \mathcal{A}^d$ if and only if $bd \in \mathcal{A}^d$. In this case, $(bd)^d = b[(ac)^d]^3dbd$.

Proof. $\Longrightarrow$ Let $aca = a', b = b', c = c'$ and $dbd = d'$. Then we have

$$\begin{aligned}
&(a'c')^2 = (ac)^4 = (db)^3ac = dbdbacac = (d'b')(a'c'),\\
&(d'b')^2 = (db)^4 = (ac)(db)^2(db) = (ac)^2(db)^2 = (a'c')(d'b'),\\
&b'(a'c')a' = b(ac)^3a = bdbdbaca = b'(d'b')a',\\
&c'(a'c')d' = c(acac)dbd = c(db)^3d = c'(d'b')d'.
\end{aligned}$$

Since $ac \in \mathcal{A}^d$, it follows by Lemma 15.1.13 that $a'c' = (ac)^2 \in \mathcal{A}^d$, In light of Theorem 6.4.11, $b'd' = (bd)^2 \in \mathcal{A}^d$. Therefore $bd \in \mathcal{A}^d$ by Lemma 15.1.13. Moreover, we verify that

$$\begin{aligned}
(bd)^d &= [(bd)^2]^dbd\\
&= (b'd')^dbd = b'[(a'c')^d]^2d'bd\\
&= b[(ac)^d]^4(db)^2d\\
&= b[(ac)^d]^4(acdb)d\\
&= b[(ac)^d]^3dbd,
\end{aligned}$$

as required.

$\Longleftarrow$ Since $db \in \mathcal{A}^d$, applying the preceding discussion, we have $ca \in \mathcal{A}^d$. Therefore $ac \in \mathcal{A}^d$, by using the Cline's formula. □

As an easy consequence,

Corollary 15.1.15. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b, c \in \mathcal{A}$. If $(aba)b = (aca)b, (aba)c = (aca)c$, then $ac \in \mathcal{A}^d$ if and only if $ba \in \mathcal{A}^d$. In this case, $(ba)^d = b[(ac)^d]^3aba$.*

Proof. This is obvious by choosing $d = a$ in Theorem 15.1.14. □

The Drazin inverse of a is the unique x in $\mathcal{A}$ satisfying $b \in comm(a), bab = b, a^{k+1}b = a^k$ for some $k \in \mathbb{N}$. We denote b by a^D. The least such integer k is called the index of a, denoted by $ind(a) = k$. Evidently, $a \in \mathcal{A}^D$ if and only if there exists some $b \in comm(a)$ such that $b = bab$ and $a - a^2b \in \mathcal{A}^{nil}$.

Theorem 15.1.16. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b, c, d \in \mathcal{A}$ satisfying*

$$\begin{aligned}
(ac)^2 &= (db)(ac),\\
(db)^2 &= (ac)(db).
\end{aligned}$$

Then $ac \in \mathcal{A}^D$ if and only if $bd \in \mathcal{A}^D$. In this case, $(bd)^D = b[(ac)^D]^3dbd$.

Proof. $\Longrightarrow$ Since $ac \in \mathcal{A}^D$, we see that $ac \in \mathcal{A}^d$. By virtue of Theorem 15.1.14, $bd \in \mathcal{A}^d$ and $(bd)^d = b[(ac)^d]^3dbd$. Let $m = i(ac)$. Then $[ac - (ac)^2(ac)^D]^m = 0$. One easily checks that

$$\begin{aligned} 1 - (bd)(bd)^d &= 1 - bdb[(ac)^D]^3dbd \\ &= 1 - bdbac[(ac)^D]^4dbd \\ &= 1 - b(ac)^2[(ac)^D]^4dbd \\ &= 1 - b[(ac)^D]^2dbd. \end{aligned}$$

Then

$$\begin{aligned} [1 - (bd)(bd)^d](bd)^3 &= [1 - b[(ac)^D]^2dbd](bd)^3 \\ &= (bd)^3 - b[(ac)^D]^2d(bd)^4 \\ &= bacdbd - b[(ac)^D]^2(ac)^3dbd \\ &= b[1 - ac(ac)^D]ac(dbd). \end{aligned}$$

By induction, we have

$$\begin{aligned} [bd - (bd)^2(bd)^d]^{m+2} &= [1 - (bd)(bd)^d](bd)^{m+2} \\ &= b[1 - ac(ac)^D](ac)^m(dbd) \\ &= b[ac - ac^2(ac)^D]^m(dbd) \\ &= 0. \end{aligned}$$

Therefore

$$(bd)^D = (bd)^d = b[(ac)^D]^3dbd,$$

as required.

$\Longleftarrow$ This is proved as in Theorem 15.1.14. □

Corollary 15.1.17. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b, c, d \in \mathcal{A}$ satisfying*

$$\begin{aligned} cac &= bac, \\ dbd &= acd. \end{aligned}$$

Then $ac \in \mathcal{A}^D$ if and only if $bd \in \mathcal{A}^D$. In this case, $(bd)^D = b[(ac)^D]^2d$.

Proof. This is obvious by Theorem 15.1.16. □

The following example illustrates that Theorem 15.1.16 is not a trivial generalization of [419, Theorem 5.2].

Example 15.1.18. Let $\mathcal{A} = M_4(\mathbb{C})$. Choose

$$a = b = c = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, d = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in M_4(\mathbb{C}).$$

Then

$$(ac)^2 = (db)(ac) = 0,$$
$$(db)^2 = (ac)(db) = 0.$$

In this case,

$$acd = \begin{pmatrix} 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix} \neq \begin{pmatrix} 0&0&0&2\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix} = dbd.$$

15.2 Additive Properties

Our main concern in this section is to find various conditions on two elements in a Banach algebra under which their sum has g-Drazin inverse. We start by recalling some elementary properties.

Lemma 15.2.1. *Let*

$$x = \begin{pmatrix} a&0\\ c&b \end{pmatrix} \text{ or } \begin{pmatrix} b&c\\ 0&a \end{pmatrix}$$

in $M_2(\mathcal{A})$. *If* $a, b \in \mathcal{A}^d$, *then* $x \in M_2(\mathcal{A})^d$.

Proof. Let $x = \begin{pmatrix} a&0\\ c&b \end{pmatrix}$ and $y = \begin{pmatrix} a^d&0\\ z&b^d \end{pmatrix}$. Here, $z = \sum\limits_{i=0}^{\infty}(b^d)^{i+2}ca^ia^\pi + \sum\limits_{i=0}^{\infty} b^ib^\pi c(a^d)^{i+2} - b^dca^d$. Then we check that

$$x - x^2y = \begin{pmatrix} 1-aa^d&0\\ *&1-bb^d \end{pmatrix} \in M_2(\mathcal{A})^{qnil}.$$

We compute that

$$\begin{aligned} ca^d + bz &= \sum_{i=0}^{\infty}(b^d)^{i+1}ca^ia^\pi + \sum_{i=0}^{\infty} b^{i+1}b^\pi c(a^d)^{i+2} + b^\pi ca^d\\ &= \sum_{i=0}^{\infty}(b^d)^{i+1}ca^ia^\pi + \sum_{i=0}^{\infty} b^ib^\pi c(a^d)^{i+1}\\ &= \sum_{i=0}^{\infty}(b^d)^{i+2}ca^{i+1}a^\pi + \sum_{i=0}^{\infty} b^ib^\pi c(a^d)^{i+1} + b^dca^\pi\\ &= b^dc + za. \end{aligned}$$

Thus $xy = yx$. In light of Theorem 15.1.3, $x \in M_2(\mathcal{A})^d$.

The other case is similar. □

Lemma 15.2.2. *Let $a, b \in \mathcal{A}$ and $ab = 0$. If $a, b \in \mathcal{A}^d$, then $a + b \in \mathcal{A}^d$.*

Proof. In view of Lemma 15.2.1, $\begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix} \in M_2(\mathcal{A})^d$. Clearly, we have

$$a + b = (1, b) \begin{pmatrix} a \\ 1 \end{pmatrix},$$
$$\begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix} = \begin{pmatrix} a \\ 1 \end{pmatrix} (1, b).$$

By using Cline's formula, $a + b \in \mathcal{A}^d$. □

We are now ready to prove:

Theorem 15.2.3. *Let $a, b \in \mathcal{A}^d$. If $aba^2 = 0, abab = 0, ab^2a = 0$ and $ab^3 = 0$, then $a + b \in \mathcal{A}^d$.*

Proof. Set

$$M = \begin{pmatrix} a^3 + a^2b + aba + ab^2 & a^3b + a^2b^2 + abab + ab^3 \\ a^2 + ab + ba + b^2 & a^2b + ab^2 + bab + b^3 \end{pmatrix}.$$

Then $M = G + F$, where

$$G = \begin{pmatrix} a^2b + aba + ab^2 & a^3b + a^2b^2 \\ 0 & a^2b + ab^2 + bab \end{pmatrix},$$
$$F = \begin{pmatrix} a^3 & 0 \\ a^2 + ab + ba + b^2 & b^3 \end{pmatrix}.$$

We compute that $G^2 = 0$. In view of Lemma 15.2.1, F has g-Drazin inverse. Since $GF = 0$, it follows by Lemma 15.2.2 that M has g-Drazin inverse.

It is easy to verify that $M = \begin{pmatrix} a & ab \\ 1 & b \end{pmatrix}^3$, and so $\begin{pmatrix} a & ab \\ 1 & b \end{pmatrix}$ has g-Drazin inverse. We observe that

$$\begin{pmatrix} a & ab \\ 1 & b \end{pmatrix} = \begin{pmatrix} a \\ 1 \end{pmatrix} (1, b),$$
$$a + b = (1, b) \begin{pmatrix} a \\ 1 \end{pmatrix}.$$

By using Cline's formula, $a + b \in \mathcal{A}^d$, as asserted. □

Corollary 15.2.4. *Let $a, b \in \mathcal{A}^d$. If $aba = 0, ab^2 = 0$, then $a + b \in \mathcal{A}^d$.*

Proof. This is obvious by Theorem 15.2.3. □

Using the previous results, we now extend [545, Theorem 2.1] and derive

Theorem 15.2.5. *Let $a, b \in \mathcal{A}^d$. If $a^2ba = 0, abab = 0, ab^2a = 0$ and $ab^3 = 0$, then $a + b \in \mathcal{A}^d$.*

Proof. Let $\alpha = b^2 + ab$ and $\beta = ba + a^2$. Since $ab^3 = 0$, it follows by Lemma 15.2.2 that α has g-Drazin inverse. Since $a^2ba = 0$, similarly, we have β has g-Drazin inverse. Set

$$M = \begin{pmatrix} b^2 + ab & 1 \\ (a+b)a(a+b)b & ba + a^2 \end{pmatrix}.$$

Then

$$\begin{aligned} M &= \begin{pmatrix} \alpha & 1 \\ 0 & \beta \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \beta\alpha & 0 \end{pmatrix} \\ &:= A + B. \end{aligned}$$

In view of Lemma 15.2.1, A has g-Drazin inverse. Obviously, $B^2 = 0$, and so it has g-Drazin inverse. We easily check that $ABAB = 0, ABA^2 = 0$ and $B^2 = 0$. In view of Theorem 15.2.3, M has g-Drazin inverse. Let $N = \left(\begin{pmatrix} 1 \\ a \end{pmatrix} (b, 1) \right)^2$. Then we have

$$\begin{aligned} M &= \begin{pmatrix} 1 & 0 \\ 0 & a+b \end{pmatrix} \begin{pmatrix} (a+b)b & 1 \\ a(a+b)b & a \end{pmatrix}, \\ N &= \begin{pmatrix} (a+b)b & 1 \\ a(a+b)b & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & a+b \end{pmatrix}, \end{aligned}.$$

By using Cline's formula, N has g-Drazin inverse. In view of Lemma 15.1.13, $\begin{pmatrix} 1 \\ a \end{pmatrix} (b, 1)$ has g-Drazin inverse. By using Cline's formula again,

$$a + b = (b, 1) \begin{pmatrix} 1 \\ a \end{pmatrix} \in \mathcal{A}^d,$$

as asserted.

Corollary 15.2.6. *Let $a, b \in \mathcal{A}^d$. If $a^3b = 0, bab = 0$ and $ba^2b = 0$, then $a + b \in \mathcal{A}^d$.*

Proof. Let $M = \begin{pmatrix} a^2 + ab & a^2b + ab^2 \\ a + b & b^2 + ab \end{pmatrix}$. Then $M = A + B$, where

$$A = \begin{pmatrix} a^2 + ab & 0 \\ a + b & b^2 + ab \end{pmatrix}, B = \begin{pmatrix} 0 & a^2b + ab^2 \\ 0 & 0 \end{pmatrix}.$$

Let $\alpha = a^2 + ab, \beta = b^2 + ab$. Clearly, we have $(ab)^2 = 0, a^2(ab) = a^3b = 0$ and $b^2(ab) = 0$. In light of Lemma 15.2.2, we have $\alpha, \beta \in \mathcal{A}^d$. By virtue

of Lemma 15.2.1, A has g-Drazin inverse. Clearly, $B^2 = 0$, and so $B^d = 0$. Since $ba^2b = 0$, we see that $A^2B = 0$ and $(AB)^2 = 0$. According to Theorem 15.2.5, M has g-Drazin inverse. Since $M = \left(\begin{pmatrix} a \\ 1 \end{pmatrix}(1, b)\right)^2$, it follows from Lemma 15.1.3 that $N := \begin{pmatrix} a \\ 1 \end{pmatrix}(1, b)$ has g-Drazin inverse. By using Cline's formula,

$$a + b = (1, b)\begin{pmatrix} a \\ 1 \end{pmatrix}$$

has g-Drazin inverse. □

For complex matrices, Drain and g-Drazin inverses coincide with each other. Thus every square complex matrix has the g-Drazin inverse.

Example 15.2.7. Let

$$a = \begin{pmatrix} 0&0&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 0&0&0&1 \end{pmatrix}, b = \begin{pmatrix} 0&-1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{pmatrix} \in M_4(\mathbb{C}).$$

Then $a^2ba = 0, abab = 0, ab^2a = 0$ and $ab^3 = 0$, but $aba^2 = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix} \neq 0.$

Theorem 15.2.8. *Let $a, b \in \mathcal{A}^d$. If $aba = 0, bab = 0, a^2b^2 = 0$ and $ab^3 = 0$, then $a + b \in \mathcal{A}^d$.*

Proof. Set

$$M = \begin{pmatrix} a^3 + a^2b + ab^2 & a^3b \\ a^2 + ab + ba + b^2 & a^2b + ab^2 + b^3 \end{pmatrix}.$$

Then

$$\begin{aligned} M &= \begin{pmatrix} a^2b + ab^2 & a^3b \\ 0 & a^2b + ab^2 \end{pmatrix} + \begin{pmatrix} a^3 & 0 \\ a^2 + ab + ba + b^2 & b^3 \end{pmatrix} \\ &:= G + F. \end{aligned}$$

We see that $G^3 = 0$, $FG^2 = 0$ and $FGF = 0$. In view of Lemma 15.2.1, F has g-Drazin inverse. According to Corollary 15.2.4, M has g-Drazin inverse.

Clearly, $M = \left(\begin{pmatrix} a \\ 1 \end{pmatrix}(1,b)\right)^3$. It follows by Lemma 15.1.13 that $\begin{pmatrix} a \\ 1 \end{pmatrix}(1,b)$ has g-Drazin inverse. By using Cline's formula, $a+b \in \mathcal{A}^d$, as asserted. □

Corollary 15.2.9. *Let $a, b, ab \in \mathcal{A}^d$ have g-Drazin inverses. If $a^2b = 0$ and $ab^2 = 0$, then $a + b \in \mathcal{A}^d$.*

Proof. Since $ab \in \mathcal{A}^d$, we see that $ba \in \mathcal{A}^d$ by Cline's formula. As $a^2(ab) = 0$, it follows by Lemma 15.2.1 that $p := a^2 + ab \in \mathcal{A}^d$. Likewise, $q := ba + b^2 \in \mathcal{A}^d$. One easily checks that

$$pqp = 0, qpq = 0, p^2q^2 = 0 \text{ and } pq^3 = 0.$$

In light of Theorem 15.2.8, $(a+b)^2 = p + q \in \mathcal{A}^d$. According to Lemma 15.1.13, $a + b \in \mathcal{A}^d$, as desired. □

Let $a, b \in \mathcal{A}^d$. If $aba = 0, bab = 0, a^2b^2 = 0$ and $a^3b = 0$, then $a+b \in \mathcal{A}^d$. This is a symmetrical result of Theorem 15.2.8 and it can be proved by a similar route.

Proposition 15.2.10. *Let $a, b \in \mathcal{A}^d$. If $ab^2 = 0, a^2ba = 0$ and $(ba)^2 = 0$, then $a + b \in \mathcal{A}^d$.*

Proof. Set

$$M = \begin{pmatrix} a^3 + a^2b + aba & a^3b + abab \\ a^2 + ab + ba + b^2 & a^2b + bab + b^3 \end{pmatrix}.$$

Then

$$\begin{aligned} M &= \begin{pmatrix} a^2b + aba & a^3b + abab \\ 0 & a^2b + bab \end{pmatrix} + \begin{pmatrix} a^3 & 0 \\ a^2 + ab + ba + b^2 & b^3 \end{pmatrix} \\ &:= G + F. \end{aligned}$$

We see that $G^4 = 0, FGF = 0$ and $FG^2 = 0$. In light of Corollary 15.2.4, M has g-Drazin inverse. Obviously, $M = \left(\begin{pmatrix} a \\ 1 \end{pmatrix}(1,b)\right)^3$. In view of Lemma 15.1.13, $\begin{pmatrix} a \\ 1 \end{pmatrix}(1,b)$ has g-Drazin inverse. By virtue of Cline's formula, $a + b \in \mathcal{A}^d$, as asserted. □

Proposition 15.2.11. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}^d$. If $ab^2 = 0, ba^2 = 0$ and $(ba)^3 = 0$, then $a + b \in \mathcal{A}^d$.*

Proof. Set

$$M = \begin{pmatrix} a^3 + a^2b + aba & a^3b + abab \\ a^2 + ab + ba + b^2 & a^2b + bab + b^3 \end{pmatrix}.$$

Then

$$\begin{aligned} M &= \begin{pmatrix} a^2b + aba & a^3b + abab \\ 0 & a^2b + bab \end{pmatrix} + \begin{pmatrix} a^3 & 0 \\ a^2 + ab + ba + b^2 & b^3 \end{pmatrix} \\ &:= G + F. \end{aligned}$$

It is easy to verify that

$$G^3 = 0, GFG = 0 \text{ and } GF^2 = 0.$$

Then $G + F$ has g-Drazin inverse. Obviously, $M = N^3$, where $N = \begin{pmatrix} a \\ 1 \end{pmatrix}(1, b)$. Therefore $a + b \in \mathcal{A}^{qni}$. □

We come now to establish additive results for the g-Drazin inverse under certain generalized communicative conditions. For future use, we now record the following.

Theorem 15.2.12. *Let $a, b \in \mathcal{A}^d$ and $c \in \mathcal{A}$. If $ca = bc$, then $ca^d = b^dc$.*

Proof. We see that

$$b^dbca^\pi = (b^d)^n b^n ca^\pi = (b^d)^n ca^n a^\pi;$$

hence,

$$\| b^dbca^\pi \|^{\frac{1}{n}} \leq \| (b^d)^n c \|^{\frac{1}{n}} \| a^n a^\pi \|^{\frac{1}{n}}.$$

Since $\lim_{n\to\infty} \| a^n a^\pi \|^{\frac{1}{n}} = 0$, we have $\lim_{n\to\infty} \| b^dbca^\pi \|^{\frac{1}{n}} = 0$. Hence $b^dbca^\pi = 0$, and so $b^dbc = b^dbcaa^d$. This implies that $b^dc = b^dcaa^d$.

Obviously, $b^\pi bca^d = b^\pi bca^{n-1}(a^d)^{n-1} = b^\pi b^n c(a^d)^{n-1}$, and so

$$\| b^\pi bca^d \|^{\frac{1}{n}} \leq \| b^\pi b^n c \|^{\frac{1}{n}} \| a(a^d)^{n-1} \|^{\frac{1}{n}}.$$

Since $\lim_{n\to\infty} \| b^\pi b^n \|^{\frac{1}{n}} = 0$, we have $\lim_{n\to\infty} \| b^\pi bca^d \|^{\frac{1}{n}} = 0$. Then $b^\pi bca^d = 0$. That is, $bca^d = b^db^2ca^d$. Thus $caa^d = b^dca^2a^d$, and so $ca^d = b^dcaa^d$. Therefore $ca^d = b^dc$, as asserted. □

Corollary 15.2.13. *Let $\mathcal{A}$ be a Banach algebra, $a, b \in \mathcal{A}^d$, $a^2b = aba$ and $b^2a = bab$. Then*

(1) $a^dab = aba^d$.
(2) $b^dba = bab^d$.

Proof. (1) Since $a(ab) = (ab)a$, it follows by Theorem 15.2.12 that $a^d(ab) = (ab)a^d$, as required.

(2) This is proved as in (1). □

Corollary 15.2.14. *Let $\mathcal{A}$ be a Banach algebra, $a, b \in \mathcal{A}^d$, $a^2b = aba$ and $b^2a = bab$. Then*

(1) $\{ab, a^db, ab^d, a^db^d\} \subseteq comm(a) \cap comm(a^d)$;
(2) $\{ba, ba^d, b^da, b^da^d\} \subseteq comm(b) \cap comm(b^d)$.

Proof. (1) By virtue of Theorem 15.2.12, we will suffice to prove that $\{ab, a^db, ab^d, a^db^d\} \subseteq comm(a)$. Since $a^2b = aba$, we have $ab \in comm(a)$. We check that $a(a^db) = (a^d)^2a^2b = (a^d)^2aba = (a^db)a$, i.e., $a^db \in comm(a)$. Since $b^dba = bab^d$, we see that $a(ab^d) = a^2b^d = (a^2b)(b^d)^2 = (aba)(b^d)^2 = a(b^d)^2(ba) = (ab^d)a$, i.e., $ab^d \in comm(a)$. Therefore $a(a^db^d) = (a^d)^2a(ab^d) = (a^d)^2(ab^d)a = (a^db^d)a$, and then $a^db^d \in comm(a)$.

According to Theorem 15.2.12, we prove that $\{ab, a^db, ab^d, a^db^d\} \subseteq comm(a^d)$.

(2) This is symmetric. □

Lemma 15.2.15. *Let $\mathcal{A}$ be a Banach algebra, $a, b \in \mathcal{A}^d$, $aba = a^2b$ and $bab = b^2a$. If $a \in \mathcal{A}^{qnil}$ or $b \in \mathcal{A}^{qnil}$, then $ab \in \mathcal{A}^{qnil}$.*

Proof. By induction, $a^ib^i = (ab)^i$ for all $i \in \mathbb{N}$. Hence,

$$\| (ab)^n \|^{\frac{1}{n}} \leq \| a^n \|^{\frac{1}{n}} \| b^n \|^{\frac{1}{n}} .$$

Since $a \in \mathcal{A}^{qnil}$ or $b \in \mathcal{A}^{qnil}$, we have $\lim\limits_{n\to\infty} \| a^n \|^{\frac{1}{n}} = 0$ or $\lim\limits_{n\to\infty} \| b^n \|^{\frac{1}{n}} = 0$. Then $\lim\limits_{n\to\infty} \| (ab)^n \|^{\frac{1}{n}} = 0$, i.e., $ab \in \mathcal{A}^{qnil}$. □

Theorem 15.2.16. *Let $\mathcal{A}$ be a Banach algebra, $a, b \in \mathcal{A}^d$, $aba = a^2b$ and $bab = b^2a$. Then $ab \in \mathcal{A}^d$.*

Proof. Let $p = aa^d$. Then we have

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}_p , b = \begin{pmatrix} b_1 & b_{12} \\ b_{21} & b_2 \end{pmatrix}_p .$$

In view of Corollary 15.2.13, $b_{12} = aa^db(1 - aa^d) = aba^d(1 - aa^d) = 0$. Then

$$ab = \begin{pmatrix} a_1b_1 & 0 \\ a_2b_{21} & a_2b_2 \end{pmatrix}_p .$$

Step 1. We compute that $a_1b_1 = aa^d(ab)aa^d = (a^2b)a^d = (aba)a^d = aa^d(ba)aa^d = b_1a_1$. Clearly, $a_1 \in \mathcal{A}^d$. Also we have $b_1 = aa^dbaa^d = aa^db$. Let $x = aa^db^d$. Then we have

$$b_1x = aa^dbaa^db^d = aa^dbb^d = aa^db^daa^db = xb_1;$$

$$b_1 - b_1^2x \in \mathcal{A}^{qnil}$$

That is, $b_1 \in \mathcal{A}^d$. Therefore $a_1b_1 \in \mathcal{A}^d$ and $(a_1b_1)^d = a_1^db_1^d$.

Step 2. Obviously, $a_2 = a - a^2a^d \in \mathcal{A}^{qnil}$ and $b_2 \in \mathcal{A}^d$. We easily check that

$$a_2b_2a_2 = a_2^2b_2, b_2^2a_2 = b_2a_2b_2.$$

In light of Lemma 15.2.15, $a_2b_2 \in \mathcal{A}^{qnil}$.

Therefore we conclude that $ab \in \mathcal{A}^d$. □

Lemma 15.2.17. *Let $a, b \in \mathcal{A}$ and $n \in \mathbb{N}$. Suppose $aba = a^2b$ and $bab = b^2a$. Then*

$$\| (a+b)^n \| \leq \sum_{i=0}^{n-1} \begin{pmatrix} n-1 \\ i \end{pmatrix} (\| a^{n-i}b^i \| + \| b^{n-i}a^i \|).$$

Proof. Since $aba^m = a^{m+1}b$ and $bab^m = b^{m+1}a$, by induction, we easily obtain this result. □

Lemma 15.2.18. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}^{qnil}$. Suppose $aba = a^2b$ and $bab = b^2a$. Then $a + b \in \mathcal{A}^{qnil}$.*

Proof. If $a = 0$, then $a + b \in \mathcal{A}^{qnil}$. We now assume that $a \neq 0$. Set $\rho = \| a \| + \| b \|, \alpha = \rho^{-1}a$ and $\beta = \rho^{-1}b$. Then $\alpha, \beta \in \mathcal{A}^{qnil}$ and $\| \alpha \|, \| \beta \| \leq 1$. Moreover, we have

$$\alpha\beta\alpha = \alpha^2\beta, \beta\alpha\beta = \beta^2\alpha.$$

For any $n \geq 3$, we can find some $k \in \mathbb{N}$ such that $2k + 1 \leq n \leq 2k + 2$. Set $c = max\big(1, \| \alpha \| + \| \beta \|\big)$. By using Lemma 15.2.17, we have

$$\begin{aligned} \| (\alpha+\beta)^n \| &\leq c \| (\alpha+\beta)^{2k+1} \| \\ &\leq c \sum_{i=0}^{2k} \begin{pmatrix} 2k \\ i \end{pmatrix} (\| \alpha^{2k+1-i}\beta^i \| + \| \beta^{2k+1-i}\alpha^i \|). \end{aligned}$$

Then

$$\begin{aligned}
&\sum_{i=0}^{2k}\binom{2k}{i}\parallel \alpha^{2k+1-i}\beta^i \parallel \\
&=\sum_{i=0}^{k}\binom{2k}{i}\parallel \alpha^{2k+1-i}\beta^i \parallel + \sum_{i=k+1}^{2k}\binom{2k}{i}\parallel \alpha^{2k+1-i}\beta^i \parallel \\
&\le \sum_{i=0}^{k}\binom{2k}{i}\parallel \alpha^{2k+1-i} \parallel + \sum_{i=k+1}^{2k}\binom{2k}{i}\parallel \beta^i \parallel \\
&\le \sum_{i=0}^{k}\binom{2k}{i}\parallel \alpha^{k} \parallel + \sum_{i=k+1}^{2k}\binom{2k}{i}\parallel \beta^k \parallel \\
&\le \sum_{i=0}^{2k}\binom{2k}{i}\big(\parallel \alpha^{k} \parallel + \parallel \beta^k \parallel\big).
\end{aligned}$$

Likewise, we have

$$\sum_{i=0}^{2k}\binom{2k}{i}\parallel \beta^{2k+1-i}\alpha^i \parallel \ \le \sum_{i=0}^{2k}\binom{2k}{i}\big(\parallel \alpha^{k} \parallel + \parallel \beta^k \parallel\big).$$

Hence

$$\begin{aligned}
&\parallel (\alpha+\beta)^n \parallel \\
&\le 2c\sum_{i=0}^{2k}\binom{2k}{i}\big(\parallel \alpha^{k} \parallel + \parallel \beta^k \parallel\big) \\
&= 2^{2k+1}c\big(\parallel \alpha^{k} \parallel + \parallel \beta^k \parallel\big).
\end{aligned}$$

Thus we get

$$\begin{aligned}
&\parallel (\alpha+\beta)^n \parallel^{\frac{1}{n}} \\
&\le 2c^{\frac{1}{n}}\big(\parallel \alpha^{k} \parallel + \parallel \beta^k \parallel\big)^{\frac{1}{n}} \\
&\le 2c^{\frac{1}{n}}\big(\parallel \alpha^{k} \parallel^{\frac{1}{n}} + \parallel \beta^k \parallel^{\frac{1}{n}}\big).
\end{aligned}$$

Accordingly, $\lim\limits_{n\to\infty}\parallel (a+b)^n \parallel^{\frac{1}{n}}=0$, as required. □

Lemma 15.2.19. *Let $e^2=e\in\mathcal{A}, a\in\mathcal{A}^d$ and $ea(1-e)=0$. If $ea\in\mathcal{A}^d$, then $a(1-e)\in\mathcal{A}^d$ and $[a(1-e)]^d=a^d(1-e)$.*

Proof. Since $ea(1-e)=0$, we have $ea=(ea)e$. In light of Theorem 15.2.12, $ea^d=(ea)^de$, and then $ea^d(1-e)=0$. We check that

$$\begin{aligned}
a(1-e)a^d(1-e) &= aa^d(1-e)=a^da(1-e)) \\
&= a^d(1-e)a(1-e), \\
(a^d(1-e))^2a(1-e) &= (a^d)^2(1-e)a(1-e)=a^d(1-e), \\
a(1-e)-(a(1-e))^2a^d(1-e) &= a(1-e)-a^2a^d(1-e) \\
&= (a-a^2a^d)(1-e)\in\mathcal{A}^{qnil}.
\end{aligned}$$

Therefore $(a(1-e))^d=a^d(1-e)$, as asserted. □

Lemma 15.2.20. *Let $\mathcal{A}$ be a Banach algebra, $a \in \mathcal{A}^d$ and $b \in \mathcal{A}^{qnil}$. If $aba = a^2b, bab = b^2a$, then $a + b \in \mathcal{A}^d$ if and only if $aa^d(a + b)aa^d \in \mathcal{A}^d$.*

Proof. Let $p = aa^d$. Then we have

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}_p, b = \begin{pmatrix} b_1 & b_{12} \\ b_{21} & b_2 \end{pmatrix}_p.$$

In view of Corollary 15.2.13, $a^dab = aba^d$, and so $b_{12} = aa^db(1 - aa^d) = aba^d(1 - aa^d) = 0$. Hence

$$a + b = \begin{pmatrix} a_1 + b_1 & 0 \\ b_{21} & a_2 + b_2 \end{pmatrix}_p.$$

Clearly, $a_2 = (1 - aa^d)a \in \mathcal{A}^{qnil}$.

We easily check that

$$(aa^d)^2b = aa^db = aba^d = aa^dbaa^d;$$
$$b^2(aa^d) = (b^2a)a^d = (bab)a^d = b(aa^d)b.$$

In light of Lemma 15.2.15, $aa^db \in \mathcal{A}^{qnil}$.

Obviously, $aa^db(1 - aa^d) = aba^d(1 - aa^d) = 0$ and $aa^db \in \mathcal{A}^d$. Then $b_2 = (1 - aa^d)b(1 - aa^d) = b(1 - aa^d) \in \mathcal{A}^d$ by Lemma 15.2.19. Moreover, we have $b_2^d = b^d(1 - aa^d) = 0$. Hence, $b_2 \in \mathcal{A}^{qnil}$.

It is easy to verify that $a_2b_2a_2 = a_2^2b_2$ and $b_2a_2b_2 = b_2^2a_2$. By using Lemma 15.2.18, we have $a_2 + b_2 \in \mathcal{A}^{qnil}$. Therefore $a + b \in \mathcal{A}^d$ if and only if $aa^d(a + b)aa^d = a_1 + b_1 \in \mathcal{A}^d$. □

We are now ready to prove:

Theorem 15.2.21. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}^d$. If*

$$aba = a^2b, bab = b^2a,$$

then the following are equivalent:

(1) $a + b \in \mathcal{A}^d$.
(2) $aa^d(a + b) \in \mathcal{A}^d$.
(3) $(a + b)bb^d \in \mathcal{A}^d$.
(4) $aa^d(a + b)bb^d \in \mathcal{A}^d$.

Proof. (1) $\Leftrightarrow$ (2) Let $p = aa^d$. Then we have

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}_p, b = \begin{pmatrix} b_1 & b_{12} \\ b_{21} & b_2 \end{pmatrix}_p.$$

As in the proof of Lemma 15.2.20, we have $b_{12} = 0$. Then

$$a + b = \begin{pmatrix} a_1 + b_1 & 0 \\ b_{21} & a_2 + b_2 \end{pmatrix}_p.$$

We claim that $a_2 + b_2 \in \mathcal{A}^d$. Obviously, $a_2 = a - a^2a^d \in \mathcal{A}^{qnil}$. Since $aa^db(1 - aa^d) = 0$ and $aa^db \in \mathcal{A}^d$, it follows by Lemma 15.2.19 that $b(1 - aa^d) \in \mathcal{A}^d$. Thus $b_2 = (1 - aa^d)b(1 - aa^d) = (1 - aa^d)b \in \mathcal{A}^d$ by Cline's formula. Moreover, we easily check that

$$a_2b_2a_2 = a_2^2b_2, b_2a_2b_2 = b_2^2a_2.$$

Let $q = b_2b_2^d$. Then we have

$$a_2 = \begin{pmatrix} c_1 & 0 \\ c_{21} & c_2 \end{pmatrix}_q, b_2 = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}_q.$$

As in the preceding discussion, we have $c_{12} = 0$. Then

$$a_2 + b_2 = \begin{pmatrix} c_1 + d_1 & 0 \\ c_{21} & c_2 + d_2 \end{pmatrix}_p.$$

Moreover, we have

$$c_1^2d_1 = c_1d_1c_1, c_2^2d_2 = c_2d_2c_2.$$

Since $a_2 \in \mathcal{A}^{qnil}$, we have $c_1, c_2 \in \mathcal{A}^{qnil}$.

Claim 1. $c_1 + d_1 \in \mathcal{A}^d$. Clearly, $d_1 = b_2^2b_2^d = (1 - aa^d)b(1 - aa^d)b(1 - aa^d)b^d = b^2(1 - aa^d)b^d = b^2b^d(1 - aa^d)$. We compute that

$$\begin{aligned}(aa^d)b^2b^d(1 - aa^d) &= aa^d(ba^d)bb^d(1 - aa^d) = aa^dbb^d(ba^d)(1 - aa^d) = 0,\\ aa^db^2b^daa^d &= aa^dbab^dba^d = a^d(ab)(a^da)bb^d = aa^db^2b^d,\\ b^2b^daa^db^2b^d &= b^2b^da^d(ab^d)b^2 = b^2b^d(ab^d)a^db^2 = b^2b^da(b^da^d)b^2\\ &= b^2b^d(ab^2)(b^da^d) = b^2(b^da)b^2b^da^d = (b^2b^d)^2aa^d.\end{aligned}$$

Then $aa^db^2b^d \in \mathcal{A}^d$, and so $d_1 \in \mathcal{A}^d$.

Claim 2. $c_2 + d_2 \in \mathcal{A}^d$. Since $d_2 = (1 - b_2b_2^d)b_2 \in \mathcal{A}^{qnil}$. It follows by Lemma 15.2.18 that $c_2 + d_2 \in \mathcal{A}^{qnil}$.

Therefore $a_2 + b_2 \in \mathcal{A}^d$. Accordingly, $a + b \in \mathcal{A}^d$ if and only if $aa^d(a + b)bb^d = a_1 + b_1 \in \mathcal{A}^d$. By using Clean's formula, $a + b \in \mathcal{A}^d$ if and only if $aa^d(a + b) \in \mathcal{A}^d$.

(1) $\Leftrightarrow$ (3) This is similar to (1) $\Leftrightarrow$ (2).

(2) $\Leftrightarrow$ (4) Clearly, $aa^d(a + b) = a^2a^d + aa^db$. By using (1) $\Leftrightarrow$ (3), $aa^d(a + b) \in \mathcal{A}^d$ if and only if $(a^2a^d + aa^db)(aa^db)(aa^db)^d \in \mathcal{A}^d$. One easily checks that

$$\begin{aligned}&(a^2a^d + aa^db)(aa^db)(aa^db)^d\\ &= a^2a^db(aa^db)^d + aa^dbaa^db(aa^db)^d\\ &= a^2a^db(aa^db^d) + aa^dbaa^db(aa^db^d)\\ &= a^2a^dbb^d + aa^dbbb^d\\ &= aa^d(a + b)bb^d,\end{aligned}$$

as required. □

As an easy consequence, we now derive

Corollary 15.2.22. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}^d$. If*

$$aba = a^2b, bab = b^2a,$$

then the following are equivalent:

(1) $a + b \in \mathcal{A}^d$.
(2) $1 + a^d b \in \mathcal{A}^d$.

Proof. (1) $\Rightarrow$ (2) Write $1 + a^d b = a_1 + b_1$, where $a_1 = a^\pi, b_1 = a^d(a + b)$. It is easy to verify that

$$(a^d)^2(a + b) = a^d(a + b)a^d, (a + b)^2a^d = (a + b)a^d(a + b).$$

It follows from Theorem 15.2.16 that $a^d(a+b) \in \mathcal{A}^d$. Clearly, $a_1b_1 = 0$. By virtue of by Lemma 15.2.2 that $a_1 + b_1 \in \mathcal{A}^d$, as desired.

(2) $\Rightarrow$ (1) We check that $aa^d(a + b) = a^2a^d[1 + a^d b]$. In view of Corollary 15.2.14, it is easy to verify that

$$\begin{aligned} a^2a^d[1 + a^d b] &= a^2a^d + aa^d b = a^2a^d + (a^d)^2(a^2b)aa^d \\ &= a^2a^d + (a^d)^2(aba)aa^d = [1 + a^d b]a^2a^d. \end{aligned}$$

Therefore $aa^d(a + b)$ has g-Drazin inverse. This completest the proof by Theorem 15.2.21. □

Corollary 15.2.23. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}^d$. If $ab = ba$, then the following are equivalent:*

(1) $a + b \in \mathcal{A}^d$.
(2) $1 + a^d b \in \mathcal{A}^d$.

Proof. Since $ab = ba$, we have $aba = a^2b, bab = b^2a$. According to Corollary 15.2.22, we complete the proof. □

15.3 Anti-triangular Matrices

The aim of this section is to investigate the g-Drazin inverse of the operator matrix $\begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}$ over a Banach algebra $\mathcal{A}$. We begin with

Theorem 15.3.1. *Let $a, b, b^\pi a \in \mathcal{A}^d$ and $b^\pi ab^d = 0$. Then the following are equivalent:*

(1) $\begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}$ *has g-Drazin inverse.*

(2) $\begin{pmatrix} b^\pi a & 1 \\ b^\pi b & 0 \end{pmatrix}$ *has g-Drazin inverse.*

(3) $\begin{pmatrix} ab^\pi & 1 \\ bb^\pi & 0 \end{pmatrix}$ *has g-Drazin inverse.*

Proof. (1) $\Rightarrow$ (2) Let $M = \begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}$ and $p = \begin{pmatrix} b^\pi & 0 \\ 0 & b^\pi \end{pmatrix}$. Since $b^\pi ab^d = 0$, we have

$$pM(1-p) = 0, M(1-p) = \begin{pmatrix} bb^d abb^d & bb^d \\ b^2 b^d & 0 \end{pmatrix}.$$

We directly verify that $[M(1-p)]^\# = \begin{pmatrix} 0 & -bb^d \\ -b^2 b^d & bb^d abb^d \end{pmatrix}$. Since M has g-Drazin inverse, it follows by Lemma 15.2.19 that $pM = \begin{pmatrix} b^\pi a & b^\pi \\ b^\pi b & 0 \end{pmatrix}$ has g-Drazin inverse.

Let $N = \begin{pmatrix} b^\pi a & 1 \\ b^\pi b & 0 \end{pmatrix}$. It is easy to verify that

$$N = \begin{pmatrix} 1 & 0 \\ 0 & b^\pi \end{pmatrix} \begin{pmatrix} b^\pi a & 1 \\ b^\pi b & 0 \end{pmatrix},$$
$$\begin{pmatrix} b^\pi a & b^\pi \\ b^\pi b & 0 \end{pmatrix} = \begin{pmatrix} b^\pi a & 1 \\ b^\pi b & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & b^\pi \end{pmatrix}.$$

In light of Cline's formula, N has g-Drazin inverse.

(2) $\Rightarrow$ (1) Let $e = \begin{pmatrix} bb^d & 0 \\ 0 & 1 \end{pmatrix}$. Then $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}_e$. Here,

$$\alpha = \begin{pmatrix} bb^d abb^d & bb^d \\ b^2 b^d & 0 \end{pmatrix}, \beta = \begin{pmatrix} bb^d ab^\pi & 0 \\ bb^\pi & 0 \end{pmatrix},$$
$$\gamma = \begin{pmatrix} 0 & b^\pi \\ 0 & 0 \end{pmatrix}, \delta = \begin{pmatrix} b^\pi a & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$\alpha^\# = \begin{pmatrix} 0 & -bb^d \\ -b^2 b^d & bb^d abb^d \end{pmatrix}, \alpha^\pi = \begin{pmatrix} b^\pi & 0 \\ 0 & b^\pi \end{pmatrix};$$
$$\beta + \gamma + \delta = \begin{pmatrix} bb^d ab^\pi + b^\pi a & b^\pi \\ bb^\pi & 0 \end{pmatrix}.$$

Hence we have

$$(\beta + \gamma + \delta)\alpha = \begin{pmatrix} bb^d ab^\pi + b^\pi a & b^\pi \\ bb^\pi & 0 \end{pmatrix} \begin{pmatrix} bb^d abb^d & bb^d \\ b^2 b^d & 0 \end{pmatrix} = 0.$$

One easily checks that

$$\begin{aligned}
\beta+\gamma+\delta &= \begin{pmatrix} bb^da+b^\pi a\ 1 \\ b\qquad 0 \end{pmatrix}\begin{pmatrix} b^\pi\ 0 \\ 0\ b^\pi \end{pmatrix},\\
\begin{pmatrix} b^\pi a\ b^\pi \\ b^\pi b\ 0 \end{pmatrix} &= \begin{pmatrix} b^\pi\ 0 \\ 0\ b^\pi \end{pmatrix}\begin{pmatrix} bb^da+b^\pi a\ 1 \\ b\qquad 0 \end{pmatrix};\\
\begin{pmatrix} b^\pi a\ b^\pi \\ b^\pi b\ 0 \end{pmatrix} &= \begin{pmatrix} b^\pi a\ 1 \\ b^\pi b\ 0 \end{pmatrix}\begin{pmatrix} 1\ 0 \\ 0\ b^\pi \end{pmatrix},\\
\begin{pmatrix} b^\pi a\ 1 \\ b^\pi b\ 0 \end{pmatrix} &= \begin{pmatrix} 1\ 0 \\ 0\ b^\pi \end{pmatrix}\begin{pmatrix} b^\pi a\ 1 \\ b^\pi b\ 0 \end{pmatrix}.
\end{aligned}$$

By hypothesis and Cline's formula, we see that $\beta+\gamma+\delta$ has g-Drazin inverse. Set $N=\begin{pmatrix} b^\pi a\ 1 \\ b^\pi b\ 0 \end{pmatrix}$. Then we have

$$\begin{aligned}
\begin{pmatrix} b^\pi a\ b^\pi \\ b^\pi b\ 0 \end{pmatrix}^d &= \begin{pmatrix} b^\pi a\ 1 \\ b^\pi b\ 0 \end{pmatrix}(N^d)^2\begin{pmatrix} 1\ 0 \\ 0\ b^\pi \end{pmatrix}\\
&= N^d\begin{pmatrix} 1\ 0 \\ 0\ b^\pi \end{pmatrix}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&(\beta+\gamma+\delta)^d\\
&= \begin{pmatrix} bb^da+b^\pi a\ 1 \\ b\qquad 0 \end{pmatrix}\left[\begin{pmatrix} b^\pi a\ b^\pi \\ b^\pi b\ 0 \end{pmatrix}^d\right]^2\begin{pmatrix} b^\pi\ 0 \\ 0\ b^\pi \end{pmatrix}\\
&= \begin{pmatrix} bb^da+b^\pi a\ 1 \\ b\qquad 0 \end{pmatrix}N^d\begin{pmatrix} 1\ 0 \\ 0\ b^\pi \end{pmatrix}N^d\begin{pmatrix} b^\pi\ 0 \\ 0\ b^\pi \end{pmatrix}.
\end{aligned}$$

By virtue of Lemma 15.2.2, M has g-Drazin inverse.

(2) $\Leftrightarrow$ (3) One directly checks that

$$\begin{aligned}
\begin{pmatrix} b^\pi a\ 1 \\ b^\pi b\ 0 \end{pmatrix} &= \begin{pmatrix} 1\ 0 \\ 0\ b^\pi \end{pmatrix}\begin{pmatrix} b^\pi a\ 1 \\ b^\pi b\ 0 \end{pmatrix},\\
\begin{pmatrix} b^\pi a\ b^\pi \\ b^\pi b\ 0 \end{pmatrix} &= \begin{pmatrix} b^\pi a\ 1 \\ b^\pi b\ 0 \end{pmatrix}\begin{pmatrix} 1\ 0 \\ 0\ b^\pi \end{pmatrix};\\
\begin{pmatrix} b^\pi a\ b^\pi \\ b^\pi b\ 0 \end{pmatrix} &= \begin{pmatrix} b^\pi\ 0 \\ 0\ b^\pi \end{pmatrix}\begin{pmatrix} a\ 1 \\ b\ 0 \end{pmatrix},\\
\begin{pmatrix} ab^\pi\ b^\pi \\ bb^\pi\ 0 \end{pmatrix} &= \begin{pmatrix} a\ 1 \\ b\ 0 \end{pmatrix}\begin{pmatrix} b^\pi\ 0 \\ 0\ b^\pi \end{pmatrix};\\
\begin{pmatrix} ab^\pi\ b^\pi \\ bb^\pi\ 0 \end{pmatrix} &= \begin{pmatrix} ab^\pi\ 1 \\ bb^\pi\ 0 \end{pmatrix}\begin{pmatrix} 1\ 0 \\ 0\ b^\pi \end{pmatrix},\\
\begin{pmatrix} ab^\pi\ 1 \\ bb^\pi\ 0 \end{pmatrix} &= \begin{pmatrix} 1\ 0 \\ 0\ b^\pi \end{pmatrix}\begin{pmatrix} ab^\pi\ 1 \\ bb^\pi\ 0 \end{pmatrix}.
\end{aligned}$$

By repeatedly using Cline's formula, we complete the proof. □

Corollary 15.3.2. *Let* $a, b, b^\pi a \in \mathcal{A}^d$. *If* $b^\pi ab^d = 0, (b^\pi a)^d b^\pi ab = 0, b^\pi ab(b^\pi a)^\pi = 0$, *then* $\begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}$ *has g-Drazin inverse.*

Proof. Let $N = \begin{pmatrix} b^\pi a & 1 \\ b^\pi b & 0 \end{pmatrix}$. Then $N^2 = \begin{pmatrix} (b^\pi a)^2 + b^\pi b & b^\pi a \\ (b^\pi b)(b^\pi a) & b^\pi b \end{pmatrix}$. Write $N^2 = P + Q$, where

$$P = \begin{pmatrix} (b^\pi a)^2 & b^\pi a \\ 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} b^\pi b & 0 \\ (b^\pi b)(b^\pi a) & b^\pi b \end{pmatrix}.$$

By hypothesis, we have

$$P^d = \begin{pmatrix} [(b^\pi a)^d]^2 & [(b^\pi a)^d]^3 \\ 0 & 0 \end{pmatrix}, P^\pi = \begin{pmatrix} (b^\pi a)^\pi & -(b^\pi a)^d \\ 0 & I \end{pmatrix}.$$

Clearly, $(b^\pi a)^d b^\pi b = (b^\pi a)^d[(b^\pi a)^d b^\pi ab]$ and $-(b^\pi b)((b^\pi a))(b^\pi a)^d + b^\pi b = bb^\pi (b^\pi a)^\pi$. Hence $P^d Q = 0, PQP^\pi = 0$. Since $P^\pi Q = Q$ has g-Drazin inverse, so has QP^π by Cline's formula. By virtue of Lemma 15.2.2, $PP^\pi + QP^\pi$ has g-Drazin inverse. Choose $p = PP^d$. Then

$$P = \begin{pmatrix} P^2P^d & 0 \\ 0 & PP^\pi \end{pmatrix}_p, Q = \begin{pmatrix} 0 & 0 \\ 0 & QP^\pi \end{pmatrix}_p.$$

Hence $N^2 = \begin{pmatrix} P^2P^d & 0 \\ 0 & PP^\pi + QP^\pi \end{pmatrix}_p$ has g-Drazin inverse. It follows by Lemma 15.1.13 that N has g-Drazin inverse. This completes the proof by Theorem 15.3.1. □

We are now ready to prove:

Theorem 15.3.3. *Let* $a, b, b^\pi a \in \mathcal{A}^d$. *If* $b^\pi(ab^2) = 0$ *and* $b^\pi(aba) = 0$, *then* $\begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}$ *has g-Drazin inverse.*

Proof. Since $b^\pi(ab^2) = 0$, we have

$$b^\pi ab^d = b^\pi ab^2(b^d)^3 = 0, b^\pi a(b^\pi b)^2 = 0, b^\pi a(b^\pi b)b^\pi a = 0.$$

Let $N = \begin{pmatrix} b^\pi a & 1 \\ b^\pi b & 0 \end{pmatrix}$. As in the proof of Corollary 15.3.2, $N^2 = P + Q$, where

$$P = \begin{pmatrix} (b^\pi a)^2 & b^\pi a \\ 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} b^\pi b & 0 \\ b^\pi bb^\pi a & b^\pi b \end{pmatrix}.$$

It is easy to verify that

$$PQ^2 = \begin{pmatrix} (b^\pi a)^2 b^\pi b & b^\pi ab \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b^\pi b & 0 \\ b^\pi bb^\pi a & b^\pi b \end{pmatrix} = 0,$$
$$PQP = \begin{pmatrix} (b^\pi a)^2 b^\pi b & b^\pi ab \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (b^\pi a)^2 & b^\pi a \\ 0 & 0 \end{pmatrix} = 0.$$

In view of Corollary 15.2.4, N^2 has g-Drazin inverse. Thus N has g-Drazin inverse. According to Theorem 15.3.1, we obtain the result. □

Corollary 15.3.4. *Let $a, b, b^\pi a \in \mathcal{A}^d$. If $b^\pi ab = 0$, then $\begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}$ has g-Drazin inverse.*

Proof. This is obvious by Theorem 15.3.3. □

Corollary 15.3.5. *Let $a, b, b^\pi a \in \mathcal{A}^d$. If $b^\pi ab = 0$, then $\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$ has g-Drazin inverse.*

Proof. Clearly, we have

$$\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -a \end{pmatrix}^{-1} \begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -a \end{pmatrix},$$

thus completing the proof by Corollary 15.3.4. □

Lemma 15.3.6. *Let $a, b \in \mathcal{A}^d$. If $ab^2 = 0$ and $aba = 0$, then $\begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}$ has g-Drazin inverse.*

Proof. Since $ab^2 = 0$, $ab^\pi = a - (ab^2)(b^d)^2 = a \in \mathcal{A}^d$. By Cline's formula, $b^\pi a \in \mathcal{A}^d$. This completes the proof by Theorem 15.3.3. □

Theorem 15.3.7. *Let $a, b \in \mathcal{A}^d$. If $ab^2 = 0, a^2ba = 0$ and $(ba)^2 = 0$, Then $\begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}$ has g-Drazin inverse.*

Proof. Let $M = \begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}$. Then $M^2 = P + Q$, where

$$P = \begin{pmatrix} a^2 & a \\ ba & 0 \end{pmatrix}, Q = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}.$$

Clearly, Q has g-Drazin inverse. We compute that

$$\begin{aligned} P &= \begin{pmatrix} 1 & 0 \\ 0 & ba \end{pmatrix} \begin{pmatrix} a^2 & a \\ 1 & 0 \end{pmatrix}, \\ \begin{pmatrix} a^2 & aba \\ 1 & 0 \end{pmatrix} &= \begin{pmatrix} a^2 & a \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & ba \end{pmatrix} \\ &= \begin{pmatrix} a^2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & aba \end{pmatrix}, \\ \begin{pmatrix} a^2 & 1 \\ aba & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & aba \end{pmatrix} \begin{pmatrix} a^2 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

In light of Lemma 15.3.6, $\begin{pmatrix} a^2 & 1 \\ aba & 0 \end{pmatrix}$ has g-Drazin inverse. By using Cline's formula, P has g-Drazin inverse. Moreover, we check that $PQP = 0$ and $PQ^2 = 0$. In light of Corollary 15.2.4, $M^2 = P + Q$ has g-Drazin inverse. By virtue of Lemma 15.1.13, M has g-Drazin inverse, as asserted. □

We come now to investigate the g-Drazin inverse for an anti-triangular matrix over a Banach algebra under certain weakly conditions.

Lemma 15.3.8. *Let $a, b \in \mathcal{A}^{qnil}$. If $ab = ba$, then $M = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \in M_2(\mathcal{A})^{qnil}$.*

Proof. Clearly, $M^2 = Ma + bI$. Let $\lambda \in \mathbb{C}$. Then $(\lambda M)a = a(\lambda M)$. Since $a \in \mathcal{A}^{qnil}$, $aI \in M_2(\mathcal{A})^{qnil}$. Thus $I - (\lambda M)a = I - \lambda(Ma)$ is invertible. Therefore $Ma \in M_2(\mathcal{A})^{qnil}$. Obviously, $bI \in M_2(\mathcal{A})^{qnil}$ and $(Ma)(bI) = (bI)(Ma)$. It follows by Lemma 15.1.1 that $M^2 \in M_2(\mathcal{A})^{qnil}$, and therefore $M \in M_2(\mathcal{A})^{qnil}$. □

Lemma 15.3.9. *Let $b \in \mathcal{A}^{qnil}$. Then $M = \begin{pmatrix} 1 & b \\ 1 & 0 \end{pmatrix} \in M_2(\mathcal{A})^d$.*

Proof. Obviously, $M = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} b$. Then $M - M^2 = Nb$, where $N = -\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. As in the proof in Lemma 15.3.8, $Nb \in M_2(\mathcal{A})^{qnil}$. In light of Lemma 15.1.2, we can find some $F^2 = F$ such that $FM = MF$ and $M - F \in M_2(\mathcal{A})^{qnil}$. Hence $M + (I - F) = I + (M - F)$ is invertible and $M(I - F) \in M_2(\mathcal{A})^{qnil}$. Therefore M has g-Drazin inverse by Theorem 6.4.8. □

Theorem 15.3.10. *Let $a, b \in \mathcal{A}^d$. If $ab = ba$, then $M = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$ has g-Drazin inverse.*

Proof. Step 1. Let $f = bb^d$. Then $M = P + Q$, where

$$P = \begin{pmatrix} a(1-f) & b(1-f) \\ 1-f & 0 \end{pmatrix}, Q = \begin{pmatrix} af & bf \\ f & 0 \end{pmatrix}.$$

Since $ab = ba$, we have $af = fa$, and so $PQ = 0$. We directly verify that

$$Q^{\#} = \begin{pmatrix} 0 & f \\ fb^d & -ab^d \end{pmatrix}.$$

Step 2. $P = R + S$, where

$$R = \begin{pmatrix} 0 & 0 \\ -f & 0 \end{pmatrix}, S = \begin{pmatrix} a(1-f) & n \\ 1 & 0 \end{pmatrix}, n = b(1 - bb^d).$$

Then $RS = 0$ and $R^d = 0$.

Step 3. $S = \begin{pmatrix} \alpha & n \\ 1 & 0 \end{pmatrix}, \alpha = a(1-f) \in \mathcal{A}^d$. Then $S = G + H$, where

$$G = \begin{pmatrix} \alpha e & en \\ e & 0 \end{pmatrix}, H = \begin{pmatrix} \alpha(1-e) & (1-e)n \\ (1-e) & 0 \end{pmatrix}, e = \alpha\alpha^d.$$

Clearly, $GH = 0$. We observe that

$$H = \begin{pmatrix} 1 & 0 \\ 0 & 1-e \end{pmatrix} \begin{pmatrix} \alpha(1-e) & (1-e)n \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \alpha(1-e) & (1-e)n \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \alpha(1-e) & (1-e)n \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1-e \end{pmatrix}.$$

In view of Lemma 15.3.8, $\begin{pmatrix} \alpha(1-e) & (1-e)n \\ 1 & 0 \end{pmatrix}$ is quasinilpotent. By using Cline's formula, H is quasinilpotent.

Step 4.

$$G = \begin{pmatrix} \alpha e & en \\ e & 0 \end{pmatrix} = \begin{pmatrix} \alpha e & 0 \\ 0 & \alpha e \end{pmatrix} \begin{pmatrix} 1 & \alpha^d n \\ \alpha^d & 0 \end{pmatrix}.$$

Clearly,

$$\begin{pmatrix} \alpha e & 0 \\ 0 & \alpha e \end{pmatrix} \begin{pmatrix} 1 & \alpha^d n \\ \alpha^d & 0 \end{pmatrix} = \begin{pmatrix} 1 & \alpha^d n \\ \alpha^d & 0 \end{pmatrix} \begin{pmatrix} \alpha e & 0 \\ 0 & \alpha e \end{pmatrix}.$$

BY virtue of Theorem 15.2.16, $\alpha e \in \mathcal{A}^d$.

Step 5. We have

$$\begin{pmatrix} 1 & \alpha^d n \\ \alpha^d & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha^d \end{pmatrix} \begin{pmatrix} 1 & \alpha^d n \\ 1 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 1 & (\alpha^d)^2 n \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & \alpha^d n \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \alpha^d \end{pmatrix}.$$

According to Lemma 15.3.9, $\begin{pmatrix} 1 & (\alpha^d)^2 n \\ 1 & 0 \end{pmatrix} \in M_2(\mathcal{A})^d$. By using Cline's formula, $\begin{pmatrix} 1 & \alpha^d n \\ \alpha^d & 0 \end{pmatrix}$ has g-Drazin inverse. Then G has g-Drazin inverse by Theorem 15.2.16. By repeatedly use Lemma 15.2.2, S and P have g-Dazin inverses. By using Lemma 15.2.2 again, M has g-Drazin inverse. □

Corollary 15.3.11. *Let $a, b, b^\pi a \in \mathcal{A}^d$. If $b^\pi a b^d = 0, abb^\pi = b^\pi ba$, then $\begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}$ has g-Drazin inverse.*

Proof. Let $N = \begin{pmatrix} ab^\pi & bb^\pi \\ 1 & 0 \end{pmatrix}$. We check that

$$(ab^\pi)(bb^\pi) = abb^\pi = b^\pi ba = b^\pi bab^\pi = (bb^\pi)(ab^\pi).$$

Hence N has g-Drazin inverse by Theorem 15.3.9. It is easy to very that

$$\begin{pmatrix} ab^\pi & 1 \\ bb^\pi & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & bb^\pi \end{pmatrix} \begin{pmatrix} ab^\pi & 1 \\ 1 & 0 \end{pmatrix},$$
$$\begin{pmatrix} ab^\pi & bb^\pi \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} ab^\pi & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & bb^\pi \end{pmatrix}.$$

By using Cline's formula, $\begin{pmatrix} ab^\pi & 1 \\ bb^\pi & 0 \end{pmatrix}$ has g-Drazin inverse. This completes the proof by Theorem 15.3.1. □

Corollary 15.3.12. *Let $a, b, b^\pi a \in \mathcal{A}^d$. If $b^\pi a b^d = 0, b^\pi ab = b^\pi ba$, then $\begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}$ has g-Drazin inverse.*

Proof. Let $N = \begin{pmatrix} b^\pi a & b^\pi b \\ 1 & 0 \end{pmatrix}$. By hypothesis, we have

$$(b^\pi a)(b^\pi b) = b^\pi ab = b^\pi ba = (b^\pi b)(b^\pi a).$$

In light of Theorem 15.3.9, N has g-Drazin inverse. As in the proof of Corollary 15.3.11, we obtain the result. □

Lemma 15.3.13. *Let $a, b \in \mathcal{A}^{qnil}$. If $a^2b = aba, b^2a = bab$, then $M = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \in M_2(\mathcal{A})^{qnil}$.*

Proof. We have $M^2 = P + Q$, where

$$P = \begin{pmatrix} a^2 & ab \\ a & 0 \end{pmatrix}, Q = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}.$$

Then we have

$$\begin{aligned} P^2Q &= \begin{pmatrix} a^2 & ab \\ a & 0 \end{pmatrix}^2 \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \\ &= \begin{pmatrix} a^4b + abab & a^3b^2 \\ a^3b & a^2b^2 \end{pmatrix}, \\ PQP &= \begin{pmatrix} a^2 & ab \\ a & 0 \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} a^2 & ab \\ a & 0 \end{pmatrix} \\ &= \begin{pmatrix} a^2ba^2 + ab^2a & a^2bab \\ aba^2 & abab \end{pmatrix}, \end{aligned}$$

and so $P^2Q = PQP$.

Also we have

$$\begin{aligned} Q^2P &= \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}^2 \begin{pmatrix} a^2 & ab \\ a & 0 \end{pmatrix} \\ &= \begin{pmatrix} b^2a^2 & b^2ab \\ b^2a & 0 \end{pmatrix}, \\ QPQ &= \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} a^2 & ab \\ a & 0 \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \\ &= \begin{pmatrix} ba^2b & bab^2 \\ bab & 0 \end{pmatrix}; \end{aligned}$$

hence, $Q^2P = QPQ$. By hypothesis, $a(ab) = (ab)a$ and $a^2(aba) = (aba)a^2$. In light of Lemma 15.2.15, $ab \in \mathcal{A}^{qnil}$. This implies that $aba \in \mathcal{A}^{qnil}$. According to Theorem 15.3.10, $P = \begin{pmatrix} a^2 & aba \\ 1 & 0 \end{pmatrix}$ is quasinilpotent. Obviously, Q is quasinilpotent. According to Lemma 15.2.18, M^2 is quasinilpotent, the result follows. □

We have accumulated all the information to prove:

Theorem 15.3.14. *Let $a, b \in \mathcal{A}$ have g-Drazin inverses. If $a^2b = aba, b^2a = bab$, then $M = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$ has g-Drazin inverse.*

Proof. Step 1. Let $f = bb^d$. Then $M = P + Q$, where

$$P = \begin{pmatrix} a(1-f) & b(1-f) \\ 1-f & 0 \end{pmatrix}, Q = \begin{pmatrix} af & bf \\ f & 0 \end{pmatrix}.$$

Since $b(ba) = b^2a = bab = (ba)b$, we have $b^d(ba) = (ba)b^d$, and so $fa(1-f) = b^d(ba)(1-bb^d) = (ba)b^d(1-bb^d) = 0$. This implies that $QP = 0$. Moreover, we have

$$\begin{aligned}(faf)(fb) &= b^d(bab)b^db \\ &= b^d(b^2a)bb^d \\ &= (fb)(faf).\end{aligned}$$

In virtue of Corollary 15.2.14, we have $a^2f = afa, fa = faf$. By virtue of Theorem 15.2.16, $faf, fb \in \mathcal{A}^d$. According to Theorem 15.3.10, $\begin{pmatrix} faf & fb \\ 1 & 0 \end{pmatrix}$ has g-Drazin inverse. Observing that

$$\begin{pmatrix} af & bf \\ f & 0 \end{pmatrix} = \begin{pmatrix} af & f \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & bf \end{pmatrix},$$
$$\begin{pmatrix} af & f \\ bf & 0 \end{pmatrix} = \begin{pmatrix} f & 0 \\ 0 & bf \end{pmatrix} \begin{pmatrix} af & f \\ 1 & 0 \end{pmatrix};$$
$$\begin{pmatrix} af & f \\ bf & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & bf \end{pmatrix} \begin{pmatrix} af & f \\ 1 & 0 \end{pmatrix},$$
$$\begin{pmatrix} af & bf \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} af & f \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & bf \end{pmatrix}.$$

By using Cline's formula repeatedly, Q has g-Drazin inverse.

Step 2. $P = R + S$, where

$$R = \begin{pmatrix} 0 & 0 \\ -f & 0 \end{pmatrix}, S = \begin{pmatrix} a(1-f) & n \\ 1 & 0 \end{pmatrix}, n = b(1-bb^d).$$

Then $RS = 0$ and $R^2 = 0$.

Step 3. $S = \begin{pmatrix} \alpha & n \\ 1 & 0 \end{pmatrix}, \alpha = a(1-f)$. Since $fa(1-f) = 0$ and $a, fa \in \mathcal{A}^d$, it follows by Lemma 15.2.19 that $\alpha \in \mathcal{A}^d$. Further, we have $S = G + H$, where

$$G = \begin{pmatrix} \alpha e & en \\ e & 0 \end{pmatrix}, H = \begin{pmatrix} \alpha(1-e) & (1-e)n \\ 1-e & 0 \end{pmatrix}, e = \alpha\alpha^d.$$

We observe that

$$\begin{aligned}\alpha ab(1-bb^d) &= a(1-bb^d)ab(1-bb^d) \\ &= (a^2b - abb^dab)(1-bb^d) \\ &= [aba - ab(b^da)b](1-bb^d) \\ &= [aba - ab^2(b^da)](1-bb^d) \\ &= ab(1-bb^d)a(1-bb^d) \\ &= ab(1-bb^d)\alpha;\end{aligned}$$

hence, $\alpha^d ab(1-bb^d) = ab(1-bb^d)\alpha^d$. Then

$$\begin{aligned} en(1-e) &= \alpha^d \alpha b(1-bb^d)(1-\alpha\alpha^d) \\ &= \alpha^d ab(1-bb^d)(1-\alpha\alpha^d) \\ &= ab(1-bb^d)\alpha^d(1-\alpha\alpha^d) \\ &= 0, \end{aligned}$$

and so $(1-e)n(1-e) = n(1-e)$. Thus we have $GH = 0$. Observing that

$$\begin{pmatrix} \alpha(1-e) & (1-e)n \\ 1-e & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1-e \end{pmatrix} \begin{pmatrix} \alpha(1-e) & (1-e)n \\ 1 & 0 \end{pmatrix},$$
$$\begin{pmatrix} \alpha(1-e) & n(1-e) \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \alpha(1-e) & (1-e)n \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1-e \end{pmatrix}.$$

We compute that

$$\begin{aligned} ((1-e)\alpha)^2[n(1-e)] &= (1-e)\alpha^2 n(1-e) \\ &= (1-e)[a(1-bb^d)ab(1-bb^d)](1-e) \\ &= (1-e)(a^2b - ab(b^d a)b)(1-bb^d)(1-e) \\ &= (1-e)(a^2b - ab^2(b^d a))(1-bb^d)(1-e) \\ &= (1-e)(aba - abb^d ba)(1-bb^d)(1-e) \\ &= (1-e)[a(1-bb^d)ba](1-bb^d)(1-e) \\ &= (1-e)\alpha n\alpha(1-e) \\ &= [(1-e)\alpha][n(1-e)][(1-e)\alpha] \\ &= [\alpha(1-e)][(1-e)n][\alpha(1-e)], \\ (n(1-e))^2[(1-e)\alpha] &= n(1-e)n\alpha(1-e) \\ &= n^2\alpha(1-e) \\ &= (1-bb^d)(b^2a)(1-bb^d)(1-e) \\ &= (1-bb^d)(bab)(1-bb^d)(1-e) \\ &= n\alpha n(1-e) \\ &= [n(1-e)][\alpha(1-e)][n(1-e)]. \end{aligned}$$

Clearly, $\alpha(1-e), n(1-e) \in \mathcal{A}^{qnil}$. In light of Lemma 15.3.8, we have $\begin{pmatrix} \alpha(1-e) & n(1-e) \\ 1 & 0 \end{pmatrix}$ is quasinilpotent. We infers that H is quasinilpotent.

Step 4. We directly compute that

$$\begin{aligned}
(\alpha e)(en) &= \alpha^d a(1-bb^d)a(1-bb^d)b(1-bb^d)\\
&= \alpha^d a(1-bb^D)ab(1-bb^d)\\
&= \alpha^d[a^2b-abb^dab](1-bb^d)\\
&= \alpha^d[a^2b-ab^d(ba)b](1-bb^d)\\
&= \alpha^d[a^2b-a(ba)b^db](1-bb^d)\\
&= \alpha^d a^2b(1-bb^d)\\
&= \alpha^d ab\alpha,\\
(en)(\alpha e) &= \alpha\alpha^d n\alpha^2\alpha^d\\
&= \alpha^d ab\alpha^2\alpha^d,\\
\alpha(ab) &= a(1-bb^d)ab\\
&= a^2b-ab(b^da)b\\
&= a^2b-ab^2(b^da)\\
&= a^2b-abb^d(ba)\\
&= a^2b-ab(ba)b^d\\
&= a^2b-a(b^2a)b^d\\
&= a^2b-a(bab)b^d\\
&= a^2b-(aba)bb^d\\
&= aba(1-bb^d)\\
&= (ab)\alpha.
\end{aligned}$$

Therefore $ab \in comm(\alpha)$, and so $ab \in comm(\alpha^d)$. This implies that

$$(\alpha e)(en) = \alpha^d ab\alpha = \alpha^d ab\alpha^2\alpha^d = (en)(\alpha e).$$

Obviously, $\alpha e, en \in \mathcal{A}^d$. In light of Theorem 15.3.10, $\begin{pmatrix} \alpha e & en \\ 1 & 0 \end{pmatrix}$ has g-Drazin inverse. We check that

$$\begin{pmatrix} \alpha e & en \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \alpha e & en \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix},$$
$$\begin{pmatrix} \alpha e & en \\ e & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix}\begin{pmatrix} \alpha e & en \\ 1 & 0 \end{pmatrix}.$$

By using Cline's formula, $G = \begin{pmatrix} \alpha e & en \\ e & 0 \end{pmatrix}$ has g-Drazin inverse. Therefore we complete the proof by applying Lemma 15.2.2 repeatedly. □

Corollary 15.3.15. *Let $a, bc \in \mathcal{A}^d$. If $a^2(bc) = a(bc)a, (bc)^2a = (bc)a(bc)$, then $M = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ has g-Drazin inverse.*

Proof. By virtue of Theorem 15.3.14, $\begin{pmatrix} a & bc \\ 1 & 0 \end{pmatrix}$ has g-Drazin inverse. We check that

$$\begin{pmatrix} a & bc \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}.$$

By virtue of Cline's formula, $\begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$ has g-Drazin inverse. That is, $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ has g-Drazin inverse, as asserted. □

Corollary 15.3.16. *Let $a, b \in \mathcal{A}$ have g-Drazin inverses. If $b^\pi ab^d = 0, b^\pi a^2 b = b^\pi aba, b^\pi b^2 a = b^\pi bab$, then $M = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$ has g-Drazin inverse.*

Proof. In view of Theorem 15.3.14, $\begin{pmatrix} ab^\pi & bb^\pi \\ 1 & 0 \end{pmatrix}$ has g-Drazin inverse. We verify that

$$\begin{pmatrix} ab^\pi & bb^\pi \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} ab^\pi & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & bb^\pi \end{pmatrix},$$
$$\begin{pmatrix} ab^\pi & 1 \\ bb^\pi & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & bb^\pi \end{pmatrix} \begin{pmatrix} ab^\pi & 1 \\ 1 & 0 \end{pmatrix}.$$

Therefore $\begin{pmatrix} ab^\pi & 1 \\ bb^\pi & 0 \end{pmatrix}$ has g-Drazin inverse by using Cline's formula. This completes the proof by Theorem 15.3.1. □

15.4 Block Operator Matrices

Let X and Y be complex Banach spaces, let $\mathcal{L}(X, Y)$ denote the Banach algebra of bounded linear operators from X to Y, and let $\mathcal{L}(X) = \mathcal{L}(X, X)$. This section is devoted to the generalized Drazin inverse for a operator matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A \in \mathcal{L}(X)^d, B \in \mathcal{L}(Y, X), C \in \mathcal{L}(X, Y)$ and $D \in \mathcal{L}(Y)^d$. Evidently, M is a bounded linear operator on $X \oplus Y$. Let $p = \begin{pmatrix} I_X & 0 \\ 0 & 0 \end{pmatrix}$. Then $M = pMp + pM(1-p) + (1-p)Mp + (1-p)M(1-p) := \begin{pmatrix} pMp & pM(1-p) \\ (1-p)Mp & (1-p)M(1-p) \end{pmatrix}_p$. Then M can be regarded as the

2×2 matrix over $\mathcal{L}(X \oplus Y)$. We always assume that $X = Y$ without loss of the generality.

Theorem 15.4.1. *If $BCA = 0, BCB = 0, DCA = 0$ and $DCB = 0$, then M has g-Drazin inverse.*

Proof. We easily see that

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = p + q,$$

where

$$p = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, q = \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}.$$

In view of Lemma 15.2.1, p and q have g-Drazin inverses. By hypothesis, we check that

$$\begin{aligned} pqp &= \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \\ &= \begin{pmatrix} BCA & BCB \\ DCA & DCB \end{pmatrix} \\ &= 0, \\ pq^2 &= \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}^2 \\ &= 0. \end{aligned}$$

This completes the proof by Corollary 15.2.4. □

Corollary 15.4.2. *If $BC = 0$ and $DC = 0$, then M has g-Drazin inverse.*

Proof. If $BC = 0$, then $BCA = 0$ and $BCB = 0$. If $DC = 0$, then $DCA = 0$ and $DCB = 0$. So we obtain the result by Theorem 15.4.1. □

Corollary 15.4.3. *If $CA = 0$ and $CB = 0$, then M has g-Drazin inverse.*

Proof. If $CA = 0$, then $BCA = 0$ and $DCA = 0$. If $CB = 0$, then $BCB = 0$ and $DCB = 0$. This completes the proof by Theorem 15.4.1. □

Theorem 15.4.4. *If $ABC = 0, ABD = 0, CBC = 0$ and $CBD = 0$, then M has g-Drazin inverse.*

Proof. By virtue of Theorem 15.4.1, $\begin{pmatrix} D & C \\ B & A \end{pmatrix}$ has g-Drazin inverse. Clearly, we verify that

$$M = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} D & C \\ B & A \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Therefore M has g-Drazin inverse. □

Corollary 15.4.5. (1) *If $AB = 0$ and $CB = 0$, then M has g-Drazin inverse.*

(2) *If $BC = 0$ and $BD = 0$, then M has g-Drazin inverse.*

Example 15.4.6. Let A, B, C, D be operators, acting on separable Hilbert space $l_2(\mathbb{N})$, defined as follows respectively:

$$\begin{aligned} A(x_1, x_2, x_3, x_4, \ldots) &= (x_1, x_1, 0, 0, \ldots), \\ B(x_1, x_2, x_3, x_4, \ldots) &= (x_1, -x_1, 0, 0, \ldots), \\ C(x_1, x_2, x_3, x_4, \ldots) &= (x_1 + x_2, x_1 - x_2, 0, 0, \ldots), \\ D(x_1, x_2, x_3, x_4, \ldots) &= (-x_2, x_2, 0, 0, \ldots). \end{aligned}$$

Set $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Then $BCA = 0, BCB = 0, DCA = 0$ and $DCB = 0$. By virtue of Theorem 15.4.1, M has g-Drazin inverse.

Theorem 15.4.7. *If $ABD = 0, CBD = 0, A^2B = 0, CAB = 0, BCBC = 0$ and $DCBC = 0$, then $M \in M_2(\mathcal{A})^d$.*

Proof. Write $M = p + q$, where $p = \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix}$, and $q = \begin{pmatrix} 0 & B \\ 0 & D \end{pmatrix}$. Then p, q have g-Drazin inverses. Moreover, we have

$$\begin{aligned} pq^2 &= \begin{pmatrix} 0 & ABD \\ 0 & CBD \end{pmatrix} = 0, \\ p^2q &= \begin{pmatrix} 0 & A^2B \\ 0 & CAB \end{pmatrix} = 0, \\ (qp)^2 &= \begin{pmatrix} BCBC & 0 \\ DCBC & 0 \end{pmatrix}. \end{aligned}$$

In light of Proposition 15.2.10, M has g-Drazin inverse. □

Corollary 15.4.8. *If $BCA = 0, DCA = 0, BDC = 0, D^2C = 0, ABCB = 0$ and $CBCB = 0$, then $M \in M_2(\mathcal{A})^d$.*

Proof. Applying Theorem 15.4.7 to the operator matrix $\begin{pmatrix} D & C \\ B & A \end{pmatrix}$, we obtain the result as the proof of Theorem 15.4.4. □

Theorem 15.4.9. *Let BC have g-Drazin inverses. If $ABC = 0, DCB = 0, A^2B = 0, D^2C = 0, BDCA = 0$ and $CABD = 0$, then M has g-Drazin inverse.*

Proof. Write $M = p + q$, where

$$p = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, q = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}.$$

By virtue of Lemma 15.2.1, p has g-Drazin inverse. Since BC has g-Drazin inverse, it follows by Cline's formula, CB has g-Drazin inverse. Hence, $q^2 = \begin{pmatrix} BC & 0 \\ 0 & CB \end{pmatrix}$ has g-Drazin inverse. In view of Lemma 15.1.13, q has g-Drazin inverse. By hypothesis, we compute that

$$\begin{aligned} pq^2 &= \begin{pmatrix} ABC & 0 \\ 0 & DCB \end{pmatrix} = 0, \\ p^2q &= \begin{pmatrix} 0 & A^2B \\ D^2C & 0 \end{pmatrix} = 0, \\ (qp)^2 &= \begin{pmatrix} BDCA & 0 \\ 0 & CABD \end{pmatrix}. \end{aligned}$$

In light of Proposition 15.2.10, M has g-Drazin inverse. □

As an immediate consequence of Theorem 15.4.9, we now derive

Corollary 15.4.10. *If $AB = 0$ and $DC = 0$ and BC has g-Drazin inverse, then M has g-Drazin inverse.*

Theorem 15.4.11. (1) *If $BCA = 0, BCB = 0, D^dC = 0$ and $BDD^\pi = 0$, then M has g-Drazin inverse.*

(2) *If $CBC = 0, CBD = 0, CA^d = 0$ and $AA^\pi B = 0$, then M has g-Drazin inverse.*

Proof. (1) Let

$$p = \begin{pmatrix} A^2A^d & B \\ 0 & D^2D^d \end{pmatrix}, q = \begin{pmatrix} AA^\pi & 0 \\ C & DD^\pi \end{pmatrix}.$$

Then p and q have g-Drazin inverses by Lemma 15.2.1. We compute that

$$\begin{aligned} pqp &= \begin{pmatrix} BCA^2A^d & BCB \\ D^2D^dCA^2A^d & D^2D^dCB \end{pmatrix}, \\ pq^2 &= \begin{pmatrix} BCAA^\pi & BD^2D^\pi \\ D^2D^dCAA^\pi & 0 \end{pmatrix}. \end{aligned}$$

By hypothesis, $pqp = 0$ and $pq^2 = 0$. In light of Corollary 15.2.4, $M = p+q$ has g-Drazin inverse, as asserted.

(2) Choosing the same p and q as in (1), we have $qpq = 0$ and $qp^2 = 0$. As p and q have g-Drazin inverse, we complete the proof by Corollary 15.2.4. □

Corollary 15.4.12. (1) *If* $BC = 0, D^dC = 0$ *and* $BDD^\pi = 0$, *then* M *has g-Drazin inverse.*

(2) *If* $CB = 0, CA^d = 0$ *and* $AA^\pi B = 0$, *then* M *has g-Drazin inverse.*

Theorem 15.4.13. *If* $BCA = 0, BCB = 0, BD^d = 0$ *and* $D^\pi DC = 0$, *then* M *has g-Drazin inverse.*

Proof. Let

$$p = \begin{pmatrix} A^2A^d & B \\ 0 & DD^\pi \end{pmatrix}, q = \begin{pmatrix} AA^\pi & 0 \\ C & D^2D^d \end{pmatrix}.$$

In view of Lemma 15.2.1, p and q have g-Drazin inverses. Moreover, we have

$$pqp = \begin{pmatrix} BCA^2A^d & BCB \\ DD^\pi CA^2A^d & DD^\pi CB \end{pmatrix},$$
$$pq^2 = \begin{pmatrix} BCAA^\pi + BD^2D^dC & BD^3D^d \\ DD^\pi CA^2A^d & 0 \end{pmatrix}.$$

By hypothesis, we get $pqp = 0$ and $pq^2 = 0$. According to Corollary 15.2.4, $M = p + q$ has g-Drazin inverse. □

Corollary 15.4.14. *If* $BC = 0, BD^d = 0$ *and* $D^\pi DC = 0$, *then* M *has g-Drazin inverse.*

We are now ready to prove:

Theorem 15.4.15. *If* $BC = 0, CB = 0$ *and* $CA = DC$, *then* M *has g-Drazin inverse.*

Proof. Write $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = p + q$, where

$$p = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, q = \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}.$$

In light of Lemma 15.2.1, p and q have g-Drazin inverse. By hypothesis, we verify that

$$\begin{aligned} pq &= \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \\ &= \begin{pmatrix} BC & 0 \\ DC & 0 \end{pmatrix} \\ &= \begin{pmatrix} BC & 0 \\ DC & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \\ &= qp. \end{aligned}$$

Obviously, $1 + q^d p = 1 \in \mathcal{A}^d$. In view of Corollary 15.2.23, the result follows. □

Corollary 15.4.16. *If $BC = 0, CB = 0$ and $AB = BD$, then M has g-Drazin inverse.*

Proof. In view of Theorem 15.4.15, $\begin{pmatrix} D & C \\ B & A \end{pmatrix}$ has g-Drazin inverse. Obviously,

$$M = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} D & C \\ B & A \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

So the corollary is true. □

Theorem 15.4.17. *If $A(BC)^2 = 0, ABCA = 0, BDC = 0$ and $BD^2 = 0$, then M has g-Drazin inverse.*

Proof. Let

$$p = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}, q = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}.$$

Then $M = p + q$. According to Lemma 15.3.6, $\begin{pmatrix} A & I \\ BC & 0 \end{pmatrix}$ has g-Drazin inverse. It is easy to check that

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = \begin{pmatrix} A & I \\ C & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix},$$
$$\begin{pmatrix} A & I \\ BC & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} A & I \\ C & 0 \end{pmatrix}.$$

By using Cline's formula, p has g-Drazin inverse. In view of Lemma 15.2.1, q has g-Drazin inverse. By hypothesis, we easily verify that

$$pqp = \begin{pmatrix} 0 & BD \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = 0,$$
$$pq^2 = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & D^2 \end{pmatrix} = 0.$$

According to Corollary 15.2.4, M has g-Drazin inverse, as asserted. □

Corollary 15.4.18. *If $D(CB)^2 = 0, DCBD = 0, CAB = 0$ and $CA^2 = 0$, then M has g-Drazin inverse.*

Proof. As in the proof of Corollary 15.4.16, we apply Theorem 15.4.17 to the matrix $\begin{pmatrix} D & C \\ B & A \end{pmatrix}$, the result follows. □

Theorem 15.4.19. *If $ABC = BCA, BDC = 0$ and $BD^2 = 0$, then M has g-Drazin inverse.*

Proof. Let

$$p = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}, q = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}.$$

Then $M = p + q$. By virtue of Corollary 15.3.11, $\begin{pmatrix} A & I \\ BC & 0 \end{pmatrix}$ has g-Drazin inverse. Similarly to Theorem 15.4.17, it follows by Cline's formula that p has g-DRazin inverse. In view of Lemma 15.2.1, q has g-Drazin inverse. By hypothesis, we easily verify that

$$pqp = \begin{pmatrix} 0 & BD \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = 0,$$
$$pq^2 = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & D^2 \end{pmatrix} = 0.$$

Therefore M has g-Drazin inverse by Corollary 15.2.4. □

Corollary 15.4.20. *If $DCB = CBD, CAB = 0$ and $CA^2 = 0$, then M has g-Drazin inverse.*

Proof. Applying Theorem 15.4.19 to $\begin{pmatrix} D & C \\ B & A \end{pmatrix}$, we complete the proof. □

Notes. Theorem 15.1.3 is a generalization of [369, Theorem 4.2]. Theorem 15.2.12 is due to Drazin [243, Theorem 2.2]. Theorem 15.2.16 was

proved by Zou et al. [253, Theorem 3.1]. Theorem 15.2.21 was proved by Zou et al. [253, Theorem 3.3]. Theorem 15.3.7 is an extension of [232, Theorem 3.1]. Theorem 15.3.10 is due to Patricio and Hartwig (see [446] and [544, Theorem 3.8]). Theorem 15.4.1 extended [231, Theorem 3] to the wider case. Theorem 15.4.15 is due to Deng et al. [231, Theorem 5]. Theorem 15.4.17 generalized [240, Theorem 2.2] from the Drazin inverse to the g-Drazin inverse.

Bibliography

[1] M.S. Abdolyousefi, N. Ashrafi and H. Chen, On 2-nil-good rings, *J. Algebra Appl.*, **17**(2018), 1850110.

[2] M.S. Abdolyousefi, N. Ashrafi and H. Chen, On unit nil-clean rings, *Mediterr. J. Math.*, **16** (2019). https://doi.org/10.1007/s00009-019-1369-z.

[3] M.S. Abdolyousefi, R. Bahmani and H. Chen, Elementary matrix reduction over certain rings, *Comm. Algebra*, **45**(2017), 2613–2618.

[4] M.S. Abdolyousefi and H. Chen, Elementary matrix reduction over J-stable rings, *Comm Algebra*, **45**(2017), 1983–1995.

[5] M.S. Abdolyousefi and H. Chen, Rings in which elements are sums of tripotents and nilpotents, *J. Algebra Appl.*, **17**(2018), 1850042.

[6] M.S. Abdolyousefi and H. Chen, Rings in which every zero divisor is the sum or difference of a nilpotent element and an idempotent, *Math. Reports*, **20(70)**(2018), 93–106.

[7] M.S. Abdolyousefi and H. Chen, Sums of tripotent and nilpotent matrices, *Bull. Korean Math. Soc.*, **55**(2018), 913–920.

[8] H. Abu-Khuzam and A. Yaqub, Periodic rings with commuting nilpotents, *Int. J. Math. Math. Sci.*, **7**(1984), 403–406.

[9] H. Abu-Khuzam and A. Yaqub, Structure of rings with certain conditions on zero divisors, *Int. J. Math. Math. Sci.*, **15**(2006), 1–6.

[10] H. Abu-Khuzam and A. Yaqub, On Boolean-like rings and related rings, *Int. J. Algebra*, **5**(2011), 275–284.

[11] A.N. Abyzov and D.T. Tapkin, When is every matrix over a ring the sum of two tripotents? *Linear Algebra Appl.*, **630**(2021), 316–325.

[12] M.S. Ahn, Weakly Clean Rings and Almost Clean rings, Ph.D. Thesis, The University of Iowa, Iowa, 2003.

[13] M.S. Ahn and D.D. Anderson, Weakly clean rings and almost clean rings, *Rocky Mountain J. Math.*, **36**(2006), 783–798.

[14] A. Alahmadi, N. Er and S. K. Jain, Modules which are invariant under monomorphisms of their injective hulls, *J. Aust. Math. Soc.*, **79**(2005), 349–360.

[15] H. Al-Ezeh, Pure ideals in commutative reduced Gelfand rings with unity, *Arch. Math.*, **53**(1989), 266–269.

[16] M. Alkan, W.K. Nicholson and A.Ç. Özcan, Strong lifting splits, *J. Pure Appl. Algebra*, **215**(2011), 1879–1888.

[17] E. Akalan and L. Vas, Classes of almost clean rings, *Algebra Represent. Theory*, **16**(2013) 843–857.

[18] R. Ameri and A. Kordi, Clean multiplicative hyperrings, *Italian J. Pure Appl. Math.*, **35**(2015), 625–636.

[19] D.D. Anderson, Generalizations of Boolean rings, Boolean-like rings and von Neumann regular rings, *Comment. Math. Univ. St. Paul.*, **35**(1986), 69–76.

[20] D.D. Anderson and V.P. Camillo, Commutative rings whose elements are a sum of a unit and idempotent, *Comm. Algebra*, **30**(2002), 3327–3336.

[21] D.F. Anderson and A. Badawi, Von Neumann regular and related elements in commutative rings, *Algebra Colloq.*, **19**(2012), 1017–1040.

[22] F.W. Anderson and K.R. Fuller, *Rings and Categories of Modules*, Springer, Berlin, 1973.

[23] T. Andreescu and D. Andrica, *Quadratic Diophantine Equations*, Springer-Verlag, New York, 2015.

[24] D. Andrica and G. Călugăreanu, A nil-clean 2×2 matrix over the integers which is not clean, *J. Algebra Appl.*, **13**(2014), 1450009 [9 pages], DOI: 10.1142/S0219498814500091.

[25] P. Ara, Strongly π–regular rings have stable range one, *Proc. Amer. Math. Soc.*, **124**(1996), 3293–3298.

[26] N. Arora and S. Kundu, Semiclean rings and rings of continuous functions, *J. Commut. Algebra*, **6**(2014), 1–16.

[27] N. Arora and S. Kundu, Commutative feebly clean rings, *J. Algebra Appl.*, **16**(2017), 1750128 (14 pages), DOI: 10.1142/S0219498817501286.

[28] P.A.G. Asensio, A. Pedro, D. Keskin Tutuncu and A.K. Srivastava, Modules invariant under monomorphisms of their envelopes, *Contemporary Mathematics*, **715**(2018), 171–179.

[29] P.A.G. Asensio, A. Pedro and A.K. Srivastava, Automorphism-invariant modules, *Contemporary Mathematics*, **634**(2015), 19–30.

[30] P.A.G. Asensio and A.K. Srivastava, Additive unit representations in endomorphism rings and an extension of a result of Dickson and Fuller, *Contemporary Mathematics*, **609**(2014), 117–121.

[31] P.A.G. Asensio and A.K. Srivastava, Automorphism-invariant modules satisfying the exchange property, *J. Algebra*, **388**(2013), 101–106.

[32] N. Ashrafi and Z. Ahmadi, Weakly $g(x)$-clean rings, *Iranian J. Math. Sci. Info.*, **7**(2012), 83–91.

[33] N. Ashrafi and Z. Ahmadi, $g(x)$-Full clean rings, *Miskolc Math. Notes*, **16**(2015), 657–666.

[34] N. Ashrafi, M. Sheibani and H. Chen, Certain factorizations of matrices over abelian rings, *Czechoslovak Math. J.*, **67**(2017), 417–425.

[35] N. Ashrafi and E. Nasibi, Strongly J-clean group rings, *Proc. Romanian Acad., Series A*, **14**(2013), 9–12.

[36] N. Ashrafi and E. Nasibi, Rings in which elements are the sum of an idempotent and a regular element, *Bull. Iranian Math. Soc.*, **39**(2013), 579–588.

[37] N. Ashrafi and N. Pouyan, The unit sum number of discrete modules, *Bull. Iranian Math. Soc.*, **37**(2011), 243–249.

[38] N. Ashrafi and P. Vámos, On the unit sum number of some rings, *Quart. J. Math.*, **56**(2005), 1–12.

[39] E.P. Armendariz, J.W. Fisher and R.L. Snider, On injective and surjective endomorphisms of finitely generated modules, *Comm. Algebra,* **6**(1978), 659–672.

[40] F. Azarpanah, When is $C(X)$ a clean ring? *Acta Math. Hungar.*, **94**(2002), 53–58.

[41] A. Badawi, π-Regular Rings, Ph.D. Thesis, University of North Texas, Texas, 1993.

[42] A. Badawi, On semicommutative π–regular rings, *Comm. Algebra,* **22** (1994), 151–157.

[43] A. Badawi, On abelian π–regular rings, *Comm. Algebra,* **25**(1997), 1009–1021.

[44] A. Badawi, A.Y.M. Chin and H.V. Chen, On rings with near idempotent elements, *Int. J. Pure Appl. Math.*, **1**(2002), 255–261.

[45] C. Bakkari, Clean property in subring retracts, *Int. Math. Forum*, **21** (2009), 1039–1043.

[46] B. Banaschewski, Gelfand and exchange rings: their spectra in point–free topology, *Arabian J. Sci. Engin.,* **25**(2000), 3–22.

[47] B. Banaschewski, Ring theory and pointfree topology, *Topology Appl.*, **137**(2004), 21–37.

[48] B. Banaschewski, pm-Rings and the prime ideal theorem, *Topology Appl.*, **158**(2011), 2340–2342.

[49] B. Banaschewski and J.C.C. Vermeulen, On the sheaf characterization of Gelfand rings, *Quaestiones Mathematicae*, **34**(2011), 137–145.

[50] D.K. Basnet and J. Bhattacharyya, On nil clean index of rings, *Int. Electronic J. Algebra*, **15**(2014) 145–156.

[51] D.K. Basnet and J. Bhattacharyya, Weak clean index of a ring, *Beitr. Algebra Geom.*, **60**(2019), 663–670.

[52] K.I. Beidara, S.K. Jain and P. Kanwar, Nonsingular CS-rings coincide with tight PP rings, *J. Algebra*, **282**(2004), 626–637.

[53] H.E. Bell, On some commutativity theorems of Herstein, *Arch. Math.*, **24**(1973), 34–38.

[54] H.E. Bell, A commutativity study for periodic rings *Pacific J. Math.*, **70**(1977), 29–36.

[55] H.E. Bell, On commutativity of periodic rings and near-rings, *Acta Math. Acad. Sci. Hungar.*, **36**(1980), 293–302.

[56] H.E. Bell, On quasi-periodic rings, *Arch. Math. (Basel)*, **36**(1981), 502–509.

[57] H.E. Bell, On commutativity and structure of periodic rings, *Math. J. Okayama Univ.*, **27**(1985), 1–3.

[58] H.E. Bell, Generalized periodic rings, *Int. J. Math. Math. Sci.*, **19**(1996), 87–92.

[59] H.E. Bell, On J-rings and periodic rings, *East-West J. Math.*, **5**(2003), 45–49.

[60] H.E. Bell and A.A. Klein, On finiteness, commutativity, and periodicity in rings, *Math. J. Okayama Univ.*, **35**(1993), 181–188.

[61] H.E. Bell, H.G. More and A. Yaqub, Generalized n-potent rings, *Math. J. Okayama Univ.*, **37**(1995), 105–111.

[62] H.E. Bell and H. Tominaga, On periodic rings and related rings, *Math. J. Okayama Univ.*, **28**(1986), 101–103.

[63] H.E. Bell and A. Yaqub, Some periodicity conditions for rings, *Math. J. Okayama Univ.*, **29**(1987), 179–183.

[64] H.E. Bell and A. Yaqub, Generalized periodic and generalized Boolean rings, *Int. J. Math. Math. Sci.*, **26**(2001), 457–465.

[65] H.E. Bell and A. Yaqub, On generalized quasi-periodic rings, *Math. Proc. Royal Irish Acad.*, **102A**(2002), 141–148.

[66] H.E. Bell and A. Yaqub, On generalized periodic-like rings, *Int. J. Math. Math. Sci.*, 2007, Article ID 29513, 5p.

[67] H.E. Bell and A. Yaqub, On commutativity semiperiodic rings, *Report Math.*, **53**(2009), 19–26.

[68] S.K. Berberian, Baer *-Rings, Springer-Verlag, Heidelberg, London, New York, 2011.

[69] P. Bhatacharjee, M.L. Knox and W.W. McGovern, The classical ring of quotients of $C_c(X)$, *Appl. Gen. Topol.*, **15**(2014), 147–154.

[70] G.F. Birkenmeier, Quasi-projective modules and the finite exchange property, *Int. J. Math. & Math. Sci.*, **12**(1989), 821–822.

[71] G.F. Birkenmeier, A survey of regularity conditions and the simplicity of prime factor rings, *Vietnam J. Math.*, **23**(1995), 29–38.

[72] G.F. Birkenmeier, H.E. Heatherly and E.K. Lee, Completely prime ideals and associated radicals, in: S.K. Jain, S.T. Rizvi (Eds.), Proceedings of the Biennial Ohio State-Denison Conference, 1992, World Scientific, Singapore, 1993, pp. 102–129.

[73] B. Blackwood, S.K. Jain, K.M. Prasad and A.K. Srivastava, Shorted operators relative to a partial order in a regular ring, *Comm. Algebra*, **37** (2009), 4141–4152.

[74] F. Borceux and G. Van den Bossche, Gelfand rings. In: Algebra in a Localic Topos with Applications to Ring Theory, *Lect. Notes Math.*, **1038**, Springer, Berlin, Heidelberg, 2006, 194–229.

[75] B.E. Borchers, Uniquely Clean Elements, Optimal Sets of Units and Counting Minimal Sets of Units, Ph.D. Thesis, University of Iowa, Iowa, 2015.

[76] G. Borooah, Clean Conditions and Incidence Rings, Ph.D. Thesis, University of California, Berkeley, 2005.

[77] G. Borooah, A.J. Diesl and T.J. Dorsey, Strongly clean triangular matrix rings over local rings, *J. Algebra*, **312**(2007), 773–797.

[78] G. Borooah, A.J. Diesl and T.J. Dorsey, Strongly clean matrix rings over commutative local rings, *J. Pure Appl. Algebra*, **212**(2008), 281–296.

[79] G. Borooah, T.J. Dorsey, T.Y. Lam and P.P. Nielsen, Medium clean rings, preprint, 2005.

[80] D.P. Bossaller, On a Generalization of Clean Rings, M.A. Thesis, Saint Louis University, Saint Louis, 2013.

[81] S. Breaz, P. Danchev and Y. Zhou, Rings in which every element is either a sum or a difference of a nilpotent and an idempotent, *J. Algebra Appl.*, **15**(2016), DOI: 10.1142/S0219498816501486.

[82] S. Breaz and G. Galugareanu, Sums of nilpotent matrices, *Linear Multilinear Algebra*, **65**(2017), 67–78.

[83] S. Breaz, G. Galugareanu, P. Danchev and T. Micu, Nil-clean matrix rings, *Linear Algebra Appl.*, **439**(2013), 3115–3119.

[84] S. Breaz and G.C. Modoi, Nil-clean companion matrices, *Linear Algebra Appl.*, **489**(2016), 50–60.

[85] C. Bu and C. Zhang, A note on the formulas for the Drazin inverse of the sum of two matrices, *Linear Algebra Appl.*, **439**(2013), 565–576.

[86] W.D. Burgess and P. Menal, On strongly π–regular rings and homomorphisms into them, *Comm. Algebra,* **16**(1988), 1701–1725.

[87] W.D. Burgess and R. Raphael, Clean classical rings of quotients of commutative rings, with applications to $C(X)$, *J. Algebra Appl.* **7**(2008), 195–209.

[88] W.D. Burgess and R. Raphael, On commutative clean rings and pm rings, *Contemp. Math.*, **480**(2009), 35–55.

[89] W.D. Burgess and R. Raphael, On embedding rings in clean rings, *Comm. Algebra*, **41**(2013), 552–564.

[90] W.D. Burgess and R. Raphael, Tychonoff spaces and a ring theoretic order on $C(X)$, *Topology Appl.*, **279**(2020), 107250.

[91] M.B. Calci, S. Halicioglu and A. Harmanci, A class of J-quasipolar rings, *Journal of Algebra and Related Topics*, **3**(2015), 1–15.

[92] M.B. Calci, B. Ungor and A. Harmanci, Central quasipolar rings, *Revista Colombiana de Matematicas*, **49**(2015), 281–292.

[93] T.P. Calci, A. Harmanci and B. Ungor, An approach to quasipolarity for rings along nilpotent elements, *Boletinn de la Sociedad Matematica Mexicana*, August 2016, DOI: 10.1007/s40590-016-0145-3.

[94] G. Călugăreanu, Rings with lattices of idempotents, *Comm. Algebra*, **38**(2010), 1050–1056.

[95] G. Călugăreanu, UU rings, *Carpathian J. Math.*, **31**(2015), 157–163.

[96] G. Călugăreanu, Tripotents: a class of strongly clean elements in rings, *An. St. Univ. Ovidius Constanta*, **26**(2018), 69–80.

[97] G. Călugăreanu, UN-rings, *J. Algebra Appl.*, **15 15**(2016), 1650182(9 pages), DOI: 10.1142/S0219498816501826.

[98] G. Călugăreanu and T.Y. Lam, Fine rings: A new class of simple rings, *J. Algebra Appl.*, **15**(2016), 1650173 (18 pages), DOI: 10.1142/S021949 8816501735.

[99] G. Călugăreanu and L. Pop, Between morphic and hopfian, *An. St. Univ. Ovidius Constanta*, **21**(2013), 51–66.

[100] V.P. Camillo, T.J. Dorsey and P. Nielsen, Dedekind-finite strongly clean rings, *Comm. Algebra*, **42**(2014), 1619–1629.

[101] V.P. Camillo and D. Khurana, A characterization of unit regular rings, *Comm. Algebra,* **29**(2001), 2293–2295.

[102] V.P. Camillo, D. Khurana, T.Y. Lam, W.K. Nicholson and Y. Zhou, Continuous modules are clean, *J. Algebra*, **304**(2006), 94–111.

[103] V.P. Camillo, D. Khurana, T.Y. Lam, W.K. Nicholson and Y. Zhou, A short proof that continuous modules are clean, *Contemporary ring theory*, **2011**, 165–169, World Sci. Publ., Hackensack, NJ, 2012.

[104] V.P. Camillo and W.K. Nicholson, Quasi-morphic rings, *J. Algebra Appl.*, **6**(2007), 789–799.

[105] V.P. Camillo, W.K. Nicholson and Z. Wang, Left quasi-morphic rings, *J. Algebra Appl.*, **7**(2008), 725–733.

[106] V. Camillo and J.J. Simón, The Nicholson–Varadarajan theorem on clean linear transformations, *Glasgow Math. J.*, **44**(2002), 365–369.

[107] V. Camillo and H.P. Yu, Exchange rings, units and idempotents, *Comm. Algebra,* **22**(1994), 4737–4749.

[108] V. Camillo and H.P. Yu, Stable range one for rings with many idempotents, *Trans. Amer. Math. Soc.*, **347**(1995), 3141–3147.

[109] R. Camps and W. Dicks, On semilocal rings, *J. Algebra,* **81**(1993), 203–211.

[110] N. Castro-González and J.Y. Vélez-Cerrada, Elements of rings and Banach algebras with related spectral idempotents, *J. Aust. Math. Soc.*, **80**(2006), 383–396.

[111] F. Cedo and L.H. Bowen, Addendum to "Examples of semiperfect rings", *Israel J. Math.*, **107**(1998), 343–348.

[112] M. Chacron, On a theorem of Herstein, *Canad. J. Math.*, **21**(1969), 1348–1353.

[113] A.W. Chatters and S.M. Khuri, Endomorphism rings of modules over nonsingular CS rings, *J. London Math. Soc.*, **21**(1980), 434–444.

[114] A.W. Chatters and S.M. Ginn, Rings generated by their regular elements, *Glasgow Math. J.*, **25**(1984), 1–5.

[115] H. Chen, Exchange rings with artinian primitive factors, *Algebra Represent. Theory*, **2**(1999), 201–207.

[116] H. Chen, Strongly π-regular Morita contexts, *Bull. Korean Math. Soc.*, **40**(2003), 91–99.

[117] H. Chen, Clean matrices over commutative rings, *Czechoslovak Math. J.*, **59**(2009), 145–158.

[118] H. Chen, On strongly J-clean rings, *Comm. Algebra*, **38**(2010), 3790–3804.

[119] H. Chen, On uniquely clean rings, *Comm. Algebra*, **39**(2011), 189–198.

[120] H. Chen, *Rings Related Stable Range Conditions*, Series in Algebra 11, World Scientific, Hackensack, NJ, 2011.

[121] H. Chen, A note on strongly clean matrices, *Int. Electronic J. Algebra*, **10**(2011), 192–204.

[122] H. Chen, Some strongly nil clean matrices over local rings, *Bull. Korean Math. Soc.*, **48** (2011), 759–767.

[123] H. Chen, Strongly J-clean matrices over local rings, *Comm. Algebra*, **40** (2012), 1352–1362.

[124] H. Chen, Strongly nil clean matrices over $R[x]/(x^2-1)$, *Bull. Korean Math. Soc.*, **49** (2012), 589–599.

[125] H. Chen, On strongly nil clean matrices, *Comm. Algebra*, **41**(2013), 1074–1086.

[126] H. Chen, On 2×2 strongly clean matrices, *Bull. Korean Math. Soc.*, **50** (2013), 125–134.

[127] H. Chen, Some classes of strongly clean rings, *Bull. Iranian Math. Soc.*, **39** (2013), 1099–1115.

[128] H. Chen, Strongly clean matrices over commutative domains, *Algebra Colloq.*, **21** (2014), 257–266.

[129] H. Chen, N. Ashrafi and M. Sheibani, Regularity and related rings, *Mediterr. J. Math.*, **15**(2018). Doi: 10.1007/s00009-018-1068-1.

[130] H. Chen and M. Chen, On strongly π-regular ideals, *J. Pure Appl. Algebra*, **195**(2005) , 21–32.

[131] H. Chen, O. Gurgun and H. Kose, Strongly clean matrices over commutative local rings, *J. Algebra Appl.*, **12**(2013), 1250126.

[132] H. Chen and A. Harmanci, Strongly *-clean properties and rings of function, *Commun.Fac.Sci.Univ.Ank.Series A1*, **67**(2018), 102–115.

[133] H. Chen, A. Harmanci and A.C. Ozcan, Strongly J-clean rings with involutions, *Contemp. Math.*, **609**(2014), 33–44.

[134] H. Chen, H. Kose and O. Gurgun, Uniquely strongly clean triangular matrice, *Turkish J. Math.*, **39**(2015), 645–649.

[135] H. Chen, H. Kose and Y. Kurtulmaz, Extensions of strongly π-regular rings, *Bull. Korean Math. Soc.*, **51**(2014), 555–565.

[136] H. Chen, H. Kose and Y. Kurtulmaz, On feckly clean rings, *J. Algebra Appl.*, **14**(2015), 1550046.

[137] H. Chen, H. Kose and Y. Kurtulmaz, Strongly clean matrices over power series, *Kyungpook Math. J.*, **56** (2016), 387–396.

[138] H. Chen, H. Kose and Y. Kurtulmaz, Strongly rad-clean matrices over commutative local rings, arXiv:2201.09276v1 [math.RA] 23 Jan 2022.

[139] H. Chen, S.S. Nabavi and M. Sheibani, Strongly Drazin Inverses in rings, *Operators and Matrices*, **13**(2019), 495–505.

[140] H. Chen and M. Sheibani, Elementary matrix reduction over Zabavsky rings, *Bull. Korean Math. Soc.*, **53**(2016), 195–204.

[141] H. Chen and M. Sheibani, Priodicity and J-clean-like rings, *Math. Reports*, **18**(68)(2016), 515–533.

[142] H. Chen and M. Sheibani, On strongly nil clean rings, *Comm. Algebra*, **45**(2017), 1719–1726.

[143] H. Chen and M. Sheibani, On strongly unit nil-clean rings, *J. Algebra Appl.*, **16**((2017), 1750115.

[144] H. Chen and M. Sheibani, On strongly 2-nil-clean rings, *J. Algebra Appl.*, **16**(2017), 1750178.

[145] H. Chen and M. Shebani, Rings in which the power of each element is the sum of an idempotent and a unit, *Publications de'l Institut Mathematique*, **102(116)** (2017), 133–148.

[146] H. Chen and M. Sheibani, Strongly weakly nil-clean rings, *J. Algebra Appl.*, **16**(2017), 1750233.
[147] H. Chen and M. Sheibani, Matrices over Zhou nil-clean rings, *Comm. Algebra*, **46**(2018), 1527–1533.
[148] H. Chen and M. Sheibani, Diagonal matrix reduction over Hermite rings, *Linear and Multilinear Algebra*, **66**(2018), 681–691.
[149] H. Chen and M. Sheibani, Structure of Zhou nil-clean rings, *Algebra Colloq.*, **25**(2018), 361–368.
[150] H. Chen and M. Sheibani, Generalized Drazin inverses in a ring, *Filomat*, **32**(2018): 5289–5295.
[151] H. Chen and M. Sheibani, Rings whose every subring is feebly clean, *Bull. Iran Math. Soc.*, **45**(2019). 257–266.
[152] H. Chen and M. Sheibani, Rings additively generated by tripotents and nilpotents, *J. Algebra Appl.*, **18**(2019), 1950050.
[153] H. Chen and M. Sheibani, Strongly 2-nil-clean rings with involutions, *Czechoslovak Math. J.*, **69 (144)**(2019), 317–330.
[154] H. Chen and M. Sheibani, On generalized Hirano inverses in rings, *Comm. Algebra*, **47**(2019), 2967–2978.
[155] H. Chen and M. Sheibani, On the matrix which is the sum of a tripotent and a quasinilpotent matrices, *Operators and Matrices*, **13**(2019), 407–419.
[156] H. Chen and M. Sheibani, Elementary matrix reduction over Bezout domains, *J. Algebra Appl.*, **18** (2019), 1950141.
[157] H. Chen and M. Sheibani, Cline's formula for g-Drazin inverses in a ring, *Filomat*, **33**(2019), 2249–2255.
[158] H. Chen and M. Sheibani, Generalized Hirano inverses in Banach algebras, *Filomat*, **33** (2019), 6239–6249.
[159] H. Chen and M. Sheibani, g-Drazin inverses for operator matrices, *Operators and Matrices*, **14**(2020), 23–31.
[160] H. Chen and M. Sheibani, Expressionns for the g-Drazin inverse in a Banach algebra, *Filomat*, **34**(2020), 3845–3854.
[161] H. Chen and M. Sheibani, The g-Hirano inverse in Banach algebras, *Linear and Multilinear Algebra*, **69**(2021), 1352–1362.
[162] H. Chen and M. Sheibani, Generalized Cline's formula and common spectral property, *J. Algebra Appl.*, **20**(2021), 2150094.
[163] H. Chen and M. Sheibani, Rings additively generated by idempotents and nilpotents, *Comm. Algebra*, **49**(2021), 1781–1787.
[164] H. Chen and M. Sheibani, Generalized Jacobson's lemma in a Banach algebra, *Comm Algebra*, **49**(2021), 3263–3272.
[165] H. Chen and M. Sheibani, The g-Drazin inverses of special operator matrices, *Operators and Matrices*, **15**(2021), 151–162.

[166] H. Chen and M. Sheibani, On generalized Zhou inverses in rings, *Comm. Algebra*, **49**(2021), 4098–4108.

[167] H. Chen and M. Sheibani, Generalized Cline's formula for g-Drazin inverse in a ring, *Filomat*, **35**(2021), 2573–2583.

[168] H. Chen and M. Sheibani, Generalized Jacobson's lemma for generalized Drazin inverses, *Filomat*, **35**(2021), 2267–2276.

[169] H. Chen and M. Sheibani, The g-Drazin inverse of the sum in Banach algebras, *Linear and Multilinear Algebra*, Jan 2020. Doi: 10.1080/03081087.2019.1710102.

[170] H. Chen and M. Sheibani, Cline's formula and Jacobson's lemma for g-Drazin inverse, *Filomat*, **35**(2021), 5083–5091.

[171] H. Chen and M. Sheibani, The g-Drazin invertibility in a Banach algebra, arXiv:2203.07568v1 [math.RA] 15 Mar 2022.

[172] H. Chen, M. Sheibani and N. Ashrafi, Rings consisting entirely of certain elements, *Czechoslovak Math. J.*, **68 (143)** (2018), 553–558.

[173] H. Chen, M. Sheibani and N. Ashrafi, Rings generalized by tripotents and nilpotents, *Czechoslovak Math. J.*, 2022 (to appear).

[174] H. Chen, M. Sheibani and H. Kose, On medium *-clean rings, *Mediterr. J. Math.*, **16** (2019). Doi: 10.1007/s00009-018-1297-3.

[175] H. Chen, M. Sheibani and H. Kose, The g-Drazin inverse involving power commutativity, *Filomat*, **34**(2020), 2961–2969.

[176] H. Chen and H. Zou, On *-DMP inverses in a ring with involution, *Comm. Algebra*, **49**(2021), 5006–5016.

[177] J. Chen and J. Cui, Two questions of L. Vaš on *-clean rings, *Bull. Australian Math. Soc.*, **88**(2013), DOI:10.1017/S0004972713000117.

[178] J. Chen, D.D. Mosi and S. Xu, On a new generalized inverse for Hilbert space operators, *Quaestiones Mathematicae*, **43**(2020), 1331–1348.

[179] J. Chen, W.K. Nicholson and Y. Zhou, Group rings in which every element is uniquely the sum of a unit and an idempotent, *J. Algebra*, **306**(2006), 453–460.

[180] J. Chen and Z. Wang, Some progress on clean rings, Marubayashi, H. (ed.) et al., Ring theory 2007. Proceedings of the 5th China-Japan-Korea conference, Tokyo, Japan, September 10–15, 2007. Hackensack, NJ: World Scientific. 38–53 (2009).

[181] J. Chen, Z. Wang and Y. Zhou, Rings in which elements are uniquely the sum of an idempotent and a unit that commute, *J. Pure Appl. Algebra*, **213**(2009), 215–223.

[182] J. Chen, X. Yang and Y. Zhou, When is the 2×2 matrix ring over a commutative local ring strongly clean? *J. Algebra*, **301**(2006), 280–293.

[183] J. Chen and Y. Zhou, Morphic rings as trivial extensions, *Glasgow Math. J.*, 47(2004), 139–148.

[184] J. Chen, X. Yang and Y. Zhou, On strongly clean matrix and triangular matrix rings, *Comm. Algebra*, **34**(2006), 3659–3674.

[185] J. Chen and Y. Zhou, Strongly clean power series rings, *Proc. Edinb. Math. Soc.*, **50**(2007), 73–85.

[186] W. Chen, *P*-clean rings, *Int. J. Math. & Math. Sci.*, 2006, No. 5, Article ID 75247, 7 p. (2006).

[187] W. Chen, A question on strongly clean rings, *Comm. Algebra*, **34**(2006), 2347–2350.

[188] W. Chen, On semiabelian π-regular rings, *Int. J. Math. & Math. Sci.*, 2007, Article ID 63171, 10 p. (2007).

[189] W. Chen and S. Cui, On clean rings and clean elements, *Southeast Asian Bull. Math.*, **32**(2008), 855–861.

[190] W. Chen, Exchange general rings with bounded indices are clean, *Thai J. Math.*, **7**(2009), 353–359.

[191] M. Chhiti, N. Mahdou and M. Tamekkante, Clean property in amalgamated algebras along an ideal, *Hacettepe J. Math. Statistics*, **44**(2015), 41–49.

[192] A.Y.M. Chin, A note on strongly π-regular rings, *Acta Math. Hungar.*, **102** (2004), 337–341.

[193] A.Y.M. Chin, Clean elements in abelian rings, *Proc. Indian Acad. Sci. (Math. Sci.)*, **119**(2009), 145–148.

[194] A.Y.M. Chin and H.V. Chen, On strongly π-regular group rings, *Southeast Asian Bull. Math.*, **26**(2002), 387–390.

[195] A.Y.M. Chin and K.T. Qua, A note on weakly clean rings, *Acta Math. Hung.*, **132**(2011), 113–116.

[196] A.Y.M. Chin and K.T. Qua, A note on n-clean group rings, *Publ. Math.*, **78**(2011), 569-574.

[197] A.Y.M. Chin and K.T. Qua, Some properties of clean rings and their generalisations, *AIP Conference Proceedings*, **1522**(2013), 915–919.

[198] A.Y.M. Chin and K.T. Qua, Some properties of n-weakly clean rings, *AIP Conference Proceedings 1065*, **644**(2014), DOI: 10.1063/1.4887665.

[199] M.Q. Chu, X.H. Xu and Y. Gan, A note on the unit π-regular rings, *J. Anhui Normal Univ.*, **35**(2012), 511-513.

[200] A. Cîmpean and P. Danchev, Weakly nil-clean index and uniquely weakly nil-clean rings, *Int. Electronic J. Algebra*, **21**(2017) 180–197.

[201] J. Clark, C. Lomp, N. Vanaja and R. Wisbauer, *Lifting Modules: Supplements and Projectivity in Module Theory*, Frontiers in Mathematics, Springer, 2006.

[202] P.M. Cohn, Some remarks on projective-free rings, *Algebra Univers.*, **49**(2003), 159–164.
[203] P.M. Cohn, Another criterion for a ring to be projective-free, *Bull. Lond. Math. Soc.*, **37**(2005), 857–859.
[204] M. Contessa, On some properties of PM-rings and MP-rings, *C. R. Math. Acad. Sci., Soc. R. Can.*, **3**(1981), 357–362.
[205] M. Contessa, On pm-rings, *Comm. Algebra*, **10**(1982), 93–108.
[206] M. Contessa, On certain classes of pm-rings, *Comm. Algebra*, **12**(1984), 1447–1469.
[207] F. Couchot, Indecomposable modules and Gelfand rings, *Comm. Algebra*, **35** (2007), 231–241.
[208] F. Couchot, Strong cleanness of matrix rings over commutative rings, *Comm. Algebra*, **36**(2008), 346–351.
[209] F. Couchot, Almost clean rings and arithmetical rings, Fontana, Marco (ed.) et al., Commutative algebra and its applications. Proceedings of the fifth international Fez conference on commutative algebra and applications, Fez, Morocco, June 23–28, 2009.
[210] F. Couchot, Finitely generated powers of prime ideals, *Palest. J. Math.*, **6**(2017), 372–379.
[211] J. Cui, A Class of Strongly Clean Rings, Ph.D. Thesis, Southeast University, Nanjing, 2011.
[212] J. Cui and J. Chen, Strongly clean 3×3 matrices over a class of local rings, *J. Nanjing Univ., Math. Biquart.*, **27**(2010), 31–40.
[213] J. Cui and J. Chen, Uniquely strongly clean triangular matrix rings, *J. Southeast Univ. (English Edition)*, **27**(2011), 463–465.
[214] J. Cui and J. Chen, When is a 2×2 matrix ring over a commutative local ring quasipolar? *Comm. Algebra*, **39**(2011), 3212–3221.
[215] J. Cui and J. Chen, Quasipolar triangular matrix rings over local rings, *Comm. Algebra*, **40**(2012), 784–794.
[216] J. Cui and J. Chen, A class of quasipolar rings, *Comm. Algebra*, **40**(2012), 4471–4482.
[217] J. Cui and J. Chen, Notes on quasipolar rings, *J. Math. Study*, **45**(2012), 167–174.
[218] J. Cui and J. Chen, Characterizations of quasipolar rings, *Comm. Algebra*, **41**(2013), 3207–3217.
[219] J. Cui and J. Chen, Pseudopolar matrix rings over local rings, *J. Algebra Appl.*, **13**(2014), 1350109 [12 pages], DOI: 10.1142/S0219498813501090.
[220] J. Cui and Z. Wang, A note on strongly *-clean rings, *J. Korean Math. Soc.*, **52**(2015), 839–851.
[221] D.S. Cvetkovic-Ilic, Some results on the $(2, 2, 0)$ Drazin inverse problem, *Linear Algebra Appl.*, **438**(2013), 4726–4741.

[222] D.S. Cvetkovic-Ilic and R. Harte, On Jacobson's lemma and Drazin invertibility, *Applied Math. Letters*, **23**(2010), 417–420.

[223] P.V. Danchev, When is an abelian weakly clean ring is clean? *J. Indones. Math. Soc.*, **21**(2015), 83–91.

[224] P.V. Danchev, A new characterization of Boolean rings with identity, *Irish Math. Soc. Bull.*, **76**(2015), 55–60.

[225] P.V. Danchev, A class of weakly nil-clean rings, *Irish Math. Soc. Bull.*, **77**(2016), 9–17.

[226] P.V. Danchev, Weakly UU rings, *Tsukuba J. Math.*, **40**(2016), 101–118.

[227] P.V. Danchev, Nil-good unital rings, *Int. J. Algebra*, **10**(2016), 239–252.

[228] P.V. Danchev, Weakly clean WUU rings, *Int. J. Algebra*, **11**(2017), 9–14.

[229] P.V. Danchev and M. Samiei, Commutative weakly nil-neat group rings, *Comm. Algebra*, **48**(2020), 2382–2387.

[230] P.V. Danchev and W.W. McGovern, Commutative weakly nil clean unital rings, *J. Algebra*, **425**(2015), 410–422.

[231] C. Deng, D.S. Cvetkovic-Ilic and Y. Wei, Some results on the generalized Drazin inverse of operator matrices, *Linear Multilinear Algebra*, **58**(2010), 503–521.

[232] C. Deng and Y. Wei, A note on the Drazin inverse for an anti-triangular matrix, *Linear Algebra Appl.*, **431**(2009), 1910–1922.

[233] A.J. Diesl, Classes of Strongly Clean Rings, Ph.D. Thesis, University of California, Berkeley, 2006.

[234] A.J. Diesl, A note on completeness in the theory of strongly clean rings, *J. Pure Appl. Algebra*, **218**(2014), 661–665.

[235] A.J. Diesl, Nil clean rings, *J. Algebra*, **383**(2013), 197–211.

[236] A.J. Diesl and T.J. Dorsey, Strongly clean matrices over arbitrary rings, *J. Algebra*, **399**(2014), 854–869.

[237] A.J. Diesl, T.J. Dorsey, W. Iberkleid, R. LaFuente-Rodriguezd and W.W. McGovern, Strongly clean triangular matrices over abelian rings, *J. Pure Appl. Algebra*, **219**(2015), 4889–4906.

[238] A.J. Diesl and D.R. Shifflet, Unifying strongly clean power series rings, *Comm. Algebra*, **46**(2018), 4448–4462.

[239] O.V. Domsha and I.S. Vasiunyk, Combining local and adequate rings, *Book of abstracts of the International Algebraic Conference*, Taras Shevchenko National University of Kyiv, Kyiv, Ukraine, 2014.

[240] E. Dopazo and M.F. Martinez-Serrano, Further results on the representation of the Drazin inverse of a 2×2 block matrix, *Linear Algebra Appl.*, **432**(2010), 1896–1904.

[241] T.J. Dorsey, Cleanness and Strong Cleanness of Rings of Matrices, Ph.D. Thesis, University of California, Berkeley, 2006.
[242] M.P. Drazin, Generalizations of Fitting's lemma in arbitrary associative rings, *Comm. Algebra*, **29**(2001), 3647–3675.
[243] M.P. Drazin, Commuting properties of generalized inverses, *Linear Multilinear Algebra*, **61**(2013), 1675–1681.
[244] X. Du, The adjoint semigroup of a ring, *Comm. Algebra*, **30**(2002), 4507–4525.
[245] D. Eisenbud, *Commutative Algebra With a View Toward Algebraic Geometry*, Graduate Texts in Mathematics 150, Springer-Verlag, 1995.
[246] E.W. Ellers and H. Ishibashi, Factorization of transformations over a local rings, *Linear Algebra Appl.*, **85**(1987), 17–27.
[247] E.G. Evans, Jr., Krull–Schmidt and cancellation over local rings, *Pacific J. Math.*, **46**(1973), 115–121.
[248] N. Er, S. Singh and A.K. Srivastava, Rings and modules which are stable under automorphisms of their injective hulls, *J. Algebra*, **379**(2013), 223–229.
[249] D.R. Estes and R.M. Guralnick, Module equivalences: local to global when primitive polynomials represent units, *J. Algebra*, **77**(1982), 138–157.
[250] L. Fan, Algebraic Analysis of Some Strongly Clean Rings and Their Generalizations, Ph.D. Thesis, Memorial University of Newfoundland, Newfoundland, 2009.
[251] L. Fan and X. Yang, On strongly clean matrix rings, *Glasgow Math. J.*, **48**(2006), 557–566.
[252] L. Fan and X. Yang, On rings whose elements are the sum of a unit and a root of a fixed polynomial, *Comm. Algebra*, **36**(2008), 269–278.
[253] L. Fan and X. Yang, Strongly clean matrix rings over commutative rings, arXiv: 0803.2176v3 [math.RA] 20 Aug 2008.
[254] L. Fan and X. Yang, On strongly $g(x)$-clean rings, arXiv: 0803.3353v1 [math.RA] 24 Mar 2008.
[255] L. Fan and X. Yang, Strongly clean property and stable range one, *Comm. Algebra*, **37**(2009), 2021–2029.
[256] L. Fan and X. Yang, A note on strongly clean matrix rings, *Comm. Algebra*, **38**(2010), 799–806.
[257] J.S. Fernandez, The Prime Spectrum of A Ring: A Survey, M.S. Thesis, Florida Atlantic University, Florida, 1991.
[258] J.W. Fisher and R.L. Snider, Rings generated by their units, *J. Algebra*, **42** (1976), 363–368.
[259] N. Ganesan, Properties of rings with a finite number of zero divisors, *Math. Ann.*, **157**(1964), 215–218.

[260] Y. Gao, J. Chen and Y. Li, Some *-clean group rings, *Algebra Colloq.*, **22**(2015), **22** 169–180.

[261] J.L. Garcia, Properties of direct summands of modules, *Comm. Algebra*, **17**(1989), 73–92.

[262] N.J. Ghandour, Structure of Certain Weakly Periodic Rings and Other Rings with Condition on Elements, M.S. Thesis, The American University of Beirut, Beirut, 2000.

[263] M. Ghanem, Some extensions of commutative clean rings, *Int. Math. Forum*, **6**(2011), 1567–1572.

[264] E. Ghashghaei and W.W. McGovern, Fusible Rings, *Comm. Algebra*, **45**(2017), 1151–1165, DOI: 10.1080/00927872.2016.1206347.

[265] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Springer-Verlag, New York, Heidelberg, Berlin, London, 1976.

[266] R. Gilmer, *Commutative Semigroup Rings*, Chicago Lect. Math., University of Chicago Press, Chicago, IL, 1984.

[267] R. Gilmer, Commutative rings with periodic multiplicative semigroups, *Comm. Algebra*, **21**(1993), 4025–4028.

[268] S. Glaz, Commutative Coherent Rings, *Lect. Notes Math.*, **1371**, Springer-Verlag, New York, 1989.

[269] B. Goldsmith, S. Pabst and A. Scott, Unit sum numbers of rings and modules, *Quart. J. Math.*, **49**(1998), 331–344.

[270] B. Goldsmith and P. Vamos, A note on clean abelian groups, *Rend. Semin. Mat. Univ. Padova*, **117**(2007), 181–191.

[271] K.R. Goodearl, Extensions of simple by simple unit-regular rings, *Lect. Notes Math.*, **825**(2980), 42–58.

[272] K.R. Goodearl, Surjective endomorphisms of finitely generated modules, *Comm. Algebra*, **15**(1987), 589–609.

[273] K.R. Goodearl, *Von Neumann Regular Rings*, Pitman, Krieger, Malabar, Fl., 1991.

[274] K.R. Goodearl, von Neumann regular rings and direct sum decomposition problems, *Facchini A. and Menini C. (eds.), Abelian Groups and Modules, Dordrecht: Kluwer Academic Publishers, Math. Appl., Dordredt*, **343**, 1995, 249–255.

[275] A.B. Gorman and D. Alexander, Ideally nil clean rings, *Comm. Algebra*, **49**(2021), 4788–4799.

[276] S. Grag, H.K. Grover and D. Khurana, Perspective rings, *J. Algebra*, **415**(2014), 1–12.

[277] H.K. Grover, D. Khurana and S. Singh, Rings with multiplicative sets of primitive idempotents, *Comm. Algebra*, **37**(2009), 2583–2590.

[278] H.K. Grover, Z. Wang, D. Khurana, J. Chen and T.Y. Lam, Sums of units in rings, *J. Algebra Appl.*, 13, 1350072 (2014)[10 pages], DOI: 10.1142/ S0219498813500722.

[279] A.N. Guèye, C.T. Guèye and M. Sangharé, A new characterization of commutative strongly π-regular rings, *J. Math. Res.*, **4**(2012), 30–33.

[280] R.N. Gupta, A. Khurana, D. Khurana and T.Y. Lam, Rings over which the transpose of every invertible matrix is invertible, *J. Algebra*, **322**(2009), 1627–1636.

[281] O. Gurgun, Extensions of quasipolar rings, *Turkish J. Math.*, January 2017, DOI: 10.3906/mat-1410-35.

[282] O. Gurgun, Properties of generalized strongly Drazin invertible elements in general rings, *J. Algebra Appl.*, 16, 1750207 (2017) [13 pages], DOI: 10.1142/S0219498817502073.

[283] O. Gurgun, S. Halicioglu and A. Harmanci, Quasipolar subrings of 3×3 matrix rings, *An. St. Univ. Ovidius Constanta*, **21**(2013), 133–146.

[284] O. Gurgun, S. Halicioglu and A. Harmanci, Nil-quasipolar rings, *Boletin de la Sociedad Matematica Mexicana*, **20**(2014), 29–38.

[285] O. Gurgun, S. Halicioglu and A. Harmanci, Strong J-cleanness of formal matrix rings, *Adv. Stud. Contemp. Math.* (Kyungshang), **24**(2014), 483–498.

[286] O. Gurgun, S. Halicioglu and A. Harmanci, Quasipolarity of special Morita context rings, *Miskolc Math. Notes*, **19**(2018), 273–289.

[287] O. Gurgun, S. Halicioglu and B. Ungor, A subclass of strongly clean rings, *Comm. Math.*, **23**(2015), 13–31.

[288] O. Gurgun and A.C. Ozczn, A class of uniquely (strongly) clean rings, *Turkish J. Math.*, **38**(2014), 40–51.

[289] S. Hadjirezaei and S. Karimzadeh, On the nil-clean matrix over a UFD, *Alg. Struc. Appl.*, **2**(2015), 49–55.

[290] S. Hadjirezaei and S. Karimzadeh, Matrix rings over a principal ideal domain in which elements are nil-clean, *J. Alg. Comb. Discrete Appl.*, **3**(2016), 91–96.

[291] A.W. Hager and C.M. Kimber, Clean rings of continuous functions, *Algebra Universalis*, **56**(2007), 77–92.

[292] A.W. Hager, C.M. Kimber and W.W. McGovern, Clean unital ℓ-groups, *Math. Slovaca*, **63**(2013), 979–992.

[293] A. Haghany, Hopficity and co-hopficity for Morita contexts, *Comm. Algebra*, **27**(1999), 477–492.

[294] F.J. Hall, R.E. Hartwig, I.J. Katz and M. Newman, Pseudo-similarity and partial unit regularity, *Czechoslovak Math. J.*, **33**(1983), 361–372.

[295] D. Han, Y. Ren and H. Zhang, On *-clean group rings over abelian groups, *J. Algebra Appl.*, **16**(2017), 1750152.

[296] J. Han and W.K. Nicholson, Notes on clean rings. In *Proceedings of the 31st Symposium on Ring Theory and Representation Theory and Japan–Korea Ring Theory and Representation Theory Seminar (Osaka, 1998),* Shinshu Univ., Matsumoto 1999, 23–32.

[297] J. Han and W.K. Nicholson, Extensions of clean rings, *Comm. Algebra,* **29**(2001), 2589–2595.

[298] A.H. Handam, $\big(n, g(x)\big)$-Clean rings, *Int. Math. Forum*, **4**(2009), 1007–1011.

[299] A.H. Handam, On f-clean rings and f-clean elements, *Proyecciones J. Math.*, **30**(2011), 277–284.

[300] D. Handelman, Perspectivity and cancellation in regular rings, *J. Algebra*, **48** (1977), 1–16.

[301] D.J. Hansen and J. Luh, J-Rings of characteristic two that are Boolean, *Int. J. Math. Math. Sci.*, **17**(1994), 807–812.

[302] F. Harary, The Structure of Boolean-like Rings, Ph.D. Thesis, University of California, Berkeley, 1949.

[303] A. Harmanci, H. Chen and A.C. Ozcan, Strongly nil-*-clean rings, *J. Alg. Comb. Discrete Appl.*, **4**(2017), 155–164.

[304] R.E. Harte, On quasinilpotents in rings, *Panam. Math. J.*, **1**(1991), 10–16.

[305] E. Hashemi, M. Hamidizadeh and A. Alhevaz, On clean and regular elements of noncommutative ring extensions, *Comm. Algebra*, **47**(2019), 1650–1661.

[306] A.M. Hassanein, Clean rings of skew Hurwitz series, *Le Matematiche*, **LXII**(2007), 47–54.

[307] M. Henriksen, On a class of regular rings that are elementary divisor rings, *Arch. Math.*, **24**(1973), 133–141.

[308] M. Henriksen, Two classes of rings generated by their units, *J. Algebra*, **31**(1974), 182–193.

[309] M. Henriksen and M. Jerison, The space of minimal prime ideals of a commutative ring, *Trans. Amer. Math. Soc.*, **115**(1965), 110–130.

[310] I.N. Herstein, An elementary proof of a theorem of Jacobson, *Duke Math. J.*, **21**(1954), 45–48.

[311] B. Herwig and M. Ziegler, A remark on sums of units, *Arch. Math.*, **79** (2002), 430–431.

[312] Y. Hirano, Some studies on strongly π–regular rings, *Math. J. Okayama Univ.*, **20**(1978), 141–149.

[313] Y. Hirano, On generalized p.p. rings, *Math. J. Okayama Univ.*, **25**(1983), 7–11.

[314] Y. Hirano, Some characterizations of π–regular rings of bounded index, *Math. J. Okayama Univ.*, **32** (1990), 97–101.

[315] Y. Hirano and H. Komatsu, A characterization of Boolean rings (III), *Math. J. Okayama Univ.*, **27**(1985), 33–34.

[316] Y. Hirano and H. Tominaga, Rings in which every element is a sum of two idempotents, *Bull. Austral. Math. Soc.*, **37**(1988), 161–164.

[317] Y. Hirano, H. Tominaga and A. Yaqub, On rings in which every element is uniquely expressible as a sum of a nilpotent element and a certain potent element, *Math. J. Okayama Univ.*, **30**(1988), 33–40.

[318] V.A. Hiremath and S. Hedge, On strongly clean rings, *Int. J. Algebra*, **5** (2011), 31–36.

[319] V.A. Hiremath and S. Hegde, On a subclass of semipotent rings, *Int. Res. J. Pure Algebra*, **2**(2012), 66–70.

[320] V.A. Hiremath and S. Hegde, Using ideals to provide a unified approach to uniquely clean rings, *J. Aust. Math. Soc.*, **96**(2014), 258–274.

[321] C.Y. Hong, N.K. Kim and Y. Lee, Exchange rings and their extensions, *J. Pure Appl. Algebra*, **179**(2003), 117–126.

[322] C.Y. Hong, N.K. Kim, T.K. Kwak and Y. Lee, On weak π-regularity of rings whose prime ideals are maximal, *J. Pure Appl. Algebra*, **146**(2000), 35–44.

[323] C.Y. Hong, N.K. Kim and T.K. Kwak, On the maximality of prime ideals in exchange rings, *Comm. Korean Math. Soc.*, **17**(2002), 409–422.

[324] S.R. Hosseini, A Study of Clean Rings, M.S. Thesis, University of Calgary, Calgary, 2011.

[325] M.H. Hosseini, A.R. Sedighi and R.A.B. Gaz, Strongly P-clean rings, *22nd Iranian Algebra Seminar*, 2012, 330–333.

[326] H. Huang, Y. Li and P. Yuan, On *-clean group rings II, *Comm. Algebra*, **44**(2016), 3171–3181.

[327] Q. Huang and G. Tang, Quasipolar property of trivial Morita contexts, *Int. J. Pure Appl. Math.*, **90**(2014), 423–431.

[328] Q. Huang, G. Tang and Y. Zhou, Quasipolar property of generalized matrix rings, *Comm. Algebra*, **42**(2014), 3883–3894.

[329] C. Huh, N.K. Kim and Y. Lee, On exchange rings with primitive factor rings artinian, *Comm. Algebra*, **28**(2000), 4989–4993.

[330] C. Huh, N.K. Kim and Y. Lee, Examples of strongly π-regular rings, *J. Pure Appl. Algebra*, **189**(2004), 195–210.

[331] D. Huylebrouck, Equivalence of von Neumann regular and idempotent matrices, *Czechoslovak Math. J.*, **41**(1991), 44–50.

[332] D. Huylebrouck and J. van Geel, Diagonalisation of idempotent matrices, *J. Algebra*, **105**(1987), 196–206.

[333] C.J. Hwang, A note on real quaternion, *Korean J. Math.*, **17**(2009), 245–248.

[334] S.U. Hwang, Y.C. Jeon and Y. Lee, Structure and topological conditions of NI rings, *J. Algebra*, **302**(2006), 186–199.

[335] W. Iberkleid and W.W. McGovern, Classes of commutative clean rings, *Ann. Fac. Sci. Toulouse Math.*, **19**(2010), 101–110.

[336] Y. Ihsen, The Hermite ring conjecture in dimension one, *J. Algebra*, **320** (2008), 437–441.

[337] N.A. Immormino, Clean Rings & Clean Group Rings, Ph.D. Thesis, Bowling Green State University, Bowling Green, 2013.

[338] N.A. Immormino and W.W. McGovern, Examples of clean commutative group rings, *J. Algebra*, **405**(2014), 168–178.

[339] A. Ismarwati, H. France-Jackson, S. Wahyuni and I. E. Wijayanti, On nice modules, *JP J. Algebra, Number Theory Appl.*, **38**(2016), 213–225.

[340] H.W. Itani, Structure of Rings with Conditions Involving Periodic, Potent, and Central Elements, M.S. Thesis, The American University of Beirut, Beirut, 2009.

[341] N. Jacobson, *Structure of Rings*, Providence, R.I.: Amer. Math. Soc. VII, 1956.

[342] M. Jarden and W. Narkiewicz, On sums of units, *Monatsh. Math.*, **150** (2007), 327–332.

[343] P.T. Johnstone, *Stone Spaces*, Cambridge Stud. Adv. Math., **3**, Cambridge Univ. Press, Cambridge, 1982.

[344] P. Kanwar, A. Leroy and J. Matczuk, Idempotents in ring extensions, *J. Algebra*, **389**(2013), 128–136.

[345] P. Kanwar, A. Leroy and J. Matczuk, Clean elements in polynomial rings, *Contemp. Math.*, **634**(2015), 197–204.

[346] I. Kaplansky, Elementary divisors and modules, *Trans. Amer. Math. Soc.*, **66**(1949), 464–491.

[347] A. Karimi-Mansoub, T. Kosan and Y. Zhou, Rings in which every units is the sum of a nilpotent and an idempotent, *Contemporary Mathematics*, **715**(2018), 189–203.

[348] A.M. Karparvar, B. Amini, A. Amini and H. Sharif, Additive decomposition of ideals, *J. Algebra Appl.*, **17**(2018), 1850085 (17 pages).

[349] H.A. Khashan and W. Burhan, Cleanness of overrings of polynomial rings, *J. Egyptian Math. Soc.*, **24**(2016), 1–4.

[350] H.A. Khashan and A.H. Handam, $g(x)$-nil clean rings, *Scienticae Mathematicae Japonicae*, **79**(2016), 145–154.

[351] R.R. Khazal, Periodic Rings, Ph.D. Thesis, University of Southwestern Louisiana, Lafayette, 1976.

[352] S.Y. Khorbatli, Structure of Certain Periodic Rings, Weakly Periodic Rings and Other Types of Rings, M.S. Thesis, The American University of Beirut, Beirut, 2006.

[353] D. Khurana, Unit-regularity of regular nilpotent elements, *Algebras Represent. Theory*, **19**(2016), 641-644.

[354] A. Khurana, D. Khurana and P.P. Nielsen, Sums of units in self-injective rings, *J. Algebra Appl.*, **13**(2014), 14500200.

[355] D. Khurana and C. Kumar, Group rings that are additively generated by idempotents and units, arXiv: 0904.0861v1 [math.RA] 6 Apr 2009.

[356] D. Khurana and T.Y. Lam, Clean matrices and unit-regular matrices, *J. Algebra*, **280**(2004), 683–698.

[357] D. Khurana and T.Y. Lam, Rings with internal cancellation, *J. Algebra*, **284** (2005), 203–235.

[358] D. Khurana and T.Y. Lam, Generalized commutators in matrix rings, *Linear Multilinear Algebra*, **60**(2012), 797–827.

[359] D. Khurana, T.Y. Lam and P.P. Nielsen, Two-sided properties of elements in exchange rings, *Algebra Represent. Theory*, **18**(2015), DOI: 10.1007/s10468- 015-9524-0.

[360] D. Khurana, T.Y. Lam and P.P. Nielsen, Exchange rings, exchange equations, and lifting properties, *Int. J. Algebra Comput.*, **26**(2016), 1177–1198.

[361] D. Khurana, T.Y. Lam and P.P. Nielsen, Exchange elements in rings, and the equation $XA - BX = I$, *Trans. Amer. Math. Soc.*, **369**(2017), 495–516.

[362] D. Khurana, T.Y. Lam, P.P. Nielsen and Y. Zhou, Uniquely clean elements in rings, *Comm. Algebra*, **43**(2015), DOI: 10.1080/00927872.2013.879158.

[363] D. Khurana, T.Y. Lam and Z. Wang, Rings of square range one, *J. Algebra*, **338**(2011), 122–143.

[364] D. Khurana and A.K. Srivastava, Right self-injective rings in which every element is a sum of two units, *J. Algebra Appl.*, **6**(2007), 281–286.

[365] D. Khurana and A.K. Srivastava, Unit sum numbers of right self-injective rings, *Bull. Aust. Math. Soc.*, **75**(2007), 355–360.

[366] D. Khurana, T.Y. Lam, P.P. Nielsen and J. Ster, Special clean elements in rings, *J. Algebra Appl.*, **19**(202), 2050208.

[367] L. Klingler, K.A. Loper, W.W. McGovern and M. Toeniskoetter, Semi-clean group rings, *J. Pure Appl. Algebra*, **225**(2021), 106744.

[368] M.L. Knox, R. Levy, W.W. McGovern and J. Shapiro, Generalizations of complemented rings with applications to rings of functions, *J. Algebra Appl.*, **8**(2009), 17–40.

[369] J.J. Koliha and P. Patrick, Elements of rings with equal spectral idempotents, *J. Austral. Math. Soc.*, **72**(2002), 137–152.

[370] M.T. Kosan, T.K. Lee and Y. Zhou, When is every matrix over a division ring a sum of an idempotent and a nilpotent? *Linear Algebra Appl.*, **450**(2014), 7–12.

[371] M.T. Kosan, T.C. Quynh and A.K. Srivastava, Rings with each right ideal automorphism-invariant, *J. Pure Applied Algebra* **220**(2016), 1525–1537.

[372] M.T. Kosan, S. Sahinkaya and Y. Zhou, On weakly clean rings, *Comm. Algebra*, **45**(2017), 3494–3502.

[373] M.T. Kosan, Z. Wang and Y. Zhou, Nil-clean and strongly nil-clean rings, *J. Pure Appl. Algebra*, **220**(2015), 633–646.

[374] M.T. Kosan, T. Yildirim and Y. Zhou, Rings whose elements are the sum of a tripotent and an element from the Jacobson radical, *Can. Math. Bull.*, **62**(2019), 810–821.

[375] M.T. Kosan and Y. Zhou, On weakly nil-clean rings, *Front. Math. China*, **11**(2011), 949–955.

[376] P.A. Krylov, Isomorphism of generalized matrix rings, *Algebra and Logic*, **47**(2008), 258–262.

[377] T.Y. Lam, *A First Course in Noncommutative Rings*, Springer-Verlag, New York, Heidelberg, Berlin, London, 2001.

[378] T.Y. Lam and R.G. Swan, Sums of alternating matrices and invertible matrices, *Contemporary Mathematics*, **419**(2006), 201–211.

[379] E.E. Lazerson, Onto inner derivations in division rings, *Bull. Amer. Math. Soc.*, **67**(1961), 356–358.

[380] T.K. Lee, Z. Yi and Y. Zhou, An example of Bergman's and the extension problem for clean rings, *Comm. Algebra*, **36**(2008), 1413–1418.

[381] T.K. Lee and Y. Zhou, A class of exchange rings, *Glasgow Math. J.*, **50** (2008), 509–522.

[382] T.K. Lee and Y. Zhou, A theorem on unit-regular rings, *Canad. Math. Bull.*, **53**(2010), 321–326.

[383] T.K. Lee and Y. Zhou, Clean index of rings, *Comm. Algebra*, **40**(2012), 807–822.

[384] T.K. Lee and Y. Zhou, Rings of clean index 4 and applications, *Comm. Algebra*, **41**(2013), 238–259.

[385] T.K. Lee and Y. Zhou, Modules which are invariant under automorphisms of their injective hulls, *J. Algebra Appl.*, 12, 1250159 (2013) [9 pages], DOI: 10.1142/S0219498812501599.

[386] T.K. Lee and Y. Zhou, From Boolean rings to clean rings, *Contemp. Math.*, **609**(2014), 223–232.

[387] Y. Lee, C. Huh and H.K. Kim, Questions on 2-primal rings, *Comm. Algebra*, **26**(1998), 595–600.

[388] Y. Lequain and A. Simis, Projective modules over $R[X_1, \ldots, X_n]$, R a Prüfer domain, *J. Pure Appl. Algebra*, **18**(1980), 165–171.

[389] J. Levitzki, On the structure of algebraic algebras and related rings, *Trans. Amer. Math. Soc.*, **74**(1953), 384–409.

[390] L.S. Levy, Almost diagonal matrices over Dedekind domains, *Math Z.*, **124**(1972), 89–99.

[391] J. Lewin, Subrings of finite index in finitely generated rings, *J. Algebra*, **5**(1967), 84–88.

[392] B. Li, Strongly clean matrix rings over noncommutative local rings, *Bull. Korean Math. Soc.*, **46**(2009), 71–78.

[393] B. Li and L. Feng, f-Clean rings and rings having many full elements, *J. Korean Math. Soc.*, **47**(2010), 247–261.

[394] C. Li and Y. Zhou, On strongly *-clean rings, *J. Algebra Appl.*, **10**(2011), 1363–1370.

[395] Y. Li, Strongly clean matrix rings over local rings, *J. Algebra*, **312**(2007), 397–404.

[396] Y. Li, M.M. Parmenter and P. Yuan, On *-clean group rings, *J. Algebra Appl.*, **14**(2015), 1550004 (11 pages), DOI: 10.1142/S0219498815500048.

[397] H. Lian and Q. Zeng, An extension of Cline's formula for a generalized Drazin inverse, *Turk. J. Math.*, **40**(2016), 161–165.

[398] Y. Liao, J. Chen and J. Cui, Cline's formula for the generalized Drazin inverse, *Bull. Malays. Math. Sci. Soc.*, **37**(2014), 37–42.

[399] X. Liu, S. Wu and Y. Yu, On the Drazin inverse of the sum of two matrices, *J. Appl. Math.*, 2011, DOI: 10.1155/2011/831892.

[400] D. Lu and W. Yu, On prime spectrums of commutative rings, *Comm. Algebra*, **34**(2006), 2667–2672.

[401] D. Lu and T. Wu, Topological characterizations of the extending property of rings, *Acta Math. Sci., Ser. B*, **30**(2010), 1086–1092.

[402] J. Luh, On the structure of J-rings, *Amer. Math. Monthly*, **74**(1967), 164–166.

[403] B. Ma, A Generalization of Weakly Periodic-like Rings and the Study of Commutativity, M.S. Thesis, Hanzghou Normal University, Hangzhou, 2014 (Chinese).

[404] D. MacHale, A remark on Boolean rings, *Math. Mag.*, **63**(1990), 248–249.

[405] N. Mahdou, A. Mimouni and M.A.S. Moutui, On PM-rings, rings of finite character and h-local rings, *J. Algebra Appl.*, **13**(2014), 1450018 [11 pages], DOI: 10.1142/S0219498814500182.

[406] G.D. Marco and A. Orsatti, Commutative rings in which every prime ideal is contained in a unique maximal ideal, *Pro. Amer. Math. Soc.*, **30**(1971), 459–466.

[407] G.T. Marks, 2-Primal Rings, Ph.D. Thesis, University of California, Berkeley, 2000.

[408] J. Matczuk, Conjugate (nil) clean rings and Köthe's problem, *J. Algebra Appl.*, May 2016, DOI: 10.1142/S0219498817500736.

[409] N.H. McCoy, Remarks on divisors of zero, *Amer. Math. Monthly*, **49**(1942), 286–295.

[410] B.R. McDonald, *Linear Algebras over Commutative Rings*, Marcel Dekker, Inc. New York & Basel, 1984.

[411] W.W. McGovern, Clean semiprime f-rings with bounded inversion, *Comm. Algebra*, **31**(2003), 3295–3304.

[412] W.W. McGovern, Neat rings, *J. Pure Appl. Algebra*, **205**(2006), 243–265.

[413] W.W. McGovern, A characterization of commutative clean rings, *Int. J. Math. Game Theory Algebra*, **15**(2006), 403–413.

[414] W.W. McGovern, Bézout rings with almost stable range 1, *J. Pure Appl. Algebra*, **212**(2008), 340–348.

[415] W.W. McGovern, S. Raja and A. Sharp, *Commutative nil clean group rings*, *J. Algebra Appl.*, **14**(2015), 1550094.

[416] W.W. McGovern and R. Raphael, Considering semi-clean rings of continuous functions, *Topology Appl.*, **190**(2015), 99–108.

[417] W.W. McGovern and F. Richman, When $R(X)$ and $R < x >$ are clean: a constructive treatment, *Comm. Algebra*, **43**(2015), 3389–3394.

[418] C. Meehan, Sums of automorphisms of free abelian groups and modules, *Proc. Roy. Irish Acad. Sect. A*, **104**(2004), 59–66.

[419] V.G. Miller and H. Zguitti, New extensions of Jacobson's lemma and Cline's formula, *Rend. Circ. Mat. Palermo, II. Ser.*, *67*(2018), 105–114.

[420] S.H. Mohamed and B.J. Müller, *Continuous and Discrete Modules*, Cambridge University Press, Cambridge, New York, Port Chester, Melbourne, Sydey, 1990.

[421] D. Mosic, Extensions of Jacobson's lemma for Drazin inverses, *Aequat. Math.*, **91**(2017), 419–428.

[422] H.G. Moore, Generalized n-like rings and commutativity, *Canad. Math. Bull.*, **23** (1980), 449–452.

[423] M.A. Mulero, Algebraic properties of rings of continuous functions, *Fundamenta Math.*, **149**(1996), 55–66.

[424] C.J. Mulvey, Representations of rings and modules, *Lect. Notes Math.*, **753**(1979), 542–585.

[425] A.R. Nasr-Isfahani, On a quotient of skew polynomial rings, *Comm. Algebra*, **41**(2013), 4520–4533.

[426] R.G. Nassif, Structure of Rings with Certain Conditions on Zero Divisors, M.S. Thesis, The American University of Beirut, Beirut, 2006.

[427] A.I. Nicholas, Clean Rings and Clean Group Rings, Ph.D. Thesis, Bowling Green State University, Bowling Green, 2013.

[428] W.K. Nicholson, Local group rings, *Canad. Math. Bull.*, **15**(1972), 137–138.

[429] W.K. Nicholson, Rings whose elements are quasi-regular or regular, *Aequations Mathematicae*, **9**(1973), 64–70.

[430] W.K. Nicholson, Lifting idempotents and exchange rings, *Trans. Amer. Math. Soc.*, **229**(1977), 269–278.

[431] W.K. Nicholson, On exchange rings, *Comm. Algebra*, **25**(1997), 1917–1918.

[432] W.K. Nicholson, Strongly clean rings and Fitting's lemma, *Comm. Algebra*, **27**(1999), 3583–3592.

[433] W.K. Nicholson and K. Varadarajan, Countable linear transformations are clean, *Proc. Am. Math. Soc.*, **126**(1998), 61–64.

[434] W.K. Nicholson, K. Varadarajan and Y. Zhou, Clean endomorphism rings, *Arch. Math.*, **83**(2004), 340–343.

[435] W.K. Nicholson and Y. Zhou, Rings in which elements are uniquely the sum of an idempotent and a unit, *Glasgow Math. J.*, **46**(2004), 227–236.

[436] W.K. Nicholson and Y. Zhou, Clean general rings, *J. Algebra*, **291**(2005), 297–311.

[437] W.K. Nicholson and Y. Zhou, Strongly lifting, *J. Algebra*, **285**(2005), 795–818.

[438] W.K. Nicholson and Y. Zhou, Clean rings: a survey, *Advances in ring theory*, 181–198, World Sci. Publ., Hackensack, NJ, 2005.

[439] W.K. Nicholson and Y. Zhou, Endomorphisms that are the sum of a unit and a root of a fixed polynomial, *Canad. Math. Bull.*, **49**(2006), 265–269.

[440] P.P. Nielsen and J. Šter, Connections between unit-regularity, regularity, cleanness, and strong cleanness of elements and rings, *Trans. Am. Math. Soc.*, **370**(2018), 1759–1782.

[441] A. Nikkhah, B. Davvaz and F.M.M. Ghaini, On 3×3 stromgly clean upper triangular matrices, *J. Math. Extension*, **10**(2016), 41–52.

[442] M. Ôhori, On strongly π–regular rings and periodic rings, *Math. J. Okayama Univ.*, **27**(1985), 49–52.

[443] H. Okamoto, J. Grosen and H. Komatsu, Some generalizations of Boolean rings, *Math. J. Okayama Univ.*, **31**(1989), 125–133.

[444] E.A. Osba, M. Henriksen and O. Alkam, Combining local and Von Neumann regular rings, *Comm. Algebra*, **32**(2004), 2639–2653.

[445] E. Pardo, Comparability, separativity, and exchange rings, *Comm. Algebra*, **24**(1996), 2915–2929.

[446] P. Patricio and R.E. Hartwig, The $(2,2,0)$ Drazin inverse problem, S*Linear Algebra Appl.*, **437**(2012), 2755–2772.

[447] K. Paykan, Cleanness of skew generalized power series rings, *Bull. Korean Math. Soc.*, **57**(2020), 1511–1528.

[448] S. Purkait, On strongly m-clean ring and m-semiperfect ring, *Comm. Algebra*, **48**(2020), 4531–4541.

[449] M.S. Putcha and A. Yaqub, Rings which are multiplicatively generated by idempotents, *Comm. Algebra*, **8**(1980), 153–159.

[450] K.N. Rajeswari and R. Aziz, Relative strong cleanness of upper triangular matrices, *Internet. J. Pure Appl. Math.*, **46**(2008), 691–695.

[451] K.N. Rajeswari and R. Aziz, A note on clean matrices in $M_2(\mathbb{Z})$, *Int. J. Algebra*, **3**(2009), 241–248.

[452] K.N. Rajeswari and R. Aziz, Strongly clean matrices in $M_2(\mathbb{Z})$: an intrinsic characterization, *Int. J. Math. Arch.*, **2**(2011), 1159–1166.

[453] K.N. Rajeswari and R. Aziz, Certain units in $M_2(\mathbb{Z})$ which are uniquely clean, *Res. J. Pure Algebra*, **1**(2011), 58–70.

[454] K.N. Rajeswari and U. Gupta, Characterization of 2×2 nil-clean matrices over integral domains, *J. Algebra Comb. Discrete Struct. Appl.*, **5**(2018), 117–127.

[455] C. Rickart, General theory of Banach algebras. The University Series in Higher Mathematics D. van Nostrand Co., Inc., Princeton, N.J.-Toronto-London-New York 1960.

[456] J. Rotman, *An Introduction to Homological Algebra*, Academic Press, New York, San Francisco, London, 1979.

[457] J. Rotman, *Galois Theory*, Springer-Verlag, New York, Berlin, Heidelberg, 1998.

[458] S.A.S. Sabeta and S. Razaghi, E-Clean matrices and unit-regular matrices, *J. Linear Topol. Algebra*, **1**(2013), 115–118.

[459] S. Sahebi and M. Jahandar, A note on uniquely (nil) clean ring, *J. Linear Topol. Algebra*, **1**(2012), 65–67.

[460] S. Sahebi and V. Rahmani, On $g(x)$-f-clean rings, *Palestine J. Math*, **3**(2014), 11–15.

[461] S. Sahinkara, G. Tang and Y. Zhou, Nil-clean group rings, *J. Algebra Appl.*, **16**(2017), 1750135 (7 pages), DOI: 10.1142/S0219498817501353.

[462] K. Samei, Clean elements in commutative reduced rings, *Comm. Algebra*, **32**(2004), 3479–3486.

[463] A.D. Sands, Radicals and Morita contexts, *J. Algebra*, **24**(1973), 335–345.

[464] R.P. Schwarz, Zero-Divisor Conditions in Commutative Group Rings, Ph.D. Thesis, University of Connecticut, 2011.

[465] M.O. Searcoid, Perturbation of linear operators by idempotents, *Irish Math. Soc. Bull.*, **39**(1997), 10–13.

[466] A. Shakoor, I. Ali, S. Wali and A. Rehman, Some formulas on the Drazin inverse for the sum of two matrices and block matrices, *Bull. Iran. Math. Soc.*, 2021, DOI: 10.1007/s41980-020-00521-3.

[467] V.P. Shchedryk, Commutative domains of elementary divisors and some properties of their elements, *Ukrainian Math. J.*, **64**(2012), 140–155.

[468] M. Sheibani and H. Chen, Elementary matrix reduction over J-stable rings, *Comm. Algebra*, **45**(2017), 1983–1995.

[469] M. Sheibani, R. Bahmani and H. Chen, Elementary matrix reduction over certain rings, *Comm. Algebra*, **45**(2017), 2613–2618.

[470] G. Shi, J. Chen, T. Li and M. Zhou, Jacobson's lemma and Cline's formula for generalized inverses in a ring with involution, *Comm. Algebra*, 2020, DOI: 10.1080/00927872.2020.1752226.

[471] D.R. Shifflet, Optimally Clean Rings, Ph.D. Thesis, Bowling Green State University, Bowling Green, 2011.

[472] J. Shiue and W.M. Chao, On the Boolean rings, *Yokohama Math. J.*, **24** (1976), 93–96.

[473] F. Siddique, Additive representations of elements in rings, Ph.D. Thesis, Saint Louis University, St. Louis, 2015.

[474] J.R. Silvester, *Introduction to Algebra K-Theory*, Campman and Hall, London and New York, 1981.

[475] A.K. Singh and B.M. Pandeya, A note on generalization of semiclean rings, *Int. J. Algebra*, **5**(2011), 1039–1047.

[476] S. Singh and A.K. Srivastava, Rings of invariant module type and automorphism-invariant modules, *Contemporary Mathematics*, **609**(2014), 299–311.

[477] G. Song and X. Guo, Diagonability of idempotent matrices over non-commutative rings, *Linear Algebra Appl.*, **297**(1999), 1–7.

[478] K.S. Sorokin, Completely decomposable abelian groups with clean endomorphism rings, *J. Math. Sci.*, **197**(2014), 655–657.

[479] K.S. Sorokin, SP-groups with clean endomorphism rings, *Vestn. Tomsk. Gos. Univ. Mat. Mekh.*, **30**(2014), 24–35.

[480] A.K. Srivastava, A survey of rings generated by units, *Ann. Fac. Sci. Toulouse Math.*, **19**(2010), 203–213.

[481] A.K. Srivastava, Rings Characterized by Properties of Direct Sums of Modules and on Rings Generated by Units, Ph.D. Thesis, Ohio University, Ohio, 2007.

[482] A.K. Srivastava, Additive representations of elements in rings: a survey, *Springer Proceedings in Mathematics & Statistics*, **174**(2016), 59–73.

[483] A. Stancu, A note on commutative weakly nil clean rings, *J. Algebra Appl.*, **15**(2016), DOI: 10.1142/S0219498816200012.

[484] A. Stancu, On some constructions of nil-clean, clean, and exchange rings, *J. Algebra Appl.*, **14**(2015), 1550101.

[485] A. Steger, Diagonability of idempotent matrices, *Pacific J. Math.*, **19** (1966), 535–542.

[486] J. Šter, Corner rings of a clean ring need not be clean, *Comm. Algebra*, **40**(2012), 1595–1604.

[487] J. Šter, Lifting units in clean rings, *J. Algebra*, **381**(2013), 200–208.

[488] J. Šter, Weakly clean rings, *J. Algebra*, **401**(2014), 1–12.

[489] J. Šter, The clean property is not a Morita invariant, *J. Algebra*, **420**(2014), 15–38.

[490] J. Šter, Nil clean involutions, arXiv: 1512.02277v2 [math.RA] 9 Dec 2015.

[491] J. Šter, Nil-clean quadratic elements, *J. Algebra Appl.*, **16**(2017), DOI: 10.1142/S0219498817501973.

[492] J. Šter, Nil-clean index of $M_n(F_2)$, *Linear Algebra Appl.*, **632**(2022), 294–307.

[493] J. Stock, On rings whose projective modules have the exchange property, *J. Algebra*, **103**(1986), 437–453.

[494] S.H. Sun, Noncommutative rings in which every prime ideal is contained in a unique maximal ideal, *J. Pure Appl. Algebra*, **76**(1991), 179–192.

[495] S.H. Sun, Rings in which every prime ideal is contained in a unique maximal right ideal, *J. Pure Appl. Algebra*, **78**(1992), 183–194.

[496] X. Sun and Y. Luo, On strongly semiclean rings, *Int. Math. Forum*, **5** (2010), 1715–1724.

[497] V. Swaminathan, On Foster's Boolean like rings, *Math. Seminar Notes*, **8**(1980), 347–367.

[498] V. Swaminathan, Structure of Boolean like rings, *Math. Seminar Notes*, **9**(1981), 471–495.

[499] V. Swaminathan, Submaximal ideals in a Boolean-like rings, *Math. Seminar Notes*, **10**(1982), 529–542.

[500] V. Swaminathan, Injective and projective Boolean-like rings, *J. Austral. Math. Soc. (Series A)*, **33**(1982), 40–49.

[501] B. Szczepara, Rings Generated by Their Units, M.S. Thesis, Concordia University, Montréal, 1979.

[502] M. Tamekkante, M. Chhiti and K. Louartiti, Generalization of almost clean rings, *Int. J. Algebra*, **5**(2011), 1213–1218.

[503] G. Tang, C. Li and Y. Zhou, Study of Morita contexts, *Comm. Algebra*, **42** (2014), 1668–1681.
[504] G. Tang and Y. Zhou, Strong cleanness of generalized matrix rings over a local ring, *Linear Algebra Appl.*, **437**(2012), 2546–2559.
[505] G. Tang and Y. Zhou, A class of formal matrix rings, *Linear Algebra Appl.*, **438**(2013), 4672–4688.
[506] G. Tang and Y. Zhou, An embedding theorem on triangular matrix rings, *Linear Multilinear Algebra*, **65**(2017), 882–890.
[507] Q.K. Tat, Weakly Clean Rings and Related Rings, Ph.D. Thesis, University of Malaya, Kualalumpur, 2015.
[508] R.F. Tichy and V. Ziegler, Units generating the ring of integers of complex cubic fields, *Colloq. Math.*, **109**(2007), 71–83.
[509] H. Tominaga, Some remarks on π-regular rings of bounded index, *Math. J. Okayama Univ.*, **4**(1955), 135–141.
[510] H. Tominaga and A. Yaqub, On generated n-like rings and related rings, *Math. J. Okayama Univ.*, **23**(1981), 199–202.
[511] M. Tousi and S. Yassemi, Tensor products of clean rings, *Glasgow Math. J.*, **47**(2005), 501–503.
[512] A.A. Tuganbaev, Semiregular, weakly regular, and π-regular rings, *J. Math. Sci.*, **109**(2002), 1509–1588.
[513] A.A. Tuganbaev, *Rings Close to Regular*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2002.
[514] A.A. Tuganbaev, Rings without infinite sets of noncentral orthogonal idempotents, *J. Math. Sci.*, **162**(2009), 730–739.
[515] A.A. Tuganbaev, Extensions of automorphisms of submodules, *J. Math. Sci.*, **206**(2015), 583–596.
[516] H.N.E. Turkey, Generalizations of Boolean rings, M.S. Thesis, American University of Beirut, Beirut, 2008.
[517] P. Vámos, 2-Good rings, *Quart. J. Math.*, **56**(2005), 417–430.
[518] P. Vámos, On rings whose nonunits are a unit multiple of a nilpotent, *J. Algebra Appl.*, **16**(2017), 1750140 (13 pages), DOI: 10.1142/S02194 98817501407.
[519] P. Vámos and S. Wiegand, Block diagonalization and 2-unit sums of matrices over Prüfer domain, *Trans. Amer. Math. Soc.*, **363**(2011), 4997–5020.
[520] K. Varadarajan, Study of Hopficity in certain classes of rings, *Comm. Algebra*, **28**(2000), 771–783.
[521] K. Varadarajan, Clean, almost clean, potent commutative rings, *J. Algebra Appl.*, **6**(2007), 671–685.
[522] L. Vaš, *-Clean rings; some clean and almost clean Baer *-rings and von Neumann algebras, *J. Algebra*, **324**(2010), 3388–3400.

[523] W.V. Vasconcelos, Injective endomorphism of finitely generated modules, *Proc. Amer. Math. Soc.*, **25** (1970), 900–901.

[524] L.N. Vaserstein, The stable range of rings and the dimension of topological spaces, *Functional Anal. Appl.*, **5**(1971), 102–110.

[525] L.N. Vaserstein and E. Wheland, Commutators and companion matrices over rings of stable rank 1, *Linear Algebra Appl.*, **142**(1990), 263–277.

[526] E.M. Vechtomov, Boolean ring, *Math. Notes*, **39**(1986), 101–103.

[527] E.M. Vechtomov, Rings of continuous functions and their maximal spectra, *Math. Notes*, **55**(1994), 32–49.

[528] X.L. Wang, Uniquely strongly clean group rings, *Comm. Math. Res.*, **28**(2012), 17–25.

[529] X.L. Wang and C. Li, Uniquely strongly *-clean rings, *J. Hubei Normal Univ.*, **38**(2016), 182–186.

[530] X.L. Wang and H. You, Cleanness of the group ring of an abelian p-group over a commutative ring, *Algebra Colloq.*, **19**(2012), 539–544.

[531] Y. Wang and Y. Ren, 2-Good rings and their extensions, *Bull. Korean Math. Soc.*, **50**(2013), 1711–1723.

[532] Z. Wang, A class of Drazin inverses in rings, *Filomat*, **31**(2017), 1781–1789.

[533] Z. Wang and J. Chen, On two open problems about strongly clean rings, *Bull. Austral. Math. Soc.*, **70**(2004), 279–282.

[534] Z. Wang and J. Chen, A note on clean rings, *Algebra Colloq.*, **14**(2007), 537–540.

[535] Z. Wang and J. Chen, 2-Clean rings, *Can. Math. Bull.*, **52**(2009), 145–153.

[536] Z. Wang and J. Chen, Pseudo Drazin inverses in associative rings and Banach algebras, *Linear Algebra Appl.*, **437**(2012), 1332–1345.

[537] Z. Wang, J. Chen and D. Khurana, Rings of idempotent stable range one, *Algebra Represent. Theory*, **15**(2012), 195–200.

[538] X. Wang and H. You, Cleanness of the group ring of an Abelian p-group over a commutative ring, *Algebra Colloq.*, **19**(2012), 539–544.

[539] W. Wanicharpichat and R. Wattanasirikoson, Idempotency, tripotency and nonsingularity of linear combinations of idempotent and square-zero matrices, *Proceedings The 8th Sciece Research Conference*, University of Phayao, May 2016.

[540] J.C. Wei, Weakly-abel rings and weakly exchange rings, *Acta Math. Hungar.*, **137**(2012), 254–262.

[541] Y. Wu, G. Tang, G. Deng and Y. Zhou, Nil-clean and unit-regular elements in certain subrings of $M_2(Z)$, *Czechoslovak Math. J.*, **69**(2019), 197–205.

[542] G. Xia, G. Tang and Y. Zhou, When is a matrix a sum of involutions or tripotents? *Commun. Algebra*, **49**(2021), 1717–1724.

[543] G. Xiao and W. Tong, n-clean rings, *Algebra Colloq.*, **13**(2006), 599–606.

[544] Q. Xu, C. Song and L. Zhang, Solvability of certain quadratic operator equations and representations of Drazin inverse, *Linear Algebra Appl.*, **439** (2013), 291–309.

[545] H. Yang and X. Liu, The Drazin inverse of the sum of two matrices and its applications, *J. Comput. Applied Math.*, **235**(2011), 1412–1417.

[546] X. Yang, Strongly Clean Rings and $g(x)$-Rings, Ph.D. Thesis, Memorial University of Newfoundland, Newfoundland, 2007.

[547] X. Yang, A survey of strongly clean rings, *Acta Appl. Math.*, **108**(2009), 157–173.

[548] X. Yang and Y. Zhou, Some families of strongly clean rings, *Linear Algebra Appl.*, **425** (2007), 119–129.

[549] X. Yang and Y. Zhou, Strong cleanness of the 2×2 matrix ring over a general local ring, *J. Algebra*, **320**(2008), 2280–2290.

[550] A. Yaqub, A note on p-like rings, *Math. Magazine*, **33**(1960), 287–290.

[551] A. Yaqub, A generalization of certain rings of A.L. Foster, *Canad. Math. Bull.*, **6**(1963), 55–60.

[552] A. Yaqub, Structure of certain periodic rings, *Canad. Math. Bull.*, **28** (1985), 120–123.

[553] A. Yaqub, Structure of weakly periodic rings with potent extended commutators, *Int. J. Math. Math. Sci.*, **25**(2001), 299–304.

[554] A. Yaqub, Weakly periodic-like rings and commutativity, *Studia Sci. Math. Hungarica*, **43**(2006), 275–284.

[555] A. Yaqub, Further generalizations of Boolean rings, *Int. Math. Forum*, **7**(2012), 2705–2714.

[556] Y. Ye, Semiclean rings, *Comm. Algebra*, **31**(2003), 5609–5625.

[557] Z.L. Ying, Clean Rings and Regular Local Rings, Ph.D. Thesis, Southeast University, Nanjing, 2008 (Chinese).

[558] Z.L. Ying and J.L Chen, On quasipolar rings, *Algebra Colloq.*, **19**(2012), 683–692.

[559] Z.L. Ying, T. Kosan and Y. Zhou, Rings in which every element is a sum of two tripotents, *Canad. Math. Bull.*, **59**(2016), 661–672.

[560] R. Yousefi and M. Dana, Generalizations of some conditions for Drazin inverses of the sum of two matrices, *Filomat*, **32**(2018), 6417–6430.

[561] H.P. Yu, On quasi-duo rings, *Glasgow Math. J.*, **37** (1995), 21–31.

[562] H.P. Yu, On Rings and Modules with the Exchange Property, Ph.D. Thesis, The University of Iowa, Iowa, 1996.

[563] H.P. Yu, On the structure of exchange rings, *Comm. Algebra*, **25**(1997), 661–670.

[564] P. Yu and X. Ou, Simple element and residue class ring of algebraic integer ring $\mathbb{Z}[\omega]$, *J. Xian Univ. Arts Sci.*, **10**(2007), 111–113 (Chinese).

[565] B.V. Zabavsky, Elementary reduction of matrices over adequate domain, *Math. Stud.*, **17**(2002), 115–116.

[566] B.V. Zabavsky, Diagonal Reduction of Matrices over Rings, Mathematical Studies Monograph Series, Vol. XVI, VNTL Publisher, 2012.

[567] B.V. Zabavsky, Diagonal reduction of matrices over finite stable range rings, *Math. Stud.*, **41**(2014), 101–108.

[568] B.V. Zabavsky, Conditions for stable range of an elementary divisor rings, *Comm. Algebra*, **45**(2017), 4062–4066.

[569] B.V. Zabavsky and S.I. Bilavska, Every zero adequate ring is an exchange ring, *J. Math. Sci.*, **187**(2012), 153–156.

[570] L. Zan, Researches and Characterizations on Some Group Rings, Ph.D. Thesis, Southeast University, Nanjing, 2007 (Chinese).

[571] D. Zelinsky, Every linear transformation is a sum of nonsingular ones, *Proc. Amer. Math. Soc.*, **5**(1954), 627–630.

[572] G. Zhuang, J. Chen and J. Cui, Jacobson's lemma for the generalized Drazin inverse, *Linear Algebra Appl.*, **436**(2012), 742–746.

[573] G. Zhang, F. Wang and W. Xu, Gelfand factor rings and weak Zariski topologies, *Comm. Algebra*, **35**(2007), 2628–2645.

[574] H. Zhang, On strongly clean modules, *Comm. Algebra*, **37**(2009), 1420–1427.

[575] H. Zhang and V. Camillo, On clean rings, *Comm. Algebra*, **44**(2016), 2475–2481.

[576] H. Zhang and W. Tong, Generalized clean rings, *J. Nanjing Univ., Math. Biquart.*, **22**(2005), 183–188.

[577] Y. Zhou, On clean group rings, Dinh Van Huynh (ed.) et al., Advances in ring theory. Papers of the conference on algebra and applications, Athens, OH, USA, June 18–21, 2008. Basel: Birkhäuser. Trends in Mathematics, 335–345(2010).

[578] Y. Zhou, Rings in which elements are sums of nilpotents, idempotents and tripotents, *J. Algebra Appl.*, **17**(2018), DOI: 10.1142/S0219498818500093.

[579] Y. Zhou and M. Ziembowski, On clean Laurent series rings, *J. Australian Math. Soc.*, **95**(2013), 421–427.

[580] H. Zhu, H. Zou and P. Patricio, Generalized inverses and their relations with clean decompositions, *J. Algebra Appl.*, **18**(2019), 1950133.

[581] V. Ziegler, The additive unit structure of complex biquadratic fields, *Glasnik Math.*, **43(63)**(2008), 293–307.

[582] B. Zimmermann-Huisgen and W. Zimmermann, Classes of modules with the exchange property, *J. Algebra*, **88**(1984), 416–434.

[583] H. Zou, D. Mosic and J. Chen, Generalized Drazin invertibility of the product and sum of two elements in a Banach algebra and its applications, *Turk. J. Math.*, **41**(2017), 548–563.

[584] H. Zou, D. Mosic and Y. Chen, The existence and representation of the Drazin inverse of a 2×2 block matrix over a ring, *J. Algebra Appl.*, 2019, 1950212 (23 pages), DOI: 10.1142/S0219498819502128.

Index

adequate range 1, 365
algebra
  algebraic, 3
  Banach, 262, 605, 609, 612, 623
  finitely dimensional, 3, 82
almost stable range 1, 365
anti-automorphism, 571
artinian primitive factor, 291
automorphism, 222, 325, 339

bimodule, 20, 35
bounded continuous functions, 538
bounded index, 196, 203
bounded linear operator, 637

Cayley-Hamilton Theorem, 145, 212
characteristic polynomial, 39, 209, 212
Cline's formula, 611, 616, 618, 627, 631
commutant, 605
  double, 605
continuous function, 493, 537
continuous map, 493

direct summand, 314
divisor, 289
domain
  algebraically closed integral, 338
  Bézout, 371
  Dedekind, 73
  integral, 16, 22, 397, 465
  principal ideal, 72
double commutant, 257
Drazin inverse, 612

element
  clean, 285, 301
  feebly clean, 565
  full, 492
  invertible, 343
  morphic, 317
  nil-clean, 375
  optimally clean, 29
  periodic, 400
  PM, 522
  pseudopolar, 257
  quasi-morphic, 317
  quasi-nilpotent, 262
  quasipolar, 263, 264
  regular, 331, 349
  special clean, 306
  strongly π-regular, 2
  strongly clean, 1
  strongly J-clean, 119
  strongly nil-clean, 175
  uniquely clean, 148
  uniquely nil-clean, 199
  unit-regular, 238
  unitary, 572
endomorphism, 8, 21, 126, 158, 209, 221, 227, 440

finite exchange property, 302
Fitting's Lemma, 220

g-Drazin inverse, 605, 614, 625
generalized Drazin inverse, 266
Generalized Nakayama's Lemma, 49
group
  abelian, 164
  cyclic, 220, 398, 400
  locally finite, 163
  solvable, 164

homomorphic image, 4, 286, 288, 404

ideal
  *-, 574
  maximal right ideal, 11
  maximal, 390, 497, 547
  maximal right (left), 184
  maximal two-sided, 185
  minimal prime, 153, 467
  Pierce, 155
  primary, 207
  prime, 122, 498, 518
  primitive, 291
idempotent, 22, 154, 164, 179, 183, 199, 289, 300, 408, 565
infinite direct product, 189
infinite residue field, 338
integral extension, 222
internal cancellation, 313
involution, 408, 571
isomorphism, 232

Jacobson radical, 4
Jordan-Chevalley decomposition, 175

lattice, 172
linear transformation, 325, 339
localization, 22, 581
locally finite p-group, 431
locally nilpotent, 4

matrix, 44
  e-clean, 294
  2-nil-clean, 384
  anti-triangular, 630
  column-finite, 326
  companion, 39, 42
  complex, 617
  cyclic, 42
  idempotent, 212
  invertible, 215, 332, 360
  operator, 625, 637
  row-finite and column-finite, 349
  strongly clean, 23, 25, 40, 61, 66, 87, 332
  strongly J-clean, 126
  strongly nil clean, 216
  strongly nil-clean, 209
  triangular, 88
  upper triangular, 109, 240
matrix unit, 165
module, 211, 220, 302, 489
  finitely generated, 231
  finitely generated projective, 12, 306, 308
  free, 333
  quasi-projective, 221
monic polynomial, 40, 211
Morita contex, 193
Morita context, 134, 167
Morita invariant, 18, 190, 306, 381
multiplicatively closed subset, 338, 447

nilpotency, 198
nilpotent, 179, 197, 199, 221, 377, 408

partial isometry, 572
partial order, 318
perspective, 314
polynomial, 12, 337
potent, 433
prime spectrum, 503
primitive criterion, 338
projection, 571, 579, 585

quasinilpotent, 605, 631

radical
  Brown-McCoy, 185

Jacobson, 22, 122, 141, 168, 193, 229, 280
nil, 186
prime, 142, 377
right T-nilpotent, 4
right iSRC factorization, 40
ring
DD_n-, 92
D^*-, 436
D_L-, 172
π-Rickart, 483
π-regular, 98, 240
gh-, 94
k-good, 331
n-clean, 345
n-regular, 240
*-, 571
*-Boolean, 585
2-Boolean, 387
2-good, 331
2-nil-clean, 331, 375
projective-free, 217
abelian, 6, 273, 426, 491, 493, 495, 497
abelian regular, 153
adequate, 289
almost clean, 480
artinian, 4, 82
Bézout, 72, 290
bleached, 80
Boolean, 171, 203, 232
Boolean-like, 196–198
clean, 285
cobleached, 259
commutative, 42, 72, 138, 141, 209, 222, 499
connected, 69
CS, 484
D-, 471
division, 11
elementary divisor, 331, 361, 368
elementary divisor ring, 363
endomorphism, 333
exchange, 4, 6, 184, 241, 278, 306, 447, 452
feckly clean, 524, 533
finite, 242
Gaussian integers, 161
generalized n-like, 441, 445
generalized matrix, 77, 194
group, 55, 67, 430, 487
Hensel, 49
Hermite, 368
Hilbert, 530
idempotent-free, 157
IU, 408
local, 11, 12, 23, 35, 57, 84, 90, 124, 139, 229, 245, 262, 282, 509
locally stable, 365
matrix, 77, 192, 205
morphic, 317, 324
nearly, 492
nil-clean, 179, 331, 375
nonsingular, 484
optimally clean, 29
perfect, 232
perfectly clean, 277
periodic, 180, 219, 441, 443, 458
perspective, 314, 349
pm, 498, 509
power series, 29, 151, 251
primitive, 441
projective-free, 66, 144, 218
pseudopolar, 257, 262
quasi-morphic, 318
quasipolar, 280, 282
regular, 308, 319
right (left) quasi-duo, 184
semilocal, 82
semipecfect, 82
separative, 308
simisimple, 4
skew power series, 158
strongly π-regular, 1, 5, 98, 200, 219, 232, 234
strongly $g(x)$-clean, 9
strongly J_n-clean, 239, 242, 253
strongly J_π-clean, 253, 254
strongly *-clean, 571, 594, 603
strongly 2-nil-clean, 380
strongly clean, 1, 6, 54, 98, 285
strongly J-clean, 119, 151

strongly nil-*-clean, 588
strongly nil-clean, 175, 179, 189, 190, 194, 195
strongly nil-clean ring, 199
strongly weakly nil-clean, 420
triangular, 205
triangular matrix, 77, 97, 134, 171
uniquely π-clean, 447
uniquely bleached, 168
uniquely clean, 148, 156, 158
uniquely D-nil-clean, 476
uniquely nil-clean, 200, 203, 208
uniquely strongly *-clean, 579, 587
uniquely strongly clean, 164, 166
uniquely weakly nil-clean, 427
unit-regular, 317, 320, 324
UU, 426
weak h-, 88
weakly bleached, 33
weakly Boolean, 413
weakly clean, 389
weakly cobleached, 280
weakly exchange, 426
weakly nil-clean, 389, 404
weakly periodic, 458, 462
root, 12, 140

self-adjoint, 601
set
*-clopen, 594
*-zero, 601
clopen, 500, 550
cozero, 602
finite orthogonal, 81
open, 499
zero, 541
space
*-, 594
Banach, 637
completely regular, 537
countably generated infinitely dimensional vector, 339
Hausdorff, 511, 537, 547
locally compact Hausdorff, 603
normal, 512
P-, 549
perfectly normal, 597
spectrum, 600
strongly *-zero-dimensional, 597
strongly zero-dimensional, 493
topological, 492, 537
Tychonoff, 537, 549
vector, 225
zero-dimensional, 500
square root, 285
stable range 1, 119, 219, 234, 338, 448
stable range 2, 361
Stone-Čech compactification, 546
strongly α-invariant, 265
subdirect product, 383, 413, 442, 462
submodule, 220, 227, 265, 302
subring, 156, 401, 549, 555, 585
subset, 559
summand, 357
summand intersection property, 357
summand sum property, 357

tripotent, 381, 386, 609
trivial extension, 576

unipotent, 408, 426
unit, 331, 426
unit closed subring, 339

vector space, 325

weak idempotent, 413

zero adequate, 290
zero-divisor, 436, 465, 468, 476
Zorn's Lemma, 155